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# Density of universal classes of series-parallel graphs

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A class of graphs  $\mathcal{C}$  ordered by the homomorphism relation is *universal* if every countable partial order can be embedded in  $\mathcal{C}$ . It was shown in [1] that the class  $\mathcal{C}_k$  of  $k$ -colorable graphs, for any fixed  $k \geq 3$ , induces a universal partial order. In [4], a surprisingly small subclass of  $\mathcal{C}_3$  which is a proper subclass of  $K_4$ -minor-free graphs ( $\mathcal{G}/K_4$ ) is shown to be universal. In another direction, a density result was given in [9], that for each rational number  $a/b \in [2, 8/3] \cup \{3\}$ , there is a  $K_4$ -minor-free graph with circular chromatic number equal to  $a/b$ . In this note we show for each rational number  $a/b$  within this interval the class  $\mathcal{K}_{a/b}$  of  $K_4$ -minor-free graphs with circular chromatic number  $a/b$  is universal if and only if  $a/b \neq 2, 5/2$  or  $3$ . This shows yet another surprising richness of the  $K_4$ -minor-free class that it contains universal classes as dense as the rational numbers.

**Keywords:** circular chromatic number, homomorphism, series-parallel graphs, universality

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## 1 Introduction

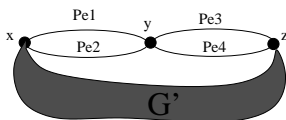
We assume graphs are finite and simple (with no loops and parallel edges). Let  $G, G'$  be graphs. A *homomorphism* from  $G$  to  $G'$  is a mapping  $f: V(G) \rightarrow V(G')$  which preserves adjacency. That is,  $\{u, v\} \in E(G)$  implies  $\{f(u), f(v)\} \in E(G')$ . We write  $G \leq G'$  if there is a homomorphism from  $G$  to  $G'$ . The notation  $G < G'$  means  $G \leq G' \not\leq G$ , whereas  $G \sim G'$  means  $G \leq G' \leq G$ . If  $G \sim G'$ , we say  $G$  and  $G'$  are *hom-equivalent*. The smallest graph  $H$  for which  $G \sim H$  is called the *core* of  $G$ . For finite graphs, the core is uniquely determined up to an isomorphism. It can also be seen that  $H$  is an induced subgraph of  $G$ . This will be denoted by  $H \subseteq G$ . Let  $\mathcal{C}$  and  $\mathcal{C}'$  be two classes of graphs. We also write  $\mathcal{C} \sim \mathcal{C}'$  if for each graph  $G \in \mathcal{C}$  there exists a  $G' \in \mathcal{C}'$  such that  $G \sim G'$  and vice versa. See [2] for introduction to graphs and their homomorphisms.

Let  $k \geq d \geq 1$  be integers. The *circular chromatic number* of  $G$ , written  $\chi_c(G)$ , is the smallest rational  $k/d$  such that  $G \leq K_{k/d}$ , where  $K_{k/d}$  is the *circular graph* with  $V(K_{k/d}) = \{0, 1, 2, \dots, k-1\}$  and

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**Fig. 1:** Unavoidable configuration of  $G$  (a minimal counterexample to Lemma 6) with odd girth  $2k+1$  and  $l_{e_1} + l_{e_2} = l_{e_3} + l_{e_4} = 2k+1$ .

$E(K_{k/d}) = \{\{i, j\} : d \leq |i - j| \leq k - d\}$ . Note that when  $d = 1$  we have the usual vertex coloring of  $G$ . Let  $\mathcal{K}_{a/b} = \{G \in \mathcal{G}/K_4 : \chi_c(G) = a/b\}$ . See [10] for some other equivalent definitions. It is trivial to see the following:

**Theorem 1**  $\mathcal{K}_2 \sim \{K_2\}$ .

It is well known that graphs in  $\mathcal{G}/K_4$  are 3-colorable. Hell and Zhu [3] have shown that triangle-free graphs in  $\mathcal{G}/K_4$  have circular chromatic number at most  $8/3$ . Hence no graph in  $\mathcal{G}/K_4$  has circular chromatic number in the interval  $(8/3, 3)$ . Hence, we have:

**Theorem 2**  $\mathcal{K}_3 \sim \{C_3\}$ .

The main results of this note are the following two theorems establishing nice dichotomy between universality and homomorphism finiteness of the class  $\mathcal{K}_{a/b}$ :

**Theorem 3**  $\mathcal{K}_{5/2} \sim \{C_5\}$ .

Somewhat surprisingly we show that Theorem 1, 2, and 3 cover all cases when  $\mathcal{K}_{a/b}$  is a finite set.

**Theorem 4**  $\mathcal{K}_{a/b}$  is universal if  $a/b \in (2, 5/2) \cup (5/2, 8/3]$ .

In section 2 we prove Theorem 3 using a folding lemma. In section 3 we prove Theorem 4.

## 2 $\mathcal{K}_{5/2}$ is equivalent to $\{C_5\}$

Let  $G$  be a graph. A *thread* in  $G$  is a path  $P \subseteq G$  such that the two endpoints of  $P$  have degree at least 3 and all internal vertices of  $P$  are degree 2 in  $G$ . We shall often use the fact that if  $P$  and  $P'$  are two edge-disjoint paths and if the lengths of  $P$  and  $P'$  have same parity such that  $P$  is a thread and has at least equal length as  $P'$ , then there is a homomorphism that maps  $P$  to  $P'$  sending the two ends of  $P$  to the two ends of  $P'$ . Such a homomorphism is said to *fold*  $P$  to  $P'$ . Let  $G$  be a graph and let  $G^s$  denote the multi-graph we obtain from  $G$  by “smoothing” all degree 2 vertices of  $G$ . For each edge  $e$  of  $G^s$ , let  $P_e$  denote the thread of  $G$  represented by  $e$  in  $G^s$ , and let  $l_e$  denote the length of  $P_e$ .

The following Folding Lemma for  $K_4$ -minor-free graphs is an analogy of the Folding Lemma of Klostermeyer and Zhang [6] for planar graphs. Its proof is easy (see [7]).

**Lemma 5 (Edge folding lemma)** *Let  $G \in \mathcal{G}/\{K_4\}$  be of odd girth  $2k+1$  and let  $e, e'$  be parallel edges in  $G^s$ , with common end vertices  $x, y$ . If  $G$  is not homomorphic to a strictly smaller graph of the same odd girth, then  $l_e + l_{e'} = 2k+1$ . Moreover,  $P_e \cup P_{e'}$  is the unique cycle of length  $2k+1$  containing both  $x$  and  $y$ .*

For short let  $K^m$  denote  $K_{(7+5m)/(3+2m)}$ . Recall that  $V(K^m) = \{0, 1, \dots, 6+5m\}$ .

**Lemma 6** Let  $G \in \mathcal{G}/\{K_4\}$  be of odd girth at least 7. Then  $\chi_c(G) \leq (7 + 5m)/(3 + 2m) < 5/2$ , for some  $m < |V(G)|/2$ .

**Proof:** Let  $G \in \mathcal{G}/\{K_4\}$  be a core of odd girth  $g \geq 7$ . It suffices to show  $G \leq K^m$  for some  $m \geq 0$ . Let  $\bar{G}^s$  be the graph we get by identifying parallel edges of  $G^s$ . Then,  $\bar{G}^s \in \mathcal{G}/K_4$  and so there exists a  $y \in V(\bar{G}^s)$  such that the degree  $\deg_{\bar{G}^s}(y) = 2$ . Then  $3 \leq \deg_{G^s}(y) \leq 4$  (here we use a parity argument that, the multiplicity of edges of  $G^s$  is at most two, as  $G$  is a core). By Lemma 5, and assuming  $G$  is a minimal counterexample we can get  $\deg_{G^s}(y) = 4$ . Hence, a configuration depicted in Figure 1 is unavoidable. Let  $G' = G - (\bigcup_{i=1}^4 P_{e_i} - \{x, z\})$ . By induction  $G' \leq K^m$ , for some  $m \geq 0$ . We can assume  $f(x) = 0$ . By investigating a few cases for values of  $f(y)$ , it is not hard to see  $G \leq K^{m+1}$ , contrary to assumption (see [7] for detailed proof).  $\square$  **Proof of Theorem 3:** Let  $G \in \mathcal{K}_{5/2}$  be

of odd girth  $g$ . Then  $G \leq C_5$ . By Lemma 6, we have  $g \leq 5$ . By Theorem 2,  $g > 3$ . Hence  $g = 5$  and so  $C_5 \leq G \leq C_5$ . Hence  $G \sim C_5$ . The converse is obvious since  $\chi_c(C_5) = 5/2$ .

### 3 Universal sets of $\mathcal{G}/K_4$ are dense in $(2, 5/2) \cup (5/2, 8/3]$

In this section we shall show that we obtain a universal class  $\mathcal{K}_{p/q} \subset \mathcal{G}/K_4$  for arbitrary  $p/q \in (2, 5/2) \cup (5/2, 8/3]$ . We use a graph  $G_{p/q}$  with  $\chi_c(G) = p/q$  as a generator of  $\mathcal{K}_{p/q}$ . We assume  $G_{p/q}$  has the following two properties:

- (P1)  $G_{p/q}$  is hom-equivalent neither to a cycle nor to a vertex.
- (P2) if  $G' \in \mathcal{G}/K_4$  satisfies (P1) and  $\chi_c(G') = p/q$ , then  $|V(G')| \geq |V(G)|$ .

**Lemma 7** Let  $G \in \mathcal{G}/K_4$  have properties (P1) and (P2). Then,  $G$  is 2-connected. Moreover,  $G$  is a core and it is not vertex-transitive.

**Proof:** Since the circular graph  $K_{k/d}$  is a vertex-transitive graph, for all  $k, d$ , we have  $\chi_c(G) = \max_i(\chi_c(H_i)), 1 \leq i \leq p$ , where each  $H_i$  is a 2-connected component of  $G$ . Here, (P2) implies that  $p = 1$  and so  $G$  is 2-connected. Next, note that any graph  $G \in \mathcal{G}/K_4$  is vertex-transitive if and only if  $G$  is an odd cycle or  $K_1$  or  $K_2$ . This is because all other 2-connected graphs in  $\mathcal{G}/K_4$  have at least one degree-2 vertex and one non-degree-2 vertex. Hence by (P1),  $G$  is not vertex-transitive. Moreover, by (P1) the core of  $G$  also is not vertex-transitive. By (P2), we deduce  $G$  is a core.  $\square$  For any rational

number  $p/q \in (2, 8/3]$ , Pan and Zhu have shown in [9] a recursive method of constructing a 2-connected graph  $G_{p/q}$  with  $\chi_c(G) = p/q$ . If  $p/q \neq (2k + 1)/k$  then  $G_{p/q}$  satisfies (P1). If  $p/q = (2k + 1)/k$ , the graph given in [9] is the cycle  $C_{2k+1}$  which is the natural candidate. Cycles do not satisfy (P1), hence we introduce a graph denoted by  $G^k$  of odd girth  $2k + 3$  as follows: Take a triangle and double each edge to obtain a multi-graph  $H$ . For  $i = 0, 1, 2$ , let  $\{e_1^i, e_2^i\}$ , be the three parallel pairs of edges of  $H$ . To obtain a thread of length  $k + 2$ , subdivide  $e_1^0$  and  $e_2^0$  each  $k + 1$  times. Next subdivide  $e_1^1$  and  $e_2^1$  each  $k$  times. Finally, subdivide  $e_1^2$  three times and  $e_2^2$ ,  $2k - 2$  times to obtain the graph  $G^k$ . We have:

**Lemma 8**  $\chi_c(G^k) = (2k + 1)/k$  for all  $k \geq 3$ .

**Proof:** It is easy to see that  $G^k \leq C_{2k+1}$ , and  $G^k \not\leq C_{2k+3}$ . Hence, we have  $(2k + 3)/(k + 1) < \chi_c(G^k) \leq (2k + 1)/k$ . Note that  $\gcd(4k + 4, 2k + 1) = 1$ . From basic number theory [8], using what is known as the *Farey sequence*, we can see that any rational strictly between  $(2k + 3)/(k + 1)$  and  $(2k + 1)/k$

has numerator  $a \geq 4k + 4$ . But then, if  $k \geq 3$  the circumference of  $G^k$  is  $4k + 3$ . It is well known [10] that the numerator  $a$  of a circular chromatic number  $a/b$  of a graph  $G$  is at most its circumference. We deduce  $\chi_c(G^k) = (2k + 1)/k$ .  $\square$

**Corollary 9** *For every rational number  $p/q \in (2, 5/2) \cup (5/2, 8/3]$  there is a graph  $G_{p/q}$  satisfying (P1) and (P2).*

Next we prove that  $\mathcal{K}_{p/q}$  inherits universality from the class  $\mathcal{P}$  of directed finite paths [5]. We take several isomorphic copies  $H_1, \dots, H_n$  of a fixed graph  $H$ , such that  $\chi_c(H) = p/q$  and construct a ‘path-like’ structured graph  $H'$  by identifying a vertex of  $H_i$  with a vertex of  $H_{i+1}$ . Then  $\chi_c(H') = \chi_c(H)$  because the circular graphs are vertex-transitive. We call such a construction  $K_1$ -concatenation. A more precise definition of ‘ $K_1$ -concatenation’ of a graph  $H$ :

Let  $P \in \mathcal{P}$  be an oriented path of length  $n \geq 1$ ,  $V(P) = \{v_1, v_2, \dots, v_{n+1}\}$ . Then either  $v_i v_{i+1}$  or  $v_{i+1} v_i \in E(P)$  (but not both). Let  $H$  be a graph and  $a, b \in V(H)$ . Let  $H_1, H_2, \dots, H_n$  be isomorphic copies of  $H$  and let  $a_i, b_i$  be the vertices of  $H_i$  corresponding to  $a$  and  $b$ . The  $K_1$ -concatenation of  $H$  by  $P$  is a graph  $P * (H, a, b)$  constructed as follows: For  $i = 1, \dots, n$ , if  $v_i v_{i+1} \in E(P)$ , choose  $b_i$  (otherwise choose  $a_i$ ). Then identify every chosen vertex of  $H_i$  with the unchosen vertex of  $H_{i+1}$ . To make the construction non-trivial, we choose  $a$  and  $b$  so that there is no automorphism sending  $a$  to  $b$  or  $b$  to  $a$ . If  $H$  satisfies (P1), then we know there exists such a pair.

**Lemma 10** *Let  $G_{p/q} \in \mathcal{G}/K_4$  satisfy (P1), (P2). Then,  $\mathcal{K}_{p/q}$  is universal.*

**Proof:** Since the class of oriented paths  $\mathcal{P}$  is universal we show for every  $P, P' \in \mathcal{P}$ , we have  $P \leq P'$  if and only if  $P * (G_{p/q}, a, b) \leq P' * (G_{p/q}, a, b)$ . This proves the lemma.

The forward implication is straightforward. To prove the reverse implication, let  $P, P' \in \mathcal{P}$  of length  $n$  and  $n'$  such that  $P$  is a core and suppose there exists a homomorphism  $f : P * (G_{p/q}, a, b) \rightarrow P' * (G_{p/q}, a, b)$ . Assume further that  $P$  is not an edge, since this case is trivial, and without loss of generality, assume that the first edge of  $P$  is directed forward. Let  $H_1, \dots, H_n$  and  $H'_1, \dots, H'_{n'}$  be isomorphic copies of  $G_{p/q}$ . First we show that  $f|_{H_i}$  is induced by an automorphism of  $G_{p/q}$  for each  $i$ . Suppose not. Then the image  $f(H_i)$  is connected. If  $f(H_i)$  is 2-connected then, it is isomorphic to  $G_{p/q}$ , since  $G_{p/q}$  is a core. Suppose  $f(H_i)$  is not 2-connected. Then each 2-connected component  $F$  of  $f(H_i)$  is a proper subgraph of  $G_{p/q}$ . By (P2), we have  $\chi_c(F) < p/q$ . Then,  $H \not\leq f(H_i)$ , a contradiction. Hence  $f|_{H_i}$  is induced by an automorphism of  $G_{p/q}$ .

Now we claim a stronger assertion that for any  $i$ ,  $f(a_i) = a'_j$  and  $f(b_i) = b'_j$ , for some  $j, 1 \leq j \leq m$ . let  $H'_j = f(H_1)$ . Since  $f|_{H_1}$  is an automorphism,  $f(b_1)$  must be in the same automorphism class of  $b'_j$ . Suppose that  $f(b_1) \neq b'_j$ , then  $f(b_1)$  is not a cut-vertex of  $P' * (G_{p/q}, a, b)$ . As  $b_1$  is a cut-vertex of  $P * (G_{p/q}, a, b)$ , it is identified with either  $a_2$  or  $b_2$ . If it were identified with  $a_2$ , then  $f|_{H_2}$  would be an automorphism of  $G_{p/q}$  such that  $f(a_2) = b'_j$ , contrary to the choice of  $a, b$  in  $V(G_{p/q})$ . Hence  $b_1$  is identified with  $b_2$ . Similarly, we get  $a_2$  is identified with  $a_3$ , and  $b_3$  with  $b_4$  and so on. This implies  $P$  is a ‘zig-zag’ which is hom-equivalent to an edge, contrary to  $P$  being a core. So  $f(b_1) = b'_j$ . The claim follows by induction on the length of  $P$ .

We define a homomorphism  $g : V(P) \rightarrow V(P')$  so that if  $f(a_i) = a'_j$  then  $g(v_i) = v'_j$  and similarly for  $b_i$ . By our construction  $g$  preserves the adjacency condition and so  $P \leq P'$ .  $\square$  **Proof of Theorem 4:** Let  $a/b \in (2, 5/2) \cup (5/2, 8/3]$ . By Lemma 9, there is a graph  $G_{a/b}$  with properties (P1),(P2). By Lemma 10,  $\mathcal{K}_{a/b}$  is universal. This concludes our result.

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