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# Largest cliques in connected supermagic graphs

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A graph  $G = (V, E)$  is said to be *magic* if there exists an integer labeling  $f : V \cup E \rightarrow [1, |V \cup E|]$  such that  $f(x) + f(y) + f(xy)$  is constant for all edges  $xy \in E$ . Enomoto, Masuda and Nakamigawa proved that there are magic graphs of order at most  $3n^2 + o(n^2)$  which contain a complete graph of order  $n$ . Bounds on Sidon sets show that the order of such a graph is at least  $n^2 + o(n^2)$ . We close the gap between those two bounds by showing that, for any given graph  $H$  of order  $n$ , there are connected magic graphs of order  $n^2 + o(n^2)$  containing  $H$  as an induced subgraph. Moreover it can be required that the graph admits a supermagic labelling  $f$ , which satisfies the additional condition  $f(V) = [1, |V|]$ .

**Keywords:** Labelings of graphs, magic graphs, Sidon sets.

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## 1 Introduction

A simple finite graph  $G = (V, E)$  is said to be *magic* if there is a bijection  $f : V \cup E \rightarrow [1, |V \cup E|]$  and a constant  $k$  such that  $f(x) + f(y) + f(xy) = k$  for each edge  $xy \in E$ . This notion was introduced by Kotzig and Rosa [8] in 1966 under the name of *magic valuations*. When  $f(V) = [1, |V|]$  then the graph is *supermagic*; see for instance [2, 10]. There are several related notions under the name of magic labellings; see the dynamic survey of Gallian [5].

An upper bound for the size of a magic graph containing a clique had been given already by Kotzig and Rosa [9], where they proved that, if  $G = (V, E)$  is a magic graph containing a complete graph of order  $n > 8$ , then

$$|V| + |E| \geq n^2 - 5n + 14.$$

This result was improved by Enomoto, Masuda and Nakamigawa [3] to

$$|V| + |E| \geq 2n^2 - O(n^{3/2}), \tag{1}$$

by using the known bound for the size of a Sidon set given in ([4]).

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Recall that a set  $A$  of integers is said to be a Sidon set if all sums of pairs of elements (non necessarily different) of  $A$ , are pairwise distinct.

In 1941 Erdős and Turan [4] proved that a Sidon set  $A \subset [1, N]$  always satisfies,

$$|A| \leq N^{1/2} + N^{1/4} + 1. \quad (2)$$

Kotzig [7] calls a set  $A \subset \mathbb{Z}$  a *well spread sequence* if all sums of distinct elements in  $A$  are pairwise different. He showed that, if  $A \subset [1, N]$  with  $N \geq 8$ , then  $N \geq 4 + \binom{|A|-1}{2}$ . Ruzsa [11] calls such a set a *weak Sidon set*. He gives a very nice short proof that a weak Sidon set in  $[1, N]$  satisfies

$$|A| \leq N^{1/2} + 4N^{1/4} + 11. \quad (3)$$

If  $A \subset V$  induces a clique in a magic graph  $G = (V, E)$  with magic labelling  $f$  then  $f(A)$  is a weak Sidon set. That is, for each pair of vertices  $x, y \in A$ , we have  $f(x) + f(y) = k - f(xy)$ , so that the sums of labels of pairs of vertices in  $A$  are pairwise distinct. Therefore  $|A|$  is bounded by (3) with  $N = |V \cup E|$ , or  $N = |V|$  if  $f$  is super magic.

We want to point out that in 1972 Kotzig using well spread sequences proved in a long paper that  $K_n$  is not magic for  $n \geq 7$ , [8]. The same result was reproved in 1999 by Craft and Tesar [1]. Note that inequality (3) shows directly this result for  $n$  large enough.

There are explicit constructions of (weak) Sidon sets whose cardinality is close to the upper bound in (3). For instance, for any prime  $p$ , Singer gives a construction of a Sidon set of cardinality  $p + 1$  in  $[1, N]$  with  $N = p^2 + p + 1$  and Bose gives one of cardinality  $p$  with  $N = p^2 - 1$ ; see for instance [6]. Ruzsa [11] gives also such a construction of a Sidon set with  $p - 1$  elements in  $[1, p^2 - p]$ . Since for each positive integer  $n$  there is a prime  $p$  such that  $p \leq n + o(n)$ , these constructions provide a Sidon set of order  $n$  in  $[1, N]$  with  $N \leq n^2 + o(n^2)$ ; see for instance [3, Lemma 3].

The existence of dense Sidon sets provide the means to obtain lower bounds for the largest possible clique in a connected magic graph. By using the construction of Singer [12] for dense difference sets Enomoto, Masuda and Nakamigawa [3] show that, for any graph  $H$  with  $n$  vertices and  $m$  edges, there is a connected supermagic graph  $G$  which contains  $H$  as an induced subgraph such that

$$|V(G)| \leq 2m + 2n^2 + o(n^2). \quad (4)$$

In particular, there are supermagic graphs  $G$  containing the complete graph  $K_n$ , such that

$$|V(G)| \leq 3n^2 + o(n^2). \quad (5)$$

On the other hand, if  $G$  is a supermagic graph which contains a clique of order  $n$ , then (1) becomes

$$|V(G)| \geq n^2 - O(n^{3/2}), \quad (6)$$

so that there is a gap between these upper and lower bounds. Our main result, which closes the gap between the bounds (5) and (6), is the following.

**Theorem 1** *Let  $s(n)$  denote the minimum order of a connected supermagic graph containing a clique of order  $n$ . Then*

$$s(n) = n^2 + o(n^2).$$

The proof of Theorem 1 can be adapted to show the following improvement of inequality (4).

**Theorem 2** *For any graph  $H$  with  $n$  vertices there is a connected supermagic graph  $G$  of order  $N = n^2 + o(n^2)$  which contains  $H$  as an induced subgraph.*

## 2 Magic graphs from Sidon sets

We have already mentioned that the existence of a connected supermagic graph of order  $N$  containing a clique of order  $n$  implies the existence of a weak Sidon set of order  $n$  in  $[1, N]$ . We will show that these two facts are actually equivalent.

Recall that a set of positive integers  $A$  is a weak Sidon set if for any different elements  $x, y, u, v \in A$ ,  $x + y \neq u + v$ . This is equivalent to say that  $x - u \neq v - y$ .

We first give a bound for a weak Sidon set  $A$  which is good for any  $|A| \geq 3$ . The proof is similar to Ruzsa [11, Theorem 4.7].

**Lemma 1** *Let  $A \subset [1, N]$  be a weak Sidon set with  $|A| \geq 3$ . Then*

$$N \geq \frac{|A|(|A| - 3)}{2} + 3 + \epsilon(|A|),$$

where  $\epsilon(n) = 1$  for  $n \geq 6$  and  $\epsilon(n) = 0$  otherwise.

The following easy Lemma gives a simple criteria for a vertex labeling to extend to a super magic labeling.

**Lemma 2** *Let  $G = (V, E)$  be a graph of order  $n$  and  $f : V \rightarrow [1, n]$  a bijection. Suppose that the edge sums of  $f$ ,  $\{f(x) + f(y), xy \in E\}$ , form a consecutive set of integers. Then  $f$  can be extended to a supermagic labeling of  $G$ .*

**Lemma 3** *Let  $G = (V, E)$  be a supermagic connected graph of order  $N$ . For each  $N' > N$  there is a supermagic connected graph  $G'$  which contains  $G$  as an induced subgraph.*

**Lemma 4** *Let  $G = (V, E)$  be a graph of order  $N$  and  $f : V \rightarrow [1, N]$  a bijection such that the edge sums  $f(x) + f(y)$ ,  $xy \in E$  are pairwise different. Then there is a super magic graph  $G'$  of order  $N$  which contains  $G$  as a spanning subgraph.*

We are now ready for the proof of our main result.

**Theorem 3** *There is a connected super magic graph of order  $N$  containing a clique of order  $n$  if and only if there is a weak Sidon set  $A \subset [1, N]$  of cardinality  $n$ .*

Theorem 1 follows from Theorem 3 and the bounds described in the Introduction. Let  $s(n)$  denote the minimum order of a supermagic graph containing a clique of order  $n$ . By (3) we have  $s(n) \geq n^2 + o(n^2)$ . On the other hand, the known constructions of dense Sidon sets together with results on the distribution of primes provide in particular weak Sidon sets of cardinality  $n$  in  $[1, N]$  with  $N = n^2 + o(n^2)$ , see for instance [3, Lemma 3]. By Theorem 3 we then have  $s(n) \leq n^2 + o(n^2)$ .

Theorem 1 can be extended to construct connected supermagic graphs which contain any given graph  $H$  as induced subgraph.

**Corollary 1** *Let  $H$  be a connected graph of order  $n$ . If there is a weak Sidon set  $A \subset [1, N]$  of cardinality  $n$  then there exists a connected super magic graph  $G$  of order  $N$  which contains  $H$  as an induced subgraph.*

As a result of Corollary 1, there are connected supermagic graphs of order  $N \leq n^2 + o(n^2)$  which contain a given graph  $H$  of order  $n$  as an induced graph. This proves Theorem 2.

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