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A vertex coloring of a graph G is *k-nonrepetitive* if one cannot find a periodic sequence with k blocks on any simple path of G. The minimum number of colors needed for such coloring is denoted by $\pi_k(G)$. This idea combines graph colorings with Thue sequences introduced at the beginning of 20th century. In particular Thue proved that if G is a simple path of any length greater than 4 then $\pi_2(G) = 3$ and $\pi_3(G) = 2$. We investigate $\pi_k(G)$ for other classes of graphs. Particularly interesting open problem is to decide if there is, possibly huge, k such that $\pi_k(G)$ is bounded for planar graphs.

Let $k \geq 2$ be a fixed integer. A coloring f of the vertices of a graph G is *k-repetitive* if there is $n \geq 1$ and a simple path $v_1v_2...v_{kn}$ of G such that $f(v_i) = f(v_j)$ whenever $i - j$ is divisible by n. Otherwise f is called *k-nonrepetitive*. The minimum number of colors needed for a k-nonrepetitive coloring of G is denoted by $\pi_k(G)$. Notice that any 2-nonrepetitive coloring must be proper in the usual sense, while this is not necessarily the case for $k \geq 3$.

By the 1906 theorem of Thue [6] $\pi_2(G) \leq 3$ and $\pi_3(G) \leq 2$ if G is a simple path of any length. Let $\pi_k(d)$ denote the supremum of $\pi_k(G)$, where G ranges over all graphs with $\Delta(G) \leq d$. A simple extension of probabilistic arguments from [2] (for $k = 2$) shows that there are absolute positive constants c_1 and c_2 such that

$$c_1 \frac{d^{k/(k-1)}}{(\log d)^{1/(k-1)}} \leq \pi_k(d) \leq c_2 d^{k/(k-1)}.$$

Moreover, one can show that for each d there exists a sufficiently large $k = k(d)$ such that $\pi_k(d) \leq d+1$. On the other hand, any $\lfloor d/2 \rfloor$-coloring of a d-regular graph of girth at least $2k + 1$ is k-repetitive. The maximum number $t(d)$ such that for each k there is a d-regular graph G with $\pi_k(G) > t(d)$ is not known for $d \geq 3$.

Kündgen and Pelsmajer [4] and Barát and Varjú [3] proved independently that $\pi_2(G)$ is bounded for graphs of bounded treewidth. By the result of Robertson and Seymour [5] it follows that if H is any fixed planar graph then $\pi_k(G)$ is bounded for graphs not containing H as a minor. However, it is still not known whether there are some constants k and c such that $\pi_k(G) \leq c$ for any planar graph G. The least possible constant c for which this could hold (with possibly huge k) is $c = 4$.

In a weaker version of the problem we ask for nonrepetitive colorings of subdivided graphs. By the result of Thue every graph has a (sufficiently large) subdivision which is nonrepetitively 5-colorable (for any $k \geq 2$). Clearly this cannot happen for all graphs if we restrict the number of vertices added to an edge. For instance, any c-coloring of the complete graph K_n, with each edge subdivided by at most r vertices, is 2-repetitive if $c < \log_r \log_2(n/r)$. The question if there are constants c, k, and r such that each planar graph G has an r-restricted subdivision S with $\pi_k(S) \leq c$, is open.

There are many interesting connections of this area to other graph coloring topics. Let $s(G)$ be the *star chromatic number* of a graph G, that is, the least number of colors in a proper coloring of the vertices of G, with additional property that every two color classes induce a star forest. It is not hard to see that $\pi_2(G) \geq s(G)$ for any graph G. Hence, by the results of Albertson et al. [1] it follows that there are planar graphs with $\pi_2(G) \geq 10$, and for each t there are graphs of treewidth t with $\pi_2(G) \geq \binom{t+1}{2}$.

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