



**HAL**  
open science

## Decomposable graphs and definitions with no quantifier alternation

Oleg Pikhurko, Joel Spencer, Oleg Verbitsky

► **To cite this version:**

Oleg Pikhurko, Joel Spencer, Oleg Verbitsky. Decomposable graphs and definitions with no quantifier alternation. 2005 European Conference on Combinatorics, Graph Theory and Applications (Euro-Comb '05), 2005, Berlin, Germany. pp.25-30, 10.46298/dmtcs.3423 . hal-01184380

**HAL Id: hal-01184380**

**<https://inria.hal.science/hal-01184380>**

Submitted on 14 Aug 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Decomposable graphs and definitions with no quantifier alternation

Oleg Pikhurko<sup>1†</sup>, Joel Spencer<sup>2</sup> and Oleg Verbitsky<sup>3‡</sup>

<sup>1</sup>Department of Math Sciences, Carnegie Mellon University, Pittsburgh, PA 15213

<sup>2</sup>Courant Institute, New York University, New York, NY 10012

<sup>3</sup>Institut für Informatik, Humboldt Universität Berlin, Berlin, D-10099

Let  $D(G)$  be the minimum quantifier depth of a first order sentence  $\Phi$  that defines a graph  $G$  up to isomorphism in terms of the adjacency and the equality relations. Let  $D_0(G)$  be a variant of  $D(G)$  where we do not allow quantifier alternations in  $\Phi$ . Using large graphs decomposable in complement-connected components by a short sequence of serial and parallel decompositions, we show examples of  $G$  on  $n$  vertices with  $D_0(G) \leq 2 \log^* n + O(1)$ . On the other hand, we prove a lower bound  $D_0(G) \geq \log^* n - \log^* \log^* n - O(1)$  for all  $G$ . Here  $\log^* n$  is equal to the minimum number of iterations of the binary logarithm needed to bring  $n$  below 1.

**Keywords:** descriptive complexity of graphs, first order logic, Ehrenfeucht game on graphs, graph decompositions

## 1 Introduction

Given a finite graph  $G$ , how succinctly can we describe it using first order logic and the laconic language consisting of the adjacency and the equality relations? A first order sentence  $\Phi$  *defines*  $G$  if  $\Phi$  is true precisely on graphs isomorphic to  $G$ . All natural succinctness measures of  $\Phi$  are of interest: the *length*  $L(\Phi)$  (a standard encoding of  $\Phi$  is supposed), the *quantifier depth*  $D(\Phi)$  which is the maximum number of nested quantifiers in  $\Phi$ , and the *width*  $W(\Phi)$  which is the number of variables used in  $\Phi$  (different occurrences of the same variable are not counted). All the three characteristics inherently arise in the analysis of the computational problem of checking if a  $\Phi$  is true on a given graph [3]. They give us a small hierarchy of descriptive complexity measures for graphs:  $L(G)$  (resp.  $D(G)$ ,  $W(G)$ ) is the minimum  $L(\Phi)$  (resp.  $D(\Phi)$ ,  $W(\Phi)$ ) of a  $\Phi$  defining  $G$ . These graph invariants will be referred to as the *logical length, depth, and width* of  $G$ . We have  $W(G) \leq D(G) \leq L(G)$ . The former number is of relevance for graph isomorphism testing, see [2].  $W(G)$  and  $D(G)$  admit a purely combinatorial characterization in terms of the Ehrenfeucht game, see [2, 8].

We here address the logical depth of a graph. We focus on the following general question: How do restrictions on logic affect the descriptive complexity of a graph? Call a first order sentence  $\Phi$  to be

<sup>†</sup>Partially supported by NSF Grant DMS 0457512.

<sup>‡</sup>Supported by an Alexander von Humboldt fellowship.

$a$ -alternation if it contains negations only in front of relation symbols and every sequence of nested quantifiers in  $\Phi$  has at most  $a$  quantifier alternations. Let  $D_a(G)$  denote a variant of  $D(G)$  for  $a$ -alternation defining sentences, so  $D(G) \leq D_{a+1}(G) \leq D_a(G)$ . The logic of 0-alternation sentences is most restrictive and provably weaker than the unbounded first order logic. Whereas the problem of deciding if a first order sentence is satisfiable by some graph is unsolvable, it becomes solvable if restricted to 0-alternation sentences (the latter due to Ramsey's logical work [7] founding the combinatorial Ramsey theory).

It is not hard to observe that  $D_0(G) \leq n + 1$  where  $n$  denotes the number of vertices in  $G$ . This bound is in general best possible as  $D(K_n) = n + 1$ . Nevertheless, it admits a non-obvious improvement under a rather small restriction on the automorphism group of  $G$ . If the latter does not contain any transposition of two vertices, then  $D_1(G) \leq (n + 5)/2$ , see [6]. No sublinear improvement is possible because of the sequence of asymmetric graphs with  $W(G) = \Omega(n)$  constructed in [2]. In [4] we prove that  $D(G) = \log_2 n - \Theta(\log_2 \log_2 n)$  and  $D_0(G) \leq (2 + o(1)) \log_2 n$  for almost all  $G$ .

After obtaining these worst-case and average-case results, we undertake a "best-case" analysis in [5]. We define the *succinctness function*  $q(n) = \min \{D(G) : G \text{ has order } n\}$  and show that its values may be superrecursively small if compared to  $n$ :  $f(q(n)) \geq n$  for no recursive  $f$ . A weaker but still surprising succinctness result is also obtained for the fragment of first order logic with no quantifier alternation. Let  $q_0(n) = \min \{D_0(G) : G \text{ has order } n\}$ .

**Theorem 1**  $q_0(n) \leq 2 \log^* n + O(1)$  for infinitely many  $n$ .

In [5] this theorem is proved by considering  $G$  in a certain class of asymmetric trees and estimating  $D_0(G)$  in terms of the radius of a tree. We here reprove this result by showing the same definability phenomenon in a different class of graphs. We consider  $G$  in a class of graphs with small complement-connected induced subgraphs and estimate  $D_0(G)$  in terms of the number of the *serial* and *parallel decompositions* [1] decomposing  $G$  in the complement-connected components.

We also present a new result complementing Theorem 1.

**Theorem 2**  $q_0(n) \geq \log^* n - \log^* \log^* n - O(1)$  for all  $n$ .

As a consequence,  $q_0(n) \leq f(q(n))$  for no recursive  $f$ , which also shows a superrecursive gap between the graph invariants  $D(G)$  and  $D_0(G)$ .

## 2 Definitions

We use the following notation:  $V(G)$  is the vertex set of a graph  $G$ ;  $\text{diam } G$  is the diameter of  $G$ ;  $\overline{G}$  is the complement of  $G$ ;  $G \sqcup H$  is the disjoint union of graphs  $G$  and  $H$ ;  $G \subset H$  means that  $G$  is isomorphic to an induced subgraph of  $H$  (we will say that  $G$  is *embeddable* in  $H$ );  $G \sqsubset H$  means that  $G$  is isomorphic to the union of some of the connected components of  $H$ .

We call  $G$  *complement-connected* if both  $G$  and  $\overline{G}$  are connected. An inclusion-maximal complement-connected induced subgraph of  $G$  will be called a *complement-connected component* of  $G$  or, for brevity, *cocomponent* of  $G$ . Cocomponents have no common vertices and partition  $V(G)$ .

The *decomposition* of  $G$ , denoted by  $\text{Dec } G$ , is the set of all connected components of  $G$  (this is a set of graphs, not just isomorphism types). Furthermore, given  $i \geq 0$ , we define the *depth  $i$  decomposition* of  $G$  by  $\text{Dec}_0 G = \text{Dec } G$  and  $\text{Dec}_{i+1} G = \bigcup_{F \in \text{Dec}_i G} \text{Dec } \overline{F}$ . Note that  $P_i = \{V(F) : F \in \text{Dec}_i G\}$  is a partition of  $V(G)$  and that  $P_{i+1}$  refines  $P_i$ . The *depth  $i$  environment* of a vertex  $v \in V(G)$ , denoted by  $\text{Env}_i(v)$ , is the  $F$  in  $\text{Dec}_i G$  containing  $v$ .

We define the *rank* of a graph  $G$ , denoted by  $rk\ G$ , inductively as follows: (1) If  $G$  is complement-connected, then  $rk\ G = 0$ . (2) If  $G$  is connected but not complement-connected, then  $rk\ G = rk\ \overline{G}$ . (3) If  $G$  is disconnected, then  $rk\ G = 1 + \max\{rk\ F : F \in Dec\ G\}$ . In other terms,  $rk\ G$  is the smallest  $k$  such that  $P_{k+1} = P_k$  or such that  $P_k$  consists of  $V(F)$  for all cocomponents  $F$  of  $G$ .

In the *Ehrenfeucht game* on two disjoint graphs  $G$  and  $H$  two players, Spoiler and Duplicator, alternately select vertices of the graphs, one vertex per move. Spoiler starts and is always free to move in any of  $G$  and  $H$ ; Then Duplicator must choose a vertex in the other graph. Let  $x_i \in V(G)$  and  $y_i \in V(H)$  denote the vertices selected by the players in the  $i$ -th round. Duplicator wins the  $k$ -round game if the component-wise correspondence between  $x_1, \dots, x_k$  and  $y_1, \dots, y_k$  is a partial isomorphism from  $G$  to  $H$ ; Otherwise the winner is Spoiler. In the *0-alternation game* Spoiler plays all the game in the same graph he selects in the first round.

Assume  $G \not\cong H$ . Let  $D(G, H)$  (resp.  $D_0(G, H)$ ) denote the minimum  $D(\Phi)$  over (resp. 0-alternation) first order sentences  $\Phi$  that are true on one of the graphs and false on the other. The Ehrenfeucht theorem relates  $D(G, H)$  and the length of the Ehrenfeucht game on  $G$  and  $H$ . We will use the following version of the theorem:  $D_0(G, H)$  is equal to the minimum  $k$  such that Spoiler has a winning strategy in the  $k$ -round 0-alternation Ehrenfeucht game on  $G$  and  $H$ . It is also useful to know that  $D_0(G) = \max\{D_0(G, H) : H \not\cong G\}$ .

We define the tower-function by  $Tower(0) = 1$  and  $Tower(i) = 2^{Tower(i-1)}$  for each subsequent  $i$ .

### 3 Upper bound: Proof of Theorem 1

**Lemma 1** *Consider the Ehrenfeucht game on graphs  $G$  and  $H$ . Let  $x, x' \in V(G)$ ,  $y, y' \in V(H)$  and assume that the pairs  $x, y$  and  $x', y'$  are selected by the players in the same rounds. Furthermore, assume that  $Env_l(x) \neq Env_l(x')$ ,  $Env_l(y) = Env_l(y')$ , and  $diam\ Env_i(y) \leq 2$  for every  $i \leq l$ . Then Spoiler can win in at most  $l + 1$  rounds ( $l$  rounds if  $G$  is connected), playing all the time in  $H$ .*

**Proof:** We proceed by induction on  $l$ . The base case is  $l = 0$  if  $G$  is disconnected and  $l = 1$  if  $G$  is connected. If  $y$  and  $y'$  are adjacent in  $Env_l(y)$ , Duplicator has already lost. Otherwise, Spoiler uses the fact that  $diam\ Env_l(y) = 2$  and selects  $y''$  adjacent in  $Env_l(y)$  to both  $y$  and  $y'$ . Duplicator cannot do so with any  $x''$  because  $x$  and  $x'$  are in different components of  $G$  if  $l = 0$  or  $\overline{G}$  if  $l = 1$ .

Assume that  $l \geq 1$ . Let  $0 \leq m \leq l$  be the minimum number such that  $x' \notin Env_m(x)$ . If  $m < l$ , Spoiler wins in the next  $m + 1 \leq l$  moves by induction. If  $m = l$ , Spoiler uses the same trick as in the base case and forces Duplicator to make a move  $x''$  outside  $Env_{l-1}(x)$ . By the induction hypothesis, Spoiler needs  $l$  extra moves to win.  $\square$

As long as Duplicator avoids meeting the conditions of Lemma 1 (in particular, selects  $x' \in Env_l(x)$  whenever Spoiler selects  $y' \in Env_l(y)$ ), we will say that she *beware of the environment threat*.

Let  $rk\ G = k$ . We call  $G$  *uniform* if  $Dec_{k-1}\ G$  contains no complement-connected graph, that is, every cocomponent appears in  $Dec_k\ G$  and no earlier. We call  $G$  *inclusion-free* if the following two conditions are true for every  $i < k$ : (1) For any  $K \in Dec_i\ G$ ,  $\overline{K}$  contains no isomorphic connected components. (2) If two elements  $K$  and  $M$  of  $Dec_i\ G$  are non-isomorphic, then neither  $\overline{K} \sqsubset \overline{M}$  nor  $\overline{M} \sqsubset \overline{K}$ .

**Lemma 2 (Main Lemma)** *Let  $G$  be a uniform inclusion-free graph. Suppose that every cocomponent of  $G$  has exactly  $c$  vertices. Then  $D_0(G) \leq 2\ rk\ G + c + 1$ .*

**Proof:** Let  $rkG = k$ . Fix a graph  $H \not\cong G$ . We will design a strategy allowing Spoiler to win the 0-alternation Ehrenfeucht game on  $G$  and  $H$  in at most  $2k + c + 1$  moves. Since  $D_0(G) = D_0(\overline{G})$ , without loss of generality we will assume that  $G$  is connected. Since the case of  $k = 0$  is trivial, we will also assume that  $k \geq 1$ .

*Case 1:  $H$  has a cocomponent  $C$  non-embeddable in any cocomponent of  $G$ .* If  $C$  has no more than  $c$  vertices, Spoiler selects all  $C$ . Otherwise he selects  $c + 1$  vertices spanning a complement-connected subgraph in  $C$  (it is not hard to show that this is always possible). If Duplicator's response  $A$  is within a cocomponent of  $G$ , then  $C \not\cong A$  by the assumption. Otherwise  $A$  is not complement-connected and Duplicator loses anyway.

In the sequel we will assume that Duplicator beware of the environment threat during all game.

*Case 2:  $G \subset H$  or there are  $l \leq k$  and  $A \in Dec_l G$  properly embeddable in some  $B \in Dec_l H$ , and not Case 1.* Spoiler plays in  $H$ . If  $G \subset H$ , set  $A = G$ ,  $B = H$ , and  $l = 0$ . Let  $H_0$  be a copy of  $A$  in  $B$ . At the first move Spoiler selects an arbitrary  $y_0 \in V(B) \setminus V(H_0)$ . Denote Duplicator's response in  $G$  by  $x_0$  and set  $G_0 = Env_l(x_0)$ . From now on Spoiler plays in  $H_0$ . Since we are not in Case 1,  $B$  is not a cocomponent of  $H$  and hence  $diam B \leq 2$ . Since Duplicator is supposed to beware of the environment threat, from now on she is forced to play in  $G_0$ .

*Subcase 2.1:  $G_0 \not\cong H_0$ .* Assume that  $l < k$  (the case of  $l = k$  will be covered by the last phase of the strategy). Since  $G_0$  and  $H_0$  are non-isomorphic copies of elements of  $Dec_l G$  and  $G$  is inclusion-free, Spoiler is able to make his next choice  $y_1$  in some  $H_1 \in Dec_l \overline{H_0}$  absent in  $Dec_l \overline{G_0}$ . Denote Duplicator's response in  $G_0$  by  $x_1$  and set  $G_1 = Env_{l+1}(x_1)$ . Note that  $G_1$  and  $H_1$  are non-isomorphic copies of elements of  $Dec_{l+1} G$ . Playing in the same fashion in the subsequent  $k - l - 1$  rounds, Spoiler finally achieves the players' moves in some non-isomorphic  $G_{k-l} \in Dec_k G$  and  $H_{k-l}$ , the latter being a copy of an element of  $Dec_k G$ . Both the graphs have  $c$  vertices. Now Spoiler selects the  $c - 1$  remaining vertices of  $H_{k-l}$  and wins whatever Duplicator's response is.

*Subcase 2.2:  $G_0 \cong H_0$ .* Though the graphs are isomorphic, the crucial fact is that  $G_0$ , unlike  $H_0$ , contains a selected vertex. By the definition of an inclusion-free graph, every automorphism of  $A \cong G_0 \cong H_0$  takes each cocomponent onto itself. Therefore every isomorphism between  $G_0$  and  $H_0$  matches cocomponents of these graphs in the same way. Let  $Y$  be the counterpart of  $Env_k(x_0)$  in  $H_0$  with respect to this matching. In the second round Spoiler selects an arbitrary  $y_1$  in  $Y$ . Denote Duplicator's answer by  $x_1$ . If  $x_1 \in Env_k(x_0)$ , Spoiler selects all  $Y$  and wins. Otherwise there is  $m \leq rk A$  such that  $Env_m(x_1)$  in  $G_0$  and  $Env_m(y_1)$  in  $H_0$  are non-isomorphic. This allows Spoiler to apply the strategy of Subcase 2.1.

*Case 3: Neither Case 1 nor Case 2.* Spoiler plays in  $G_0 = G$ . His first move  $x_0$  is arbitrary. Denote Duplicator's response in  $H$  by  $y_0$  and set  $H_0 = Env_0(y_0)$ . Since we are not in Case 2,  $G_0 \not\subset H_0$ . As  $G_0$  is inclusion-free,  $\overline{G_0}$  has a connected component  $G_1$  with no isomorphic copy in  $\overline{H_0}$ . Spoiler selects  $x_1$  arbitrarily in  $G_1$ . Let Duplicator respond with  $y_1$  somewhere in  $H_0$  and denote  $H_1 = Env_1(y_1)$ . Thus  $G_1 \not\cong H_1$  and  $G_1 \not\subset H_1$ , the latter again because we are not in Case 2. In the next round Spoiler again selects a vertex in a component  $G_2$  of  $\overline{G_1}$  absent in  $\overline{H_1}$ . Continuing in the same fashion, Spoiler finally forces playing the game on some  $G_m \in Dec_m G_0$  and  $H_m \in Dec_m H_0$  with  $G_m \not\subset H_m$  under one of the two terminal conditions: (1)  $m < k$  and  $H_m$  (or its complement) is a cocomponent of  $H$ . (2)  $m = k$ . In the first case note that, as we are not in Case 1,  $H_m$  is embeddable in some cocomponent of  $G$  (or its complement) and hence has at most  $c$  vertices. Therefore it suffices for Spoiler to select altogether  $c + 1$  vertices in  $G_m$  to win (recall the assumption that Duplicator beware of the environment threat and hence cannot move outside  $H_m$ ). In the second case  $G_m$  is a cocomponent of  $G$  and hence has  $c$  vertices. Spoiler selects all  $G_m$ . Since Duplicator's response must be complement-connected, she is forced to play

within a cocomponent of  $H_m$  and hence loses.

*Length of the game.* The above strategy allows Spoiler to win in at most  $k + c$  moves under the condition that Duplicator beware of the environment threat. If Duplicator ignores this threat, Spoiler needs  $k + 1$  additional moves according to Lemma 1.  $\square$

Let  $R_0$  consist of all complement-connected graphs of order 5. Assume that  $R_{i-1}$  is already specified. Fix  $F_i$  to be the family of all  $\lfloor |R_{i-1}|/2 \rfloor$ -element subsets of  $R_{i-1}$ . Define  $R_i$  to be the set of the complements of  $\bigsqcup_{G \in S} G$  for all  $S$  in  $F_i$ . Note that  $R_i$  consists of inclusion-free uniform graphs of rank  $i$  whose cocomponents all have 5 vertices. All graphs in  $R_i$  have the same order; Denote it by  $N_i$ . Let  $M_i = |R_i|$ . By the construction,

$$M_{i+1} = \binom{M_i}{\lfloor M_i/2 \rfloor} = \sqrt{\frac{2 + o(1)}{\pi M_i}} 2^{M_i} \quad \text{and} \quad N_{i+1} = \lfloor M_i/2 \rfloor N_i > M_i.$$

A simple estimation shows that  $N_i \geq \text{Tower}(i - O(1))$ . To complete the proof of Theorem 1, choose  $G_i$  in  $R_i$ . Using Main Lemma, we obtain  $q_0(N_i) \leq D_0(G_i) \leq 2i + 6 \leq 2 \log^* N_i + O(1)$ .

## 4 Lower bound: Proof-sketch of Theorem 2

Let  $L_a(G)$  denote the minimum length of an  $a$ -alternation sentence defining  $G$ .

**Lemma 3**  $L_a(G) \leq \text{Tower}(D_a(G) + \log^* D_a(G) + O(1))$ .

An analog of this lemma for  $L(G)$  and  $D(G)$  appears in [5] but its proof does not work under restrictions on the alternation number. The proof of Lemma 3 will appear in the full version.

Given  $n$ , denote  $k = q_0(n)$  and fix a graph  $G$  on  $n$  vertices such that  $D_0(G) = k$ . By Lemma 3,  $G$  is definable by a 0-alternation  $\Phi$  of length at most  $\text{Tower}(k + \log^* k + O(1))$ . Using the standard reduction, we convert  $\Phi$  to an equivalent prenex  $\exists^* \forall^*$ -sentence  $\Psi$  (i.e. existential quantifiers in  $\Psi$  all precede universal quantifiers). Since the reduction does not increase the total number of quantifiers,  $D(\Psi) \leq L(\Phi)$ . It is well known and easy to prove that, if a prenex  $\exists^* \forall^*$ -sentence  $\Psi$  is true on some structure, then it is true on some structure of order at most  $D(\Psi)$ . Since the  $\Psi$  is true only on  $G$ , we have  $n \leq D(\Psi) \leq L(\Phi) \leq \text{Tower}(k + \log^* k + O(1))$ , which proves the theorem.

## References

- [1] A. Brandstädt, V. B. Le, J. P. Spinrad. *Graph classes: a survey*. SIAM Monographs (1999).
- [2] J.-Y. Cai, M. Fürer, N. Immerman. An optimal lower bound on the number of variables for graph identification. *Combinatorica* 12:389–410 (1992).
- [3] E. Grädel. Finite model theory and descriptive complexity. In: *Finite Model Theory and Its Applications*. Texts in Theoretical Computer Science, an EATCS Series. Springer (2005).
- [4] J. H. Kim, O. Pikhurko, J. Spencer, O. Verbitsky. How complex are random graphs in first order logic? *Random Structures and Algorithms* 26:119–145 (2005).
- [5] O. Pikhurko, J. Spencer, O. Verbitsky. Succinct definitions in the first order theory of graphs. Accepted for publication in *Annals of Pure and Applied Logic*. ([arxiv.org/abs/math.LO/0401307](https://arxiv.org/abs/math.LO/0401307))

- [6] O. Pikhurko, H. Veith, O. Verbitsky. First order definability of graphs: tight bounds on quantifier rank. *Submitted* (2004) ([arxiv.org/abs/math.CO/0311041](https://arxiv.org/abs/math.CO/0311041))
- [7] F. Ramsey. On a problem of formal logic. *Proc. London Math. Soc. 2-nd series* 30:264–286 (1930).
- [8] J. Spencer. *The strange logic of random graphs*. Springer Verlag (2001).