

Excluded subposets in the Boolean lattice

Gyula O.H. Katona

► **To cite this version:**

Gyula O.H. Katona. Excluded subposets in the Boolean lattice. Stefan Felsner. 2005 European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), 2005, Berlin, Germany. Discrete Mathematics and Theoretical Computer Science, DMTCS Proceedings vol. AE, European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), pp.229-230, 2005, DMTCS Proceedings. <hal-01184366>

HAL Id: hal-01184366

<https://hal.inria.fr/hal-01184366>

Submitted on 14 Aug 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Excluded subposets in the Boolean lattice

Gyula O.H. Katona^{1†}

¹ Rényi Institute, Budapest, Pf. 127, 1364, Hungary, e-mail: ohkatona@renyi.hu

We are looking for the maximum number of subsets of an n -element set not containing 4 distinct subsets satisfying $A \subset B, C \subset B, C \subset D$. It is proved that this number is at least the number of the $\lfloor \frac{n}{2} \rfloor$ -element sets times $1 + \frac{2}{n}$, on the other hand an upper bound is given with 4 replaced by the value 2.

Keywords: extremal problems, families of subsets

Let $[n] = \{1, 2, \dots, n\}$ be a finite set, families \mathcal{F}, \mathcal{G} , etc. of its subsets will be investigated. $\binom{[n]}{k}$ denotes the family of all k -element subsets of $[n]$. Let P be a poset. The goal of the present investigations is to determine the maximum size of a family $\mathcal{F} \subseteq 2^{[n]}$ which does not contain P as a (non-necessarily induced) subposet. This maximum is denoted by $\text{La}(n, P)$. In some cases two posets, say P_1, P_2 could be excluded. The maximum number of subsets is denoted by $\text{La}(n, P_1, P_2)$ in this case.

The easiest example is the case when P consist of two comparable elements. Then we are actually looking for the largest family without inclusion that is without two distinct members $F, G \in \mathcal{F}$ such that $F \subset G$. The well-known Sperner theorem ([4]) gives the answer, the maximum is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

We say that the distinct sets A, B_1, \dots, B_r form an r -fork if they satisfy $A \subset B_1, \dots, B_r$. A is called the *handle*, B_i s are called the *prongs* of the fork. On the other hand, the distinct sets A, B_1, \dots, B_r form an r -brush if they satisfy $B_1, \dots, B_r \subset A$. The r -forks and the r -brush are denoted by $F(r), B(r)$, respectively. An old theorem solves the problem when the 2-fork and the 2-brush are excluded.

Theorem 1 [3]

$$\text{La}(n, F(2), B(2)) = 2 \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

The optimal construction is the family

$$\mathcal{F} = \left\{ F : F \in \binom{[n-1]}{\lfloor \frac{n-1}{2} \rfloor} \right\} \cup \left\{ F \cup \{n\} : F \in \binom{[n-1]}{\lfloor \frac{n-1}{2} \rfloor} \right\}.$$

We have proved the following theorem in a paper appearing soon.

Theorem 2 [2] *Let $n \geq 3$. If the family $\mathcal{F} \subseteq 2^{[n]}$ contains no four distinct sets A, B, C, D such that $A \subset C, A \subset D, B \subset C, B \subset D$, then $|\mathcal{F}|$ cannot exceed the sum of the two largest binomial coefficients of order n , i.e., $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$.*

[†]The work was supported by the Hungarian National Foundation for Scientific Research (OTKA), grant numbers T037846 and T034702.

Following the suggestion of J.R. Griggs, such a family could be called a *butterfly-free meadow*. The optimal construction here is obvious, one can take all the subsets of sizes $\lfloor n/2 \rfloor$ and $\lfloor n/2 \rfloor + 1$.

In all of these cases the maximum size of the family is exactly determined. This is not true when the r -fork is excluded. In a paper under preparation A. De Bonis and the present author proved the following theorem.

Theorem 3 [1]

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{r}{n} + O\left(\frac{1}{n^2}\right) \right) \leq \text{La}(F(r+1)) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + 2\frac{r}{n} + O\left(\frac{\log n}{n^{3/2}}\right) \right).$$

A weaker version of the upper bound in this theorem was obtained in [5]: the constant in the second term was larger. There is still a gap between the lower and upper bounds in the second term: a factor 2. This however seems to be a serious difficulty. The best construction (lower bound) contains all sets in one level and a thinned next level.

Let the poset N consist of 4 elements illustrated here with 4 distinct sets satisfying $A \subset B, C \subset B, C \subset D$. We were not able to determine $\text{La}(n, N)$ for a long time. Recently, a new method jointly developed by J.R. Griggs, helped us to prove the following theorem.

Theorem 4

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{2}{n} + o\left(\frac{1}{n}\right) \right) \leq \text{La}(n, N) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{4}{n} + o\left(\frac{1}{n}\right) \right).$$

References

- [1] A. De Bonis and G.O.H. Katona, Excluded posets in the Boolean lattice, paper under preparation.
- [2] A. De Bonis, G.O.H. Katona, K.J. Swanepoel, Largest family without $A \cup B \subseteq C \cup D$, appearing in *J. Combin. Theory Ser. A*.
- [3] G.O.H. Katona and T. Tarján, Extremal problems with excluded subgraphs in the n -cube, *Lecture Notes in Math.* **1018**, 84-93.
- [4] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, *Math. Z.* **27**(1928), 544-548.
- [5] Hai Tran Thanh, An extremal problem with excluded subposets in the Boolean lattice, *Order* **15**(1998) 51-57.