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# Monotone Boolean Functions with $s$ Zeros Farthest from Threshold Functions

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Let  $T_t$  denote the  $t$ -threshold function on the  $n$ -cube:  $T_t(x) = 1$  if  $|\{i : x_i = 1\}| \geq t$ , and 0 otherwise. Define the distance between Boolean functions  $g$  and  $h$ ,  $d(g, h)$ , to be the number of points on which  $g$  and  $h$  disagree. We consider the following extremal problem: Over a monotone Boolean function  $g$  on the  $n$ -cube with  $s$  zeros, what is the maximum of  $d(g, T_t)$ ? We show that the following monotone function  $p_s$  maximizes the distance: For  $x \in \{0, 1\}^n$ ,  $p_s(x) = 0$  if and only if  $N(x) < s$ , where  $N(x)$  is the integer whose  $n$ -bit binary representation is  $x$ . Our result generalizes the previous work for the case  $t = \lceil n/2 \rceil$  and  $s = 2^{n-1}$  by Blum, Burch, and Langford [BBL98-FOCS98], who considered the problem to analyze the behavior of a learning algorithm for monotone Boolean functions, and the previous work for the same  $t$  and  $s$  by Amano and Maruoka [AM02-ALT02].

## 1 Introduction and Overview

For a Boolean function  $h$  and a class  $\mathcal{C}$  of Boolean functions, we consider the following extremal problem: what is the maximum distance between  $g \in \mathcal{C}$  and  $h$ ? Equivalently, under the uniform distribution on  $\{0, 1\}^n$ , how small can the correlation between  $g \in \mathcal{C}$  and  $h$  be? The distance between Boolean functions  $g$  and  $h$ ,  $d(g, h)$ , is defined to be the number of points on which  $g$  and  $h$  disagree. A Boolean function  $g : \{0, 1\}^n \rightarrow \{0, 1\}$  is *monotone* if, for  $x, y \in \{0, 1\}^n$ ,  $x \leq y \Rightarrow g(x) \leq g(y)$ , where for  $x, y \in \{0, 1\}^n$ ,  $x \leq y$  if and only if  $x_i \leq y_i$  for all  $i = 1, \dots, n$ . A Boolean function is *fair* if it outputs 1 on exactly half of its inputs. The starting point of our work is the fact that *among all fair monotone Boolean functions, a single variable function  $g(x) = x_i$  is farthest from the majority function*; this was conjectured by Blum, Burch, and Langford [BBL98], and was proved by Amano and Maruoka [AM02].

The main concern of the work of Blum, Burch, and Langford is *learning* of monotone Boolean functions. They gave the following simple algorithm for weakly learning a monotone Boolean function under the uniform distribution: Given samples  $(x_1, g(x_1)), (x_2, g(x_2)), \dots, (x_m, g(x_m))$ , output, as a hypothesis, a function that is most correlated with those samples among *three* functions  $\{0, 1, \text{Majority}\}$ , where 0 and 1 are constant functions. With high probability the output of the algorithm has correlation at least  $\Omega(1/\sqrt{n})$  with  $g$ . Blum, Burch, and Langford showed, using the Kruskal-Katona theorem about the minimum size of a shadow, that any fair monotone Boolean function  $g$  has correlation at least  $\Omega(1/\sqrt{n})$  with Majority. They conjectured that in fact a single variable function  $g(x) = x_i$  is a fair monotone function

that is farthest from Majority. Amano and Maruoka proved this conjecture also using the Kruskal-Katona theorem.

In this paper we give a generalization in which we consider any threshold function, not just Majority, and any monotone function with a prescribed number of zeros, not just a fair one. Our proof is self-contained; we do not use the Kruskal-Katona theorem.

Let  $T_t$  denote the  $t$ -threshold function: for  $x \in \{0, 1\}^n$ ,  $T_t(x) = 1$  if  $|\{i : x_i = 1\}| \geq t$ , and 0 otherwise. Throughout the paper  $t$  is an integer; it will be convenient to allow  $t$  to be negative; for  $t \leq 0$ ,  $T_t$  is the constant 1 function on the  $n$ -cube. The majority function is function  $T_t$  with  $t = \lceil n/2 \rceil$ . For a Boolean function  $g$ , let  $\#_0(g)$  and  $\#_1(g)$  respectively denote the number of points on which  $g = 0$  and on which  $g = 1$ . Similarly, for Boolean functions  $g$  and  $h$ , and  $a, b \in \{0, 1\}$ , let  $\#_{ab}(g, h)$  denote the number of points  $x$  such that  $g(x) = a$  and  $h(x) = b$ .

The problem we consider is the following: Among all monotone  $g : \{0, 1\}^n \rightarrow \{0, 1\}$  with  $\#_0(g) = s$  ( $0 \leq s < 2^n$ ), what is the maximum of  $d(g, T_t)$ ? Maximizing  $d(g, h)$  for a fixed function  $h$  and a function  $g$  with  $\#_0(g) = s$  is equivalent to maximizing  $\#_{01}(g, h)$  since for any  $g$  and  $h$ ,  $d(g, h) = \#_0(h) - \#_0(g) + 2\#_{01}(g, h)$ . For  $0 \leq s < 2^n$ , define the monotone function  $p_s : \{0, 1\}^n \rightarrow \{0, 1\}$  as follows. For  $x \in \{0, 1\}^n$ ,  $p_s(x) = 0$  if and only if  $N(x) < s$ , where  $N(x)$  is the integer whose  $n$ -bit binary representation is  $x$ .

**Theorem** Let  $g : \{0, 1\}^n \rightarrow \{0, 1\}$  be monotone with  $\#_0(g) = s$ . Then, for any integer  $t$ ,

$$\#_{01}(g, T_t) \leq \#_{01}(p_s, T_t), \quad \text{and hence} \quad d(g, T_t) \leq d(p_s, T_t).$$

## 2 Proof of Theorem

For an integer  $i \geq 0$ , let  $b(i)$  be the number of 1's in the binary representation of the integer  $i$ . Let  $0 \leq l \leq m$ . For an integer  $t$ , define  $f_t(l, m)$  to be the number of integers  $i$  such that  $l \leq i < m$  and  $b(i) \geq t$ . Note that for  $t \leq 0$ ,  $f_t(l, m) = f_0(l, m) = m - l$ . Also note that the following hold:  $f_t(l, m) \leq f_{t-1}(l, m)$ ; for  $0 \leq k < 2^r$ ,  $f_t(0, k) = f_{t-1}(2^r, 2^r + k)$ ;  $\#_{01}(p_s, T_t) = f_t(0, s)$ .

It turns out that the main work we do to prove the theorem is an analysis of  $f_t(l, m)$ . This aspect of our proof is somewhat similar to a proof of the edge-isoperimetric inequality on the Boolean cube explained in the book by Bollobas [Bo86, § 16 Isoperimetric Problems]. We now state a key lemma, Lemma 1, and two auxiliary lemmas, Lemmas 2 and 3. We will give a proof of the theorem using Lemma 1, and then give proofs of the three lemmas.

**Lemma 1** For  $0 \leq l \leq m$  and any integer  $t$ ,

$$f_t(0, m) + f_{t-1}(0, l) \leq f_t(0, m + l).$$

**Lemma 2** For  $k, l \geq 0$  and any integer  $t$ ,

$$f_t(0, k) \leq f_t(l, l + k).$$

**Lemma 3** For  $k, l, q \geq 0$  such that  $l + k \leq 2^q$  and for any integer  $t$ ,

$$f_t(l, l + k) \leq f_t(2^q - k, 2^q).$$

**Proof of Theorem using Lemma 1.** The proof is by induction on  $n$ . The base case  $n = 1$  is trivial. For induction assume that  $n > 1$  and that the assertion holds for  $n - 1$ . Let  $g_0$  and  $g_1$  be the Boolean functions on the  $(n - 1)$ -cube obtained from  $g$  by fixing the first bit to be 0 and 1 respectively; i.e., for  $e = 0, 1$ , and for  $x \in \{0, 1\}^{n-1}$ ,  $g_e(x) = g(ex)$ . Let  $m = \#_0(g_0)$  and  $l = \#_0(g_1)$ . Clearly,  $m + l = s$ , and since  $g$  is monotone,  $m \geq l$ . Thus we have

$$\begin{aligned} \#_{01}(g, T_t) &= \#_{01}(g_0, T_t) + \#_{01}(g_1, T_{t-1}) \\ &\leq \#_{01}(p_m, T_t) + \#_{01}(p_l, T_{t-1}) \\ &= f_t(0, m) + f_{t-1}(0, l) \\ &\leq f_t(0, m + l) \\ &= \#_{01}(p_s, T_t), \end{aligned}$$

where the first inequality is by the inductive assumption and the second inequality is by Lemma 1.  $\square$

**Proofs of Lemmas 2 and 3.** For  $0 \leq i < 2^q$ , the  $q$ -bit binary representation of  $i$  has bit 1 at position  $j$  ( $1 \leq j \leq q$ ) if and only if the  $q$ -bit binary representation of  $2^q - 1 - i$  has bit 0 at position  $j$ . Hence Lemma 2 readily yields Lemma 3.

Now we prove Lemma 2. The assertion is trivial when  $l = k = 0$ . Assume that  $l + k \geq 1$  and let  $r = \lfloor \log_2(l + k) \rfloor$  so that we have  $2^r \leq l + k < 2^{r+1}$ . The proof is by induction on  $r$ ; more precisely, we prove Lemma 2 by inductively assuming that the assertion of Lemma 2 holds *and* the corresponding assertion of Lemma 3 holds.

In the base case when  $r = 0$ , we have  $l + k = 1$  and thus either (i)  $l = 0, k = 1$  or (ii)  $l = 1, k = 0$ ; in both cases the claim is immediate. For induction assume that  $r > 0$  and that for  $r - 1$  the assertion holds, and hence the corresponding assertion of Lemma 3 also holds.

CASE 1:  $2^r \leq l$ : In this case  $2^r \leq l \leq l + k < 2^{r+1}$  and

$$f_t(l, l + k) = f_{t-1}(l - 2^r, l + k - 2^r) \geq f_{t-1}(0, k) \geq f_t(0, k),$$

where the first inequality is by the inductive assumption.

CASE 2:  $l < 2^r$  and  $k < 2^r$ :

$$\begin{aligned} f_t(l, l + k) &= f_t(l, 2^r) + f_t(2^r, l + k) \\ &= f_t(l, 2^r) + f_{t-1}(0, l + k - 2^r) \\ &\geq f_t(l + k - 2^r, k) + f_{t-1}(0, l + k - 2^r) \\ &\geq f_t(l + k - 2^r, k) + f_t(0, l + k - 2^r) \\ &= f_t(0, k), \end{aligned}$$

where the first inequality is by Lemma 3.

CASE 3:  $l < 2^r$  and  $k \geq 2^r$ : In this case  $l < 2^r \leq 2^r + l \leq l + k$  and

$$\begin{aligned}
f_t(l, l+k) &= [f_t(l, 2^r) + f_t(2^r, 2^r + l)] + f_t(2^r + l, l+k) \\
&= [f_t(l, 2^r) + f_{t-1}(0, l)] + f_t(2^r + l, l+k) \\
&\geq [f_t(0, l) + f_t(l, 2^r)] + f_t(2^r + l, l+k) \\
&= f_t(0, 2^r) + f_t(2^r + l, l+k) \\
&= f_t(0, 2^r) + f_{t-1}(l, l+k-2^r) \\
&\geq f_t(0, 2^r) + f_{t-1}(0, k-2^r) \\
&= f_t(0, 2^r) + f_t(2^r, k) \\
&= f_t(0, k),
\end{aligned}$$

where the second inequality is by the inductive assumption.  $\square$

**Proof of Lemma 1.** The assertion is trivial when  $l = m = 0$ . Assume that  $m \geq 1$  and let  $r = \lfloor \log_2 m \rfloor$  so that we have  $2^r \leq m < 2^{r+1}$ . The proof is by induction on  $r$ . In the base case when  $r = 0$  we have  $m = 1$ , and  $l = 0$  or  $l = 1$ ; in both cases the claim is immediate. For induction assume that  $r > 0$  and that the claim holds for  $r - 1$ .

CASE 1:  $2^r \leq l$ :

$$\begin{aligned}
f_t(0, m) + f_{t-1}(0, l) &= f_t(0, 2^r) + f_t(2^r, m) + f_{t-1}(0, 2^r) + f_{t-1}(2^r, l) \\
&= f_t(0, 2^r) + f_{t-1}(0, m-2^r) + f_{t-1}(0, 2^r) + f_{t-2}(0, l-2^r) \\
&= f_t(0, 2^r) + f_{t-1}(0, 2^r) + f_{t-1}(0, m-2^r) + f_{t-2}(0, l-2^r) \\
&\leq f_t(0, 2^r) + f_{t-1}(0, 2^r) + f_{t-1}(0, m+l-2^{r+1}) \\
&= f_t(0, m+l),
\end{aligned}$$

where the inequality is by the inductive assumption.

CASE 2:  $l < 2^r$ ,  $(m - 2^r) + l \leq 2^r$ :

$$\begin{aligned}
f_t(0, m) + f_{t-1}(0, l) &= f_t(0, 2^r) + f_t(2^r, m) + f_{t-1}(0, l) \\
&= f_t(0, 2^r) + f_{t-1}(0, m-2^r) + f_{t-1}(0, l) \\
&\leq f_t(0, 2^r) + f_{t-1}(0, m-2^r) + f_{t-1}(m-2^r, m-2^r+l) \\
&= f_t(0, 2^r) + f_{t-1}(0, m-2^r+l) \\
&= f_t(0, 2^r) + f_t(2^r, m+l) \\
&= f_t(0, m+l),
\end{aligned}$$

where the inequality is by Lemma 2.

CASE 3:  $l < 2^r$ ,  $m + l > 2^{r+1}$ : We have the following derivation where Lemma 3 is used in the form (1) for the inequality below.

$$f_{t-1}((m+l) - 2^{r+1}, l) \leq f_{t-1}(m - 2^r, 2^r) \quad (1)$$

$$\begin{aligned} & f_t(0, m) + f_{t-1}(0, l) \\ = & f_t(0, 2^r) + f_t(2^r, m) + f_{t-1}(0, (m+l) - 2^{r+1}) + f_{t-1}((m+l) - 2^{r+1}, l) \\ = & f_t(0, 2^r) + f_{t-1}(0, m - 2^r) + f_{t-1}(0, (m+l) - 2^{r+1}) + f_{t-1}((m+l) - 2^{r+1}, l) \\ \leq & f_t(0, 2^r) + f_{t-1}(0, m - 2^r) + f_{t-1}(m - 2^r, 2^r) + f_{t-1}(0, (m+l) - 2^{r+1}) \\ = & f_t(0, 2^r) + f_{t-1}(0, 2^r) + f_{t-1}(0, (m+l) - 2^{r+1}) \\ = & f_t(0, 2^r) + f_t(2^r, 2^{r+1}) + f_t(2^{r+1}, m+l) \\ = & f_t(0, m+l). \quad \square \end{aligned}$$

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