

Largest cliques in connected supermagic graphs

Anna Lladó

► **To cite this version:**

Anna Lladó. Largest cliques in connected supermagic graphs. Stefan Felsner. 2005 European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), 2005, Berlin, Germany. Discrete Mathematics and Theoretical Computer Science, DMTCS Proceedings vol. AE, European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), pp.219-222, 2005, DMTCS Proceedings. <hal-01184371>

HAL Id: hal-01184371

<https://hal.inria.fr/hal-01184371>

Submitted on 14 Aug 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Largest cliques in connected supermagic graphs

A. Lladó[†]

Universitat Politècnica de Catalunya
Jordi Girona, 1, E-08034 Barcelona, Spain
allado@mat.upc.es

A graph $G = (V, E)$ is said to be *magic* if there exists an integer labeling $f : V \cup E \rightarrow [1, |V \cup E|]$ such that $f(x) + f(y) + f(xy)$ is constant for all edges $xy \in E$. Enomoto, Masuda and Nakamigawa proved that there are magic graphs of order at most $3n^2 + o(n^2)$ which contain a complete graph of order n . Bounds on Sidon sets show that the order of such a graph is at least $n^2 + o(n^2)$. We close the gap between those two bounds by showing that, for any given graph H of order n , there are connected magic graphs of order $n^2 + o(n^2)$ containing H as an induced subgraph. Moreover it can be required that the graph admits a supermagic labelling f , which satisfies the additional condition $f(V) = [1, |V|]$.

Keywords: Labelings of graphs, magic graphs, Sidon sets.

1 Introduction

A simple finite graph $G = (V, E)$ is said to be *magic* if there is a bijection $f : V \cup E \rightarrow [1, |V \cup E|]$ and a constant k such that $f(x) + f(y) + f(xy) = k$ for each edge $xy \in E$. This notion was introduced by Kotzig and Rosa [8] in 1966 under the name of *magic valuations*. When $f(V) = [1, |V|]$ then the graph is *supermagic*; see for instance [2, 10]. There are several related notions under the name of magic labellings; see the dynamic survey of Gallian [5].

An upper bound for the size of a magic graph containing a clique had been given already by Kotzig and Rosa [9], where they proved that, if $G = (V, E)$ is a magic graph containing a complete graph of order $n > 8$, then

$$|V| + |E| \geq n^2 - 5n + 14.$$

This result was improved by Enomoto, Masuda and Nakamigawa [3] to

$$|V| + |E| \geq 2n^2 - O(n^{3/2}), \tag{1}$$

by using the known bound for the size of a Sidon set given in ([4]).

[†]Supported by the Ministry of Science and Technology of Spain, and the European Regional Development Fund (ERDF) under project-BFM-2002-00412 and by the Catalan Research Council under grant 2001SGR-00258.

Recall that a set A of integers is said to be a Sidon set if all sums of pairs of elements (non necessarily different) of A , are pairwise distinct.

In 1941 Erdős and Turan [4] proved that a Sidon set $A \subset [1, N]$ always satisfies,

$$|A| \leq N^{1/2} + N^{1/4} + 1. \quad (2)$$

Kotzig [7] calls a set $A \subset \mathbb{Z}$ a *well spread sequence* if all sums of distinct elements in A are pairwise different. He showed that, if $A \subset [1, N]$ with $N \geq 8$, then $N \geq 4 + \binom{|A|-1}{2}$. Ruzsa [11] calls such a set a *weak Sidon set*. He gives a very nice short proof that a weak Sidon set in $[1, N]$ satisfies

$$|A| \leq N^{1/2} + 4N^{1/4} + 11. \quad (3)$$

If $A \subset V$ induces a clique in a magic graph $G = (V, E)$ with magic labelling f then $f(A)$ is a weak Sidon set. That is, for each pair of vertices $x, y \in A$, we have $f(x) + f(y) = k - f(xy)$, so that the sums of labels of pairs of vertices in A are pairwise distinct. Therefore $|A|$ is bounded by (3) with $N = |V \cup E|$, or $N = |V|$ if f is super magic.

We want to point out that in 1972 Kotzig using well spread sequences proved in a long paper that K_n is not magic for $n \geq 7$, [8]. The same result was reproved in 1999 by Craft and Tesar [1]. Note that inequality (3) shows directly this result for n large enough.

There are explicit constructions of (weak) Sidon sets whose cardinality is close to the upper bound in (3). For instance, for any prime p , Singer gives a construction of a Sidon set of cardinality $p + 1$ in $[1, N]$ with $N = p^2 + p + 1$ and Bose gives one of cardinality p with $N = p^2 - 1$; see for instance [6]. Ruzsa [11] gives also such a construction of a Sidon set with $p - 1$ elements in $[1, p^2 - p]$. Since for each positive integer n there is a prime p such that $p \leq n + o(n)$, these constructions provide a Sidon set of order n in $[1, N]$ with $N \leq n^2 + o(n^2)$; see for instance [3, Lemma 3].

The existence of dense Sidon sets provide the means to obtain lower bounds for the largest possible clique in a connected magic graph. By using the construction of Singer [12] for dense difference sets Enomoto, Masuda and Nakamigawa [3] show that, for any graph H with n vertices and m edges, there is a connected supermagic graph G which contains H as an induced subgraph such that

$$|V(G)| \leq 2m + 2n^2 + o(n^2). \quad (4)$$

In particular, there are supermagic graphs G containing the complete graph K_n , such that

$$|V(G)| \leq 3n^2 + o(n^2). \quad (5)$$

On the other hand, if G is a supermagic graph which contains a clique of order n , then (1) becomes

$$|V(G)| \geq n^2 - O(n^{3/2}), \quad (6)$$

so that there is a gap between these upper and lower bounds. Our main result, which closes the gap between the bounds (5) and (6), is the following.

Theorem 1 *Let $s(n)$ denote the minimum order of a connected supermagic graph containing a clique of order n . Then*

$$s(n) = n^2 + o(n^2).$$

The proof of Theorem 1 can be adapted to show the following improvement of inequality (4).

Theorem 2 *For any graph H with n vertices there is a connected supermagic graph G of order $N = n^2 + o(n^2)$ which contains H as an induced subgraph.*

2 Magic graphs from Sidon sets

We have already mentioned that the existence of a connected supermagic graph of order N containing a clique of order n implies the existence of a weak Sidon set of order n in $[1, N]$. We will show that these two facts are actually equivalent.

Recall that a set of positive integers A is a weak Sidon set if for any different elements $x, y, u, v \in A$, $x + y \neq u + v$. This is equivalent to say that $x - u \neq v - y$.

We first give a bound for a weak Sidon set A which is good for any $|A| \geq 3$. The proof is similar to Ruzsa [11, Theorem 4.7].

Lemma 1 *Let $A \subset [1, N]$ be a weak Sidon set with $|A| \geq 3$. Then*

$$N \geq \frac{|A|(|A| - 3)}{2} + 3 + \epsilon(|A|),$$

where $\epsilon(n) = 1$ for $n \geq 6$ and $\epsilon(n) = 0$ otherwise.

The following easy Lemma gives a simple criteria for a vertex labeling to extend to a super magic labeling.

Lemma 2 *Let $G = (V, E)$ be a graph of order n and $f : V \rightarrow [1, n]$ a bijection. Suppose that the edge sums of f , $\{f(x) + f(y), xy \in E\}$, form a consecutive set of integers. Then f can be extended to a supermagic labeling of G .*

Lemma 3 *Let $G = (V, E)$ be a supermagic connected graph of order N . For each $N' > N$ there is a supermagic connected graph G' which contains G as an induced subgraph.*

Lemma 4 *Let $G = (V, E)$ be a graph of order N and $f : V \rightarrow [1, N]$ a bijection such that the edge sums $f(x) + f(y)$, $xy \in E$ are pairwise different. Then there is a super magic graph G' of order N which contains G as a spanning subgraph.*

We are now ready for the proof of our main result.

Theorem 3 *There is a connected super magic graph of order N containing a clique of order n if and only if there is a weak Sidon set $A \subset [1, N]$ of cardinality n .*

Theorem 1 follows from Theorem 3 and the bounds described in the Introduction. Let $s(n)$ denote the minimum order of a supermagic graph containing a clique of order n . By (3) we have $s(n) \geq n^2 + o(n^2)$. On the other hand, the known constructions of dense Sidon sets together with results on the distribution of primes provide in particular weak Sidon sets of cardinality n in $[1, N]$ with $N = n^2 + o(n^2)$, see for instance [3, Lemma 3]. By Theorem 3 we then have $s(n) \leq n^2 + o(n^2)$.

Theorem 1 can be extended to construct connected supermagic graphs which contain any given graph H as induced subgraph.

Corollary 1 *Let H be a connected graph of order n . If there is a weak Sidon set $A \subset [1, N]$ of cardinality n then there exists a connected super magic graph G of order N which contains H as an induced subgraph.*

As a result of Corollary 1, there are connected supermagic graphs of order $N \leq n^2 + o(n^2)$ which contain a given graph H of order n as an induced graph. This proves Theorem 2.

References

- [1] D. Craft, E. H. Tesar, On a question of Erdős about edge-magic graphs, *Disc. Math.* **207** (1999) 271–276.
- [2] H. Enomoto, A.S. Lladó, T. Nakamigawa, G. Ringel, Super edge-magic graphs, *SUT J. Math.* **34** (1998) 105–109.
- [3] H. Enomoto, K. Masuda, T. Nakamigawa, Induced Graph Theorem on Magic Valuations, *Ars Combin.* **56** (2000), 25–32.
- [4] P. Erdős, P. Turán, On a problem of Sidon in additive number theory and some related problems, *J. of london Math. Soc.* **16** (1941), 212–215.
- [5] J.A. Gallian, A Dynamic Survey on Graph Labeling, *The Electronic Journal of Combinatorics*, DS6 (2000).
- [6] H. Halberstam, K.F. Roth, *Sequences*, Springer-Verlag, New York, 1983.
- [7] A. Kotzig, On well spread sets of integers, *Publications du CRM-161*, (1972) (83 pages)
- [8] A. Kotzig, A. Rosa, Magic valuations of finite graphs, *Canadian Mathematical Bulletin* **13** (4) (1970), 451–461.
- [9] A. Kotzig, A. Rosa, Magic Valuations of Complete Graphs, *Publications du CRM-175*, (1972) (8 pages)
- [10] G. Ringel, A.S. Lladó, Another Tree Conjecture, *Bull. Inst. Combin. Appl.* **18** (1996), 83–85.
- [11] I. Z. Ruzsa, Solving a linear equation in a set of integers, *Acta Arithmetica* LXV.3 (1993) 259–282.
- [12] J. Singer, A theorem in finite projective geometry and some applications to number theory, *Trans. Am. Math. Soc.* **43** (1938), 377–385.