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# Excluded subposets in the Boolean lattice

Gyula O.H. Katona<sup>1†</sup>

<sup>1</sup> Rényi Institute, Budapest, Pf. 127, 1364, Hungary, e-mail: ohkatona@renyi.hu

We are looking for the maximum number of subsets of an  $n$ -element set not containing 4 distinct subsets satisfying  $A \subset B, C \subset B, C \subset D$ . It is proved that this number is at least the number of the  $\lfloor \frac{n}{2} \rfloor$ -element sets times  $1 + \frac{2}{n}$ , on the other hand an upper bound is given with 4 replaced by the value 2.

**Keywords:** extremal problems, families of subsets

Let  $[n] = \{1, 2, \dots, n\}$  be a finite set, families  $\mathcal{F}, \mathcal{G}$ , etc. of its subsets will be investigated.  $\binom{[n]}{k}$  denotes the family of all  $k$ -element subsets of  $[n]$ . Let  $P$  be a poset. The goal of the present investigations is to determine the maximum size of a family  $\mathcal{F} \subseteq 2^{[n]}$  which does not contain  $P$  as a (non-necessarily induced) subposet. This maximum is denoted by  $\text{La}(n, P)$ . In some cases two posets, say  $P_1, P_2$  could be excluded. The maximum number of subsets is denoted by  $\text{La}(n, P_1, P_2)$  in this case.

The easiest example is the case when  $P$  consist of two comparable elements. Then we are actually looking for the largest family without inclusion that is without two distinct members  $F, G \in \mathcal{F}$  such that  $F \subset G$ . The well-known Sperner theorem ([4]) gives the answer, the maximum is  $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ .

We say that the distinct sets  $A, B_1, \dots, B_r$  form an  $r$ -fork if they satisfy  $A \subset B_1, \dots, B_r$ .  $A$  is called the *handle*,  $B_i$ s are called the *prongs* of the fork. On the other hand, the distinct sets  $A, B_1, \dots, B_r$  form an  $r$ -brush if they satisfy  $B_1, \dots, B_r \subset A$ . The  $r$ -forks and the  $r$ -brush are denoted by  $F(r), B(r)$ , respectively. An old theorem solves the problem when the 2-fork and the 2-brush are excluded.

**Theorem 1** [3]

$$\text{La}(n, F(2), B(2)) = 2 \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

The optimal construction is the family

$$\mathcal{F} = \left\{ F : F \in \binom{[n-1]}{\lfloor \frac{n-1}{2} \rfloor} \right\} \cup \left\{ F \cup \{n\} : F \in \binom{[n-1]}{\lfloor \frac{n-1}{2} \rfloor} \right\}.$$

We have proved the following theorem in a paper appearing soon.

**Theorem 2** [2] *Let  $n \geq 3$ . If the family  $\mathcal{F} \subseteq 2^{[n]}$  contains no four distinct sets  $A, B, C, D$  such that  $A \subset C, A \subset D, B \subset C, B \subset D$ , then  $|\mathcal{F}|$  cannot exceed the sum of the two largest binomial coefficients of order  $n$ , i.e.,  $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$ .*

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Following the suggestion of J.R. Griggs, such a family could be called a *butterfly-free meadow*. The optimal construction here is obvious, one can take all the subsets of sizes  $\lfloor n/2 \rfloor$  and  $\lfloor n/2 \rfloor + 1$ .

In all of these cases the maximum size of the family is exactly determined. This is not true when the  $r$ -fork is excluded. In a paper under preparation A. De Bonis and the present author proved the following theorem.

**Theorem 3** [1]

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \left( 1 + \frac{r}{n} + O\left(\frac{1}{n^2}\right) \right) \leq \text{La}(F(r+1)) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left( 1 + 2\frac{r}{n} + O\left(\frac{\log n}{n^{3/2}}\right) \right).$$

A weaker version of the upper bound in this theorem was obtained in [5]: the constant in the second term was larger. There is still a gap between the lower and upper bounds in the second term: a factor 2. This however seems to be a serious difficulty. The best construction (lower bound) contains all sets in one level and a thinned next level.

Let the poset  $N$  consist of 4 elements illustrated here with 4 distinct sets satisfying  $A \subset B, C \subset B, C \subset D$ . We were not able to determine  $\text{La}(n, N)$  for a long time. Recently, a new method jointly developed by J.R. Griggs, helped us to prove the following theorem.

**Theorem 4**

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \left( 1 + \frac{2}{n} + o\left(\frac{1}{n}\right) \right) \leq \text{La}(n, N) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left( 1 + \frac{4}{n} + o\left(\frac{1}{n}\right) \right).$$

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