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A sufficient condition for bicolored hypergraphs

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In this note we prove Sterboul's conjecture, that provides a sufficient condition for the bicoloredability of hypergraphs.

Keywords: hypergraphs, coloring, Sterboul's conjecture

In [2], Fournier and Las Vergnas gave a sufficient condition for the bicoloredability of hypergraphs. Their theorem was a weaker form of a conjecture due to Sterboul, that we prove here. These facts are reproduced in [1] and [3].

A hypergraph is a pair $H = (V, \mathcal{E})$, where the elements of V are the *vertices*, and the elements of \mathcal{E} are subsets of V and are called the *edges*. A function $c : V \rightarrow \{1, 2\}$ is called a *bipartition* of V , and for $x \in V$, we call $c(x)$ the *color* of x . If c is such that for any $e \in \mathcal{E}$ with $|e| \geq 2$ both colors occur, then c is called a *bicoloration* of H . If only one color occurs, the edge is said to be *monochromatic*. If a hypergraph admits a bicoloration we say that it is *bicolored*.

A sequence $(x_1, e_1, x_2, \dots, e_k, x_1)$, where the e_i 's are distinct edges, the x_i 's are distinct vertices, and $k \geq 3$, is said to be a *cycle* if $x_i \in e_{i-1} \cap e_i$ for $i = 2, \dots, k$ and $x_1 \in e_1 \cap e_k$. A cycle is said to be *odd* if it has an odd number of edges.

An odd cycle $(x_1, e_1, x_2, \dots, e_k, x_1)$ such that two non-consecutive edges are disjoint and $|e_i \cap e_{i+1}| = 1$ for $i = 1, 2, \dots, k - 1$, is called a *Sterboul cycle*. If a hypergraph H has no Sterboul cycle, it is said to be a *Sterboul hypergraph*.

Then we can word Sterboul's conjecture as follows:

Theorem 1 *If H is a Sterboul hypergraph, then H is bicolored.*

Proof: The proof works by induction on the number of edges.

When the hypergraph has no edge, the theorem clearly holds.

The general step assumes that we have a hypergraph $H = (V, \mathcal{E})$ and $e_0 \in \mathcal{E}$ such that $H \setminus e_0 = (V, \mathcal{E} \setminus e_0)$ has a bicoloration $c : V \rightarrow \{1, 2\}$.

We can assume that e_0 has size at least 2, and that c leaves e_0 monochromatic, or else we have nothing to do.

Now we use the following algorithm to transform the bipartition c into a bicolouration of H . The algorithm switches successively the colors of some vertices of H that are contained in a monochromatic edge in the current bipartition. It constructs an arborescence $G_0 = (V_0, E_0)$ and a mapping $g : V_0 \rightarrow \mathcal{E}$ that keep track of the running of the algorithm: the vertices of G_0 are those whose colors were switched, and g associates a vertex with the monochromatic edge that caused its color switch.

The vertices are chosen with a DFS (Depth-First Search) method, and to do so the algorithm uses a LIFO (Last In First Out) stack \mathcal{P} that contains the set of vertices whose colors have been switched, and so that might be in a monochromatic edge. $\text{top}(\mathcal{P})$ returns the last vertex entered in \mathcal{P} ; $\text{drop}(\mathcal{P})$ removes $\text{top}(\mathcal{P})$ out of \mathcal{P} ; and $\text{put}(x, \mathcal{P})$ enters a new vertex x in \mathcal{P} .

INPUT: A hypergraph $H = (V, \mathcal{E})$ and a bipartition c such that $e_0 \in \mathcal{E}$ is the only monochromatic edge.

OUTPUT: A bicolouration of H or an Error message only if H is not a Sterboul hypergraph.

```

let  $x_0 \in e_0$ 
 $G_0 := (\{x_0\}, \emptyset)$ 
 $g(x_0) := e_0$ 
switch  $c(x_0)$ 
put( $x_0, \mathcal{P}$ )
While  $\mathcal{P} \neq \emptyset$  do
  let  $v = \text{top}(\mathcal{P})$ 
  If there exists  $e \in \mathcal{E}$ ,  $|e| \geq 2$ , monochromatic such that  $v \in e$  then
    If  $e \setminus V_0 = \emptyset$  then
      return Error
    else
      let  $w \in e \setminus V_0$ 
       $V_0 := V_0 \cup w$  ;  $E_0 := E_0 \cup (vw)$ 
       $g(w) := e$ 
      switch  $c(w)$ 
      put( $w, \mathcal{P}$ )
    end If
  else
    drop( $\mathcal{P}$ )
  end If
end While

```

First we remark that G_0 is indeed an arborescence since the end point of each new arc of G_0 is a new vertex. Then for a given $x \in V_0$ there is a unique path in G_0 from x_0 to x . Moreover when x is at the top of \mathcal{P} , then \mathcal{P} contains exactly the vertices of that path (because \mathcal{P} is a LIFO stack).

We can also remark that if the algorithm does not return `ERROR`, then at each iteration either a new vertex is put into \mathcal{P} , or a vertex is dropped out of \mathcal{P} . Since a vertex appears at most once in G_0 and thus can be put at most once in \mathcal{P} , we have at most $2|V|$ iterations, and the algorithm ends.

We note $\mathcal{P}^{(i)}$, $G_0^{(i)} = (V_0^{(i)}, E_0^{(i)})$, $c^{(i)}$, $g^{(i)}$ the values of \mathcal{P} , $G_0 = (V_0, E_0)$, c , g (respectively) at the beginning of the i -th iteration. We also note $c^{(0)}$ the original bipartition (which is different from $c^{(1)}$ because of the switch of $c(x_0)$).

To prove the validity of the algorithm, we have to prove that:

- `ERROR` cannot be returned if H is a Sterboul hypergraph.
- The output of the algorithm if no `ERROR` occurs is a bicolouration.

Before proving those points, we claim the following:

Claim 1 *Suppose that H is a Sterboul hypergraph. Consider the beginning of the i -th iteration. Let $\mathcal{P}^{(i)} = (x_k \dots x_0)$, and $e_j = g^{(i)}(x_j)$ for $j = 0, \dots, k$. Then we have:*

- (a) *For each $j = 0, \dots, k$, x_j is the only vertex of its color in e_j .*
- (b) *For each $j = 0, \dots, k-1$ we have $e_j \cap e_{j+1} = \{x_j\}$.*
- (c) *Two non-consecutive edges are disjoint.*

Proof: The proof works by induction on i .

For $i = 1$ the claim clearly holds since $\mathcal{P}^{(1)} = (x_0)$.

We now consider $i \geq 1$ and we suppose the claim holds at iteration i . We are going to prove that it also holds at iteration $i+1$. Let $\mathcal{P}^{(i)} = (x_k \dots x_0)$, and $e_j = g^{(i)}(x_j)$ for $j = 0, \dots, k$.

If during the i -th iteration the algorithm dropped x_k out of \mathcal{P} (that is $\mathcal{P}^{(i+1)} = (x_{k-1} \dots x_0)$), the claim clearly holds at iteration $i+1$. Thus we assume that the algorithm found $e_{k+1} \in \mathcal{E}$ with $x_k \in e_{k+1}$ that is monochromatic for $c^{(i)}$, and $x_{k+1} \in e_{k+1} \setminus V_0^{(i)}$ (because we assume that there is a $(i+1)$ -th iteration or else the claim is true) so that $\mathcal{P}^{(i+1)} = (x_{k+1} x_k \dots x_0)$.

Since (a) holds at iteration i , we know that if $w \in e_k \setminus x_k$ then $c^{(i)}(w) \neq c^{(i)}(x_k)$. As $x_k \in e_{k+1}$ and e_{k+1} is monochromatic for $c^{(i)}$, then $e_k \cap e_{k+1} = \{x_k\}$ and (b) holds at iteration $i+1$.

Suppose $j_0 = \max\{0 \leq j \leq k-1 \mid e_j \cap e_{k+1} \neq \emptyset\}$ exists, and let $y \in e_{j_0} \cap e_{k+1}$. If $k - j_0$ is odd, then $(y, e_{j_0}, x_{j_0}, \dots, e_{k+1}, y)$ is a Sterboul cycle (because (c) holds at iteration i and (b) holds at iteration $i+1$), so $k - j_0$ is even. But then since (a) holds at iteration i , we have $c^{(i)}(y) \neq c^{(i)}(x_{j_0})$ ($y \neq x_{j_0}$ by definition of j_0), $c^{(i)}(x_{j_0}) \neq c^{(i)}(x_{j_0+1})$, ..., $c^{(i)}(x_{k-1}) \neq c^{(i)}(x_k)$ and then $c^{(i)}(y) \neq c^{(i)}(x_k)$, which is impossible because e_{k+1} is monochromatic for $c^{(i)}$. Hence j_0 does not exist, and (c) holds at iteration $i+1$.

Thus $x_{k+1} \notin e_j$ for all $j = 0, \dots, k$. Since the only color switch done during the i -th iteration concerns x_{k+1} , then (a) holds at iteration $i+1$.

This achieves to prove the claim. □

Now we are able to prove the validity of the algorithm.

- Suppose that H is a Sterboul hypergraph. Consider an iteration i , and let $\mathcal{P}^{(i)} = (x_k \dots x_0)$. If $k = 1$ then from (b) of the claim we have $x_1 \notin e_0$. If $k \geq 2$ then from (c) of the claim we also have $x_k \notin e_0$. This proves that we always have $e_0 \cap V_0 = \{x_0\}$.

If `ERROR` is returned, it means that at a given iteration i_0 , the algorithm found an edge e monochromatic for $c^{(i_0)}$ such that $e \setminus V_0^{(i_0)} = \emptyset$. Then e was also monochromatic for $c^{(0)}$, but e_0 was the only such edge,

so we have a contradiction because we have just seen that we must have $e_0 \cap V_0^{(i_0)} = \{x_0\}$. Thus if H is a Sterboul hypergraph, `ERROR` cannot be returned.

- Finally, if the bipartition obtained by the algorithm is not a bicolouration then we have some $e \in \mathcal{E}$ that is monochromatic. Consider i_0 the last iteration during which the algorithm dropped a vertex of e out of \mathcal{P} (i_0 exists or else e was monochromatic with $c^{(0)}$, but e_0 was the only such edge and e_0 is not monochromatic at the end of the algorithm), and let y_0 be that vertex. Then e was not monochromatic with $c^{(i_0)}$ or else the algorithm would have considered e during the iteration i_0 instead of dropping y_0 . But since no color switch concerning a vertex of e occurs afterwards (by choice of i_0), we have a contradiction. Thus the final bipartition is a bicolouration.

So the algorithm is correct and the theorem is proved. □

We can slightly modify the algorithm so that it gives a Sterboul cycle instead of just returning `ERROR` when the hypergraph is not Sterboul (in order to have a certificate that the hypergraph is not Sterboul).

To do so we just have to check at each iteration that the properties of Claim 1 still hold. If not it means that the monochromatic edge considered intersects the path induced by the stack, and a Sterboul cycle can be easily found.

Our algorithm finds a bicolouration for Sterboul hypergraphs in polynomial time. However it cannot be used to recognize bicolorable hypergraphs (since a Sterboul hypergraph may be bicolorable) and neither to recognize Sterboul hypergraphs (since it may happen that it gives a bicolouration for a hypergraph that is not Sterboul).

The problem of recognizing bicolorable hypergraphs is well-known to be NP-complete [4]. But we leave the following question open: what is the complexity of recognizing Sterboul hypergraphs?

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