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# Decomposable graphs and definitions with no quantifier alternation

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Let  $D(G)$  be the minimum quantifier depth of a first order sentence  $\Phi$  that defines a graph  $G$  up to isomorphism in terms of the adjacency and the equality relations. Let  $D_0(G)$  be a variant of  $D(G)$  where we do not allow quantifier alternations in  $\Phi$ . Using large graphs decomposable in complement-connected components by a short sequence of serial and parallel decompositions, we show examples of  $G$  on  $n$  vertices with  $D_0(G) \leq 2 \log^* n + O(1)$ . On the other hand, we prove a lower bound  $D_0(G) \geq \log^* n - \log^* \log^* n - O(1)$  for all  $G$ . Here  $\log^* n$  is equal to the minimum number of iterations of the binary logarithm needed to bring  $n$  below 1.

**Keywords:** descriptive complexity of graphs, first order logic, Ehrenfeucht game on graphs, graph decompositions

## 1 Introduction

Given a finite graph  $G$ , how succinctly can we describe it using first order logic and the laconic language consisting of the adjacency and the equality relations? A first order sentence  $\Phi$  *defines*  $G$  if  $\Phi$  is true precisely on graphs isomorphic to  $G$ . All natural succinctness measures of  $\Phi$  are of interest: the *length*  $L(\Phi)$  (a standard encoding of  $\Phi$  is supposed), the *quantifier depth*  $D(\Phi)$  which is the maximum number of nested quantifiers in  $\Phi$ , and the *width*  $W(\Phi)$  which is the number of variables used in  $\Phi$  (different occurrences of the same variable are not counted). All the three characteristics inherently arise in the analysis of the computational problem of checking if a  $\Phi$  is true on a given graph [3]. They give us a small hierarchy of descriptive complexity measures for graphs:  $L(G)$  (resp.  $D(G)$ ,  $W(G)$ ) is the minimum  $L(\Phi)$  (resp.  $D(\Phi)$ ,  $W(\Phi)$ ) of a  $\Phi$  defining  $G$ . These graph invariants will be referred to as the *logical length, depth, and width* of  $G$ . We have  $W(G) \leq D(G) \leq L(G)$ . The former number is of relevance for graph isomorphism testing, see [2].  $W(G)$  and  $D(G)$  admit a purely combinatorial characterization in terms of the Ehrenfeucht game, see [2, 8].

We here address the logical depth of a graph. We focus on the following general question: How do restrictions on logic affect the descriptive complexity of a graph? Call a first order sentence  $\Phi$  to be

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$a$ -alternation if it contains negations only in front of relation symbols and every sequence of nested quantifiers in  $\Phi$  has at most  $a$  quantifier alternations. Let  $D_a(G)$  denote a variant of  $D(G)$  for  $a$ -alternation defining sentences, so  $D(G) \leq D_{a+1}(G) \leq D_a(G)$ . The logic of 0-alternation sentences is most restrictive and provably weaker than the unbounded first order logic. Whereas the problem of deciding if a first order sentence is satisfiable by some graph is unsolvable, it becomes solvable if restricted to 0-alternation sentences (the latter due to Ramsey's logical work [7] founding the combinatorial Ramsey theory).

It is not hard to observe that  $D_0(G) \leq n + 1$  where  $n$  denotes the number of vertices in  $G$ . This bound is in general best possible as  $D(K_n) = n + 1$ . Nevertheless, it admits a non-obvious improvement under a rather small restriction on the automorphism group of  $G$ . If the latter does not contain any transposition of two vertices, then  $D_1(G) \leq (n + 5)/2$ , see [6]. No sublinear improvement is possible because of the sequence of asymmetric graphs with  $W(G) = \Omega(n)$  constructed in [2]. In [4] we prove that  $D(G) = \log_2 n - \Theta(\log_2 \log_2 n)$  and  $D_0(G) \leq (2 + o(1)) \log_2 n$  for almost all  $G$ .

After obtaining these worst-case and average-case results, we undertake a "best-case" analysis in [5]. We define the *succinctness function*  $q(n) = \min \{D(G) : G \text{ has order } n\}$  and show that its values may be superrecursively small if compared to  $n$ :  $f(q(n)) \geq n$  for no recursive  $f$ . A weaker but still surprising succinctness result is also obtained for the fragment of first order logic with no quantifier alternation. Let  $q_0(n) = \min \{D_0(G) : G \text{ has order } n\}$ .

**Theorem 1**  $q_0(n) \leq 2 \log^* n + O(1)$  for infinitely many  $n$ .

In [5] this theorem is proved by considering  $G$  in a certain class of asymmetric trees and estimating  $D_0(G)$  in terms of the radius of a tree. We here reprove this result by showing the same definability phenomenon in a different class of graphs. We consider  $G$  in a class of graphs with small complement-connected induced subgraphs and estimate  $D_0(G)$  in terms of the number of the *serial* and *parallel decompositions* [1] decomposing  $G$  in the complement-connected components.

We also present a new result complementing Theorem 1.

**Theorem 2**  $q_0(n) \geq \log^* n - \log^* \log^* n - O(1)$  for all  $n$ .

As a consequence,  $q_0(n) \leq f(q(n))$  for no recursive  $f$ , which also shows a superrecursive gap between the graph invariants  $D(G)$  and  $D_0(G)$ .

## 2 Definitions

We use the following notation:  $V(G)$  is the vertex set of a graph  $G$ ;  $\text{diam } G$  is the diameter of  $G$ ;  $\overline{G}$  is the complement of  $G$ ;  $G \sqcup H$  is the disjoint union of graphs  $G$  and  $H$ ;  $G \subset H$  means that  $G$  is isomorphic to an induced subgraph of  $H$  (we will say that  $G$  is *embeddable* in  $H$ );  $G \sqsubset H$  means that  $G$  is isomorphic to the union of some of the connected components of  $H$ .

We call  $G$  *complement-connected* if both  $G$  and  $\overline{G}$  are connected. An inclusion-maximal complement-connected induced subgraph of  $G$  will be called a *complement-connected component* of  $G$  or, for brevity, *cocomponent* of  $G$ . Cocomponents have no common vertices and partition  $V(G)$ .

The *decomposition* of  $G$ , denoted by  $\text{Dec } G$ , is the set of all connected components of  $G$  (this is a set of graphs, not just isomorphism types). Furthermore, given  $i \geq 0$ , we define the *depth  $i$  decomposition* of  $G$  by  $\text{Dec}_0 G = \text{Dec } G$  and  $\text{Dec}_{i+1} G = \bigcup_{F \in \text{Dec}_i G} \text{Dec } \overline{F}$ . Note that  $P_i = \{V(F) : F \in \text{Dec}_i G\}$  is a partition of  $V(G)$  and that  $P_{i+1}$  refines  $P_i$ . The *depth  $i$  environment* of a vertex  $v \in V(G)$ , denoted by  $\text{Env}_i(v)$ , is the  $F$  in  $\text{Dec}_i G$  containing  $v$ .

We define the *rank* of a graph  $G$ , denoted by  $rk\ G$ , inductively as follows: (1) If  $G$  is complement-connected, then  $rk\ G = 0$ . (2) If  $G$  is connected but not complement-connected, then  $rk\ G = rk\ \overline{G}$ . (3) If  $G$  is disconnected, then  $rk\ G = 1 + \max\{rk\ F : F \in Dec\ G\}$ . In other terms,  $rk\ G$  is the smallest  $k$  such that  $P_{k+1} = P_k$  or such that  $P_k$  consists of  $V(F)$  for all cocomponents  $F$  of  $G$ .

In the *Ehrenfeucht game* on two disjoint graphs  $G$  and  $H$  two players, Spoiler and Duplicator, alternately select vertices of the graphs, one vertex per move. Spoiler starts and is always free to move in any of  $G$  and  $H$ ; Then Duplicator must choose a vertex in the other graph. Let  $x_i \in V(G)$  and  $y_i \in V(H)$  denote the vertices selected by the players in the  $i$ -th round. Duplicator wins the  $k$ -round game if the component-wise correspondence between  $x_1, \dots, x_k$  and  $y_1, \dots, y_k$  is a partial isomorphism from  $G$  to  $H$ ; Otherwise the winner is Spoiler. In the *0-alternation game* Spoiler plays all the game in the same graph he selects in the first round.

Assume  $G \not\cong H$ . Let  $D(G, H)$  (resp.  $D_0(G, H)$ ) denote the minimum  $D(\Phi)$  over (resp. 0-alternation) first order sentences  $\Phi$  that are true on one of the graphs and false on the other. The Ehrenfeucht theorem relates  $D(G, H)$  and the length of the Ehrenfeucht game on  $G$  and  $H$ . We will use the following version of the theorem:  $D_0(G, H)$  is equal to the minimum  $k$  such that Spoiler has a winning strategy in the  $k$ -round 0-alternation Ehrenfeucht game on  $G$  and  $H$ . It is also useful to know that  $D_0(G) = \max\{D_0(G, H) : H \not\cong G\}$ .

We define the tower-function by  $Tower(0) = 1$  and  $Tower(i) = 2^{Tower(i-1)}$  for each subsequent  $i$ .

### 3 Upper bound: Proof of Theorem 1

**Lemma 1** Consider the Ehrenfeucht game on graphs  $G$  and  $H$ . Let  $x, x' \in V(G)$ ,  $y, y' \in V(H)$  and assume that the pairs  $x, y$  and  $x', y'$  are selected by the players in the same rounds. Furthermore, assume that  $Env_l(x) \neq Env_l(x')$ ,  $Env_l(y) = Env_l(y')$ , and  $diam\ Env_i(y) \leq 2$  for every  $i \leq l$ . Then Spoiler can win in at most  $l + 1$  rounds ( $l$  rounds if  $G$  is connected), playing all the time in  $H$ .

**Proof:** We proceed by induction on  $l$ . The base case is  $l = 0$  if  $G$  is disconnected and  $l = 1$  if  $G$  is connected. If  $y$  and  $y'$  are adjacent in  $Env_l(y)$ , Duplicator has already lost. Otherwise, Spoiler uses the fact that  $diam\ Env_l(y) = 2$  and selects  $y''$  adjacent in  $Env_l(y)$  to both  $y$  and  $y'$ . Duplicator cannot do so with any  $x''$  because  $x$  and  $x'$  are in different components of  $G$  if  $l = 0$  or  $\overline{G}$  if  $l = 1$ .

Assume that  $l \geq 1$ . Let  $0 \leq m \leq l$  be the minimum number such that  $x' \notin Env_m(x)$ . If  $m < l$ , Spoiler wins in the next  $m + 1 \leq l$  moves by induction. If  $m = l$ , Spoiler uses the same trick as in the base case and forces Duplicator to make a move  $x''$  outside  $Env_{l-1}(x)$ . By the induction hypothesis, Spoiler needs  $l$  extra moves to win.  $\square$

As long as Duplicator avoids meeting the conditions of Lemma 1 (in particular, selects  $x' \in Env_l(x)$  whenever Spoiler selects  $y' \in Env_l(y)$ ), we will say that she *beware of the environment threat*.

Let  $rk\ G = k$ . We call  $G$  *uniform* if  $Dec_{k-1}\ G$  contains no complement-connected graph, that is, every cocomponent appears in  $Dec_k\ G$  and no earlier. We call  $G$  *inclusion-free* if the following two conditions are true for every  $i < k$ : (1) For any  $K \in Dec_i\ G$ ,  $\overline{K}$  contains no isomorphic connected components. (2) If two elements  $K$  and  $M$  of  $Dec_i\ G$  are non-isomorphic, then neither  $\overline{K} \sqsubset \overline{M}$  nor  $\overline{M} \sqsubset \overline{K}$ .

**Lemma 2 (Main Lemma)** Let  $G$  be a uniform inclusion-free graph. Suppose that every cocomponent of  $G$  has exactly  $c$  vertices. Then  $D_0(G) \leq 2\ rk\ G + c + 1$ .

**Proof:** Let  $rkG = k$ . Fix a graph  $H \not\cong G$ . We will design a strategy allowing Spoiler to win the 0-alternation Ehrenfeucht game on  $G$  and  $H$  in at most  $2k + c + 1$  moves. Since  $D_0(G) = D_0(\overline{G})$ , without loss of generality we will assume that  $G$  is connected. Since the case of  $k = 0$  is trivial, we will also assume that  $k \geq 1$ .

*Case 1:  $H$  has a cocomponent  $C$  non-embeddable in any cocomponent of  $G$ .* If  $C$  has no more than  $c$  vertices, Spoiler selects all  $C$ . Otherwise he selects  $c + 1$  vertices spanning a complement-connected subgraph in  $C$  (it is not hard to show that this is always possible). If Duplicator's response  $A$  is within a cocomponent of  $G$ , then  $C \not\cong A$  by the assumption. Otherwise  $A$  is not complement-connected and Duplicator loses anyway.

In the sequel we will assume that Duplicator beware of the environment threat during all game.

*Case 2:  $G \subset H$  or there are  $l \leq k$  and  $A \in Dec_l G$  properly embeddable in some  $B \in Dec_l H$ , and not Case 1.* Spoiler plays in  $H$ . If  $G \subset H$ , set  $A = G$ ,  $B = H$ , and  $l = 0$ . Let  $H_0$  be a copy of  $A$  in  $B$ . At the first move Spoiler selects an arbitrary  $y_0 \in V(B) \setminus V(H_0)$ . Denote Duplicator's response in  $G$  by  $x_0$  and set  $G_0 = Env_l(x_0)$ . From now on Spoiler plays in  $H_0$ . Since we are not in Case 1,  $B$  is not a cocomponent of  $H$  and hence  $diam B \leq 2$ . Since Duplicator is supposed to beware of the environment threat, from now on she is forced to play in  $G_0$ .

*Subcase 2.1:  $G_0 \not\cong H_0$ .* Assume that  $l < k$  (the case of  $l = k$  will be covered by the last phase of the strategy). Since  $G_0$  and  $H_0$  are non-isomorphic copies of elements of  $Dec_l G$  and  $G$  is inclusion-free, Spoiler is able to make his next choice  $y_1$  in some  $H_1 \in Dec_l \overline{H_0}$  absent in  $Dec_l \overline{G_0}$ . Denote Duplicator's response in  $G_0$  by  $x_1$  and set  $G_1 = Env_{l+1}(x_1)$ . Note that  $G_1$  and  $H_1$  are non-isomorphic copies of elements of  $Dec_{l+1} G$ . Playing in the same fashion in the subsequent  $k - l - 1$  rounds, Spoiler finally achieves the players' moves in some non-isomorphic  $G_{k-l} \in Dec_k G$  and  $H_{k-l}$ , the latter being a copy of an element of  $Dec_k G$ . Both the graphs have  $c$  vertices. Now Spoiler selects the  $c - 1$  remaining vertices of  $H_{k-l}$  and wins whatever Duplicator's response is.

*Subcase 2.2:  $G_0 \cong H_0$ .* Though the graphs are isomorphic, the crucial fact is that  $G_0$ , unlike  $H_0$ , contains a selected vertex. By the definition of an inclusion-free graph, every automorphism of  $A \cong G_0 \cong H_0$  takes each cocomponent onto itself. Therefore every isomorphism between  $G_0$  and  $H_0$  matches cocomponents of these graphs in the same way. Let  $Y$  be the counterpart of  $Env_k(x_0)$  in  $H_0$  with respect to this matching. In the second round Spoiler selects an arbitrary  $y_1$  in  $Y$ . Denote Duplicator's answer by  $x_1$ . If  $x_1 \in Env_k(x_0)$ , Spoiler selects all  $Y$  and wins. Otherwise there is  $m \leq rk A$  such that  $Env_m(x_1)$  in  $G_0$  and  $Env_m(y_1)$  in  $H_0$  are non-isomorphic. This allows Spoiler to apply the strategy of Subcase 2.1.

*Case 3: Neither Case 1 nor Case 2.* Spoiler plays in  $G_0 = G$ . His first move  $x_0$  is arbitrary. Denote Duplicator's response in  $H$  by  $y_0$  and set  $H_0 = Env_0(y_0)$ . Since we are not in Case 2,  $G_0 \not\subset H_0$ . As  $G_0$  is inclusion-free,  $\overline{G_0}$  has a connected component  $G_1$  with no isomorphic copy in  $\overline{H_0}$ . Spoiler selects  $x_1$  arbitrarily in  $G_1$ . Let Duplicator respond with  $y_1$  somewhere in  $H_0$  and denote  $H_1 = Env_1(y_1)$ . Thus  $G_1 \not\cong H_1$  and  $G_1 \not\subset H_1$ , the latter again because we are not in Case 2. In the next round Spoiler again selects a vertex in a component  $G_2$  of  $\overline{G_1}$  absent in  $\overline{H_1}$ . Continuing in the same fashion, Spoiler finally forces playing the game on some  $G_m \in Dec_m G_0$  and  $H_m \in Dec_m H_0$  with  $G_m \not\subset H_m$  under one of the two terminal conditions: (1)  $m < k$  and  $H_m$  (or its complement) is a cocomponent of  $H$ . (2)  $m = k$ . In the first case note that, as we are not in Case 1,  $H_m$  is embeddable in some cocomponent of  $G$  (or its complement) and hence has at most  $c$  vertices. Therefore it suffices for Spoiler to select altogether  $c + 1$  vertices in  $G_m$  to win (recall the assumption that Duplicator beware of the environment threat and hence cannot move outside  $H_m$ ). In the second case  $G_m$  is a cocomponent of  $G$  and hence has  $c$  vertices. Spoiler selects all  $G_m$ . Since Duplicator's response must be complement-connected, she is forced to play

within a cocomponent of  $H_m$  and hence loses.

*Length of the game.* The above strategy allows Spoiler to win in at most  $k + c$  moves under the condition that Duplicator beware of the environment threat. If Duplicator ignores this threat, Spoiler needs  $k + 1$  additional moves according to Lemma 1.  $\square$

Let  $R_0$  consist of all complement-connected graphs of order 5. Assume that  $R_{i-1}$  is already specified. Fix  $F_i$  to be the family of all  $\lfloor |R_{i-1}|/2 \rfloor$ -element subsets of  $R_{i-1}$ . Define  $R_i$  to be the set of the complements of  $\bigsqcup_{G \in S} G$  for all  $S$  in  $F_i$ . Note that  $R_i$  consists of inclusion-free uniform graphs of rank  $i$  whose cocomponents all have 5 vertices. All graphs in  $R_i$  have the same order; Denote it by  $N_i$ . Let  $M_i = |R_i|$ . By the construction,

$$M_{i+1} = \binom{M_i}{\lfloor M_i/2 \rfloor} = \sqrt{\frac{2 + o(1)}{\pi M_i}} 2^{M_i} \quad \text{and} \quad N_{i+1} = \lfloor M_i/2 \rfloor N_i > M_i.$$

A simple estimation shows that  $N_i \geq \text{Tower}(i - O(1))$ . To complete the proof of Theorem 1, choose  $G_i$  in  $R_i$ . Using Main Lemma, we obtain  $q_0(N_i) \leq D_0(G_i) \leq 2i + 6 \leq 2 \log^* N_i + O(1)$ .

## 4 Lower bound: Proof-sketch of Theorem 2

Let  $L_a(G)$  denote the minimum length of an  $a$ -alternation sentence defining  $G$ .

**Lemma 3**  $L_a(G) \leq \text{Tower}(D_a(G) + \log^* D_a(G) + O(1))$ .

An analog of this lemma for  $L(G)$  and  $D(G)$  appears in [5] but its proof does not work under restrictions on the alternation number. The proof of Lemma 3 will appear in the full version.

Given  $n$ , denote  $k = q_0(n)$  and fix a graph  $G$  on  $n$  vertices such that  $D_0(G) = k$ . By Lemma 3,  $G$  is definable by a 0-alternation  $\Phi$  of length at most  $\text{Tower}(k + \log^* k + O(1))$ . Using the standard reduction, we convert  $\Phi$  to an equivalent prenex  $\exists^* \forall^*$ -sentence  $\Psi$  (i.e. existential quantifiers in  $\Psi$  all precede universal quantifiers). Since the reduction does not increase the total number of quantifiers,  $D(\Psi) \leq L(\Phi)$ . It is well known and easy to prove that, if a prenex  $\exists^* \forall^*$ -sentence  $\Psi$  is true on some structure, then it is true on some structure of order at most  $D(\Psi)$ . Since the  $\Psi$  is true only on  $G$ , we have  $n \leq D(\Psi) \leq L(\Phi) \leq \text{Tower}(k + \log^* k + O(1))$ , which proves the theorem.

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