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# A sufficient condition for bicolored hypergraphs

David Défossez<sup>1</sup>

<sup>1</sup>Laboratoire Leibniz-Imag, 46 avenue Félix Viallet, 38031 Grenoble Cédex, France. E-mail: david.defossez@imag.fr

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In this note we prove Sterboul's conjecture, that provides a sufficient condition for the bicoloredability of hypergraphs.

**Keywords:** hypergraphs, coloring, Sterboul's conjecture

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In [2], Fournier and Las Vergnas gave a sufficient condition for the bicoloredability of hypergraphs. Their theorem was a weaker form of a conjecture due to Sterboul, that we prove here. These facts are reproduced in [1] and [3].

A hypergraph is a pair  $H = (V, \mathcal{E})$ , where the elements of  $V$  are the *vertices*, and the elements of  $\mathcal{E}$  are subsets of  $V$  and are called the *edges*. A function  $c : V \rightarrow \{1, 2\}$  is called a *bipartition* of  $V$ , and for  $x \in V$ , we call  $c(x)$  the *color* of  $x$ . If  $c$  is such that for any  $e \in \mathcal{E}$  with  $|e| \geq 2$  both colors occur, then  $c$  is called a *bicoloration* of  $H$ . If only one color occurs, the edge is said to be *monochromatic*. If a hypergraph admits a bicoloration we say that it is *bicolored*.

A sequence  $(x_1, e_1, x_2, \dots, e_k, x_1)$ , where the  $e_i$ 's are distinct edges, the  $x_i$ 's are distinct vertices, and  $k \geq 3$ , is said to be a *cycle* if  $x_i \in e_{i-1} \cap e_i$  for  $i = 2, \dots, k$  and  $x_1 \in e_1 \cap e_k$ . A cycle is said to be *odd* if it has an odd number of edges.

An odd cycle  $(x_1, e_1, x_2, \dots, e_k, x_1)$  such that two non-consecutive edges are disjoint and  $|e_i \cap e_{i+1}| = 1$  for  $i = 1, 2, \dots, k - 1$ , is called a *Sterboul cycle*. If a hypergraph  $H$  has no Sterboul cycle, it is said to be a *Sterboul hypergraph*.

Then we can word Sterboul's conjecture as follows:

**Theorem 1** *If  $H$  is a Sterboul hypergraph, then  $H$  is bicolored.*

**Proof:** The proof works by induction on the number of edges.

When the hypergraph has no edge, the theorem clearly holds.

The general step assumes that we have a hypergraph  $H = (V, \mathcal{E})$  and  $e_0 \in \mathcal{E}$  such that  $H \setminus e_0 = (V, \mathcal{E} \setminus e_0)$  has a bicoloration  $c : V \rightarrow \{1, 2\}$ .

We can assume that  $e_0$  has size at least 2, and that  $c$  leaves  $e_0$  monochromatic, or else we have nothing to do.

Now we use the following algorithm to transform the bipartition  $c$  into a bicolouration of  $H$ . The algorithm switches successively the colors of some vertices of  $H$  that are contained in a monochromatic edge in the current bipartition. It constructs an arborescence  $G_0 = (V_0, E_0)$  and a mapping  $g : V_0 \rightarrow \mathcal{E}$  that keep track of the running of the algorithm: the vertices of  $G_0$  are those whose colors were switched, and  $g$  associates a vertex with the monochromatic edge that caused its color switch.

The vertices are chosen with a DFS (Depth-First Search) method, and to do so the algorithm uses a LIFO (Last In First Out) stack  $\mathcal{P}$  that contains the set of vertices whose colors have been switched, and so that might be in a monochromatic edge.  $\text{top}(\mathcal{P})$  returns the last vertex entered in  $\mathcal{P}$ ;  $\text{drop}(\mathcal{P})$  removes  $\text{top}(\mathcal{P})$  out of  $\mathcal{P}$ ; and  $\text{put}(x, \mathcal{P})$  enters a new vertex  $x$  in  $\mathcal{P}$ .

INPUT: A hypergraph  $H = (V, \mathcal{E})$  and a bipartition  $c$  such that  $e_0 \in \mathcal{E}$  is the only monochromatic edge.

OUTPUT: A bicolouration of  $H$  or an Error message only if  $H$  is not a Sterboul hypergraph.

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let  $x_0 \in e_0$ 
 $G_0 := (\{x_0\}, \emptyset)$ 
 $g(x_0) := e_0$ 
switch  $c(x_0)$ 
put( $x_0, \mathcal{P}$ )
While  $\mathcal{P} \neq \emptyset$  do
  let  $v = \text{top}(\mathcal{P})$ 
  If there exists  $e \in \mathcal{E}$ ,  $|e| \geq 2$ , monochromatic such that  $v \in e$  then
    If  $e \setminus V_0 = \emptyset$  then
      return Error
    else
      let  $w \in e \setminus V_0$ 
       $V_0 := V_0 \cup w$  ;  $E_0 := E_0 \cup (vw)$ 
       $g(w) := e$ 
      switch  $c(w)$ 
      put( $w, \mathcal{P}$ )
    end If
  else
    drop( $\mathcal{P}$ )
  end If
end While

```

First we remark that  $G_0$  is indeed an arborescence since the end point of each new arc of  $G_0$  is a new vertex. Then for a given  $x \in V_0$  there is a unique path in  $G_0$  from  $x_0$  to  $x$ . Moreover when  $x$  is at the top of  $\mathcal{P}$ , then  $\mathcal{P}$  contains exactly the vertices of that path (because  $\mathcal{P}$  is a LIFO stack).

We can also remark that if the algorithm does not return `ERROR`, then at each iteration either a new vertex is put into  $\mathcal{P}$ , or a vertex is dropped out of  $\mathcal{P}$ . Since a vertex appears at most once in  $G_0$  and thus can be put at most once in  $\mathcal{P}$ , we have at most  $2|V|$  iterations, and the algorithm ends.

We note  $\mathcal{P}^{(i)}$ ,  $G_0^{(i)} = (V_0^{(i)}, E_0^{(i)})$ ,  $c^{(i)}$ ,  $g^{(i)}$  the values of  $\mathcal{P}$ ,  $G_0 = (V_0, E_0)$ ,  $c$ ,  $g$  (respectively) at the beginning of the  $i$ -th iteration. We also note  $c^{(0)}$  the original bipartition (which is different from  $c^{(1)}$  because of the switch of  $c(x_0)$ ).

To prove the validity of the algorithm, we have to prove that:

- `ERROR` cannot be returned if  $H$  is a Sterboul hypergraph.
- The output of the algorithm if no `ERROR` occurs is a bicolouration.

Before proving those points, we claim the following:

**Claim 1** *Suppose that  $H$  is a Sterboul hypergraph. Consider the beginning of the  $i$ -th iteration. Let  $\mathcal{P}^{(i)} = (x_k \dots x_0)$ , and  $e_j = g^{(i)}(x_j)$  for  $j = 0, \dots, k$ . Then we have:*

- (a) *For each  $j = 0, \dots, k$ ,  $x_j$  is the only vertex of its color in  $e_j$ .*
- (b) *For each  $j = 0, \dots, k-1$  we have  $e_j \cap e_{j+1} = \{x_j\}$ .*
- (c) *Two non-consecutive edges are disjoint.*

**Proof:** The proof works by induction on  $i$ .

For  $i = 1$  the claim clearly holds since  $\mathcal{P}^{(1)} = (x_0)$ .

We now consider  $i \geq 1$  and we suppose the claim holds at iteration  $i$ . We are going to prove that it also holds at iteration  $i+1$ . Let  $\mathcal{P}^{(i)} = (x_k \dots x_0)$ , and  $e_j = g^{(i)}(x_j)$  for  $j = 0, \dots, k$ .

If during the  $i$ -th iteration the algorithm dropped  $x_k$  out of  $\mathcal{P}$  (that is  $\mathcal{P}^{(i+1)} = (x_{k-1} \dots x_0)$ ), the claim clearly holds at iteration  $i+1$ . Thus we assume that the algorithm found  $e_{k+1} \in \mathcal{E}$  with  $x_k \in e_{k+1}$  that is monochromatic for  $c^{(i)}$ , and  $x_{k+1} \in e_{k+1} \setminus V_0^{(i)}$  (because we assume that there is a  $(i+1)$ -th iteration or else the claim is true) so that  $\mathcal{P}^{(i+1)} = (x_{k+1} x_k \dots x_0)$ .

Since (a) holds at iteration  $i$ , we know that if  $w \in e_k \setminus x_k$  then  $c^{(i)}(w) \neq c^{(i)}(x_k)$ . As  $x_k \in e_{k+1}$  and  $e_{k+1}$  is monochromatic for  $c^{(i)}$ , then  $e_k \cap e_{k+1} = \{x_k\}$  and (b) holds at iteration  $i+1$ .

Suppose  $j_0 = \max\{0 \leq j \leq k-1 \mid e_j \cap e_{k+1} \neq \emptyset\}$  exists, and let  $y \in e_{j_0} \cap e_{k+1}$ . If  $k - j_0$  is odd, then  $(y, e_{j_0}, x_{j_0}, \dots, e_{k+1}, y)$  is a Sterboul cycle (because (c) holds at iteration  $i$  and (b) holds at iteration  $i+1$ ), so  $k - j_0$  is even. But then since (a) holds at iteration  $i$ , we have  $c^{(i)}(y) \neq c^{(i)}(x_{j_0})$  ( $y \neq x_{j_0}$  by definition of  $j_0$ ),  $c^{(i)}(x_{j_0}) \neq c^{(i)}(x_{j_0+1})$ , ...,  $c^{(i)}(x_{k-1}) \neq c^{(i)}(x_k)$  and then  $c^{(i)}(y) \neq c^{(i)}(x_k)$ , which is impossible because  $e_{k+1}$  is monochromatic for  $c^{(i)}$ . Hence  $j_0$  does not exist, and (c) holds at iteration  $i+1$ .

Thus  $x_{k+1} \notin e_j$  for all  $j = 0, \dots, k$ . Since the only color switch done during the  $i$ -th iteration concerns  $x_{k+1}$ , then (a) holds at iteration  $i+1$ .

This achieves to prove the claim. □

Now we are able to prove the validity of the algorithm.

- Suppose that  $H$  is a Sterboul hypergraph. Consider an iteration  $i$ , and let  $\mathcal{P}^{(i)} = (x_k \dots x_0)$ . If  $k = 1$  then from (b) of the claim we have  $x_1 \notin e_0$ . If  $k \geq 2$  then from (c) of the claim we also have  $x_k \notin e_0$ . This proves that we always have  $e_0 \cap V_0 = \{x_0\}$ .

If `ERROR` is returned, it means that at a given iteration  $i_0$ , the algorithm found an edge  $e$  monochromatic for  $c^{(i_0)}$  such that  $e \setminus V_0^{(i_0)} = \emptyset$ . Then  $e$  was also monochromatic for  $c^{(0)}$ , but  $e_0$  was the only such edge,

so we have a contradiction because we have just seen that we must have  $e_0 \cap V_0^{(i_0)} = \{x_0\}$ . Thus if  $H$  is a Sterboul hypergraph, `ERROR` cannot be returned.

- Finally, if the bipartition obtained by the algorithm is not a bicolouration then we have some  $e \in \mathcal{E}$  that is monochromatic. Consider  $i_0$  the last iteration during which the algorithm dropped a vertex of  $e$  out of  $\mathcal{P}$  ( $i_0$  exists or else  $e$  was monochromatic with  $c^{(0)}$ , but  $e_0$  was the only such edge and  $e_0$  is not monochromatic at the end of the algorithm), and let  $y_0$  be that vertex. Then  $e$  was not monochromatic with  $c^{(i_0)}$  or else the algorithm would have considered  $e$  during the iteration  $i_0$  instead of dropping  $y_0$ . But since no color switch concerning a vertex of  $e$  occurs afterwards (by choice of  $i_0$ ), we have a contradiction. Thus the final bipartition is a bicolouration.

So the algorithm is correct and the theorem is proved. □

We can slightly modify the algorithm so that it gives a Sterboul cycle instead of just returning `ERROR` when the hypergraph is not Sterboul (in order to have a certificate that the hypergraph is not Sterboul).

To do so we just have to check at each iteration that the properties of Claim 1 still hold. If not it means that the monochromatic edge considered intersects the path induced by the stack, and a Sterboul cycle can be easily found.

Our algorithm finds a bicolouration for Sterboul hypergraphs in polynomial time. However it cannot be used to recognize bicolorable hypergraphs (since a Sterboul hypergraph may be bicolorable) and neither to recognize Sterboul hypergraphs (since it may happen that it gives a bicolouration for a hypergraph that is not Sterboul).

The problem of recognizing bicolorable hypergraphs is well-known to be NP-complete [4]. But we leave the following question open: what is the complexity of recognizing Sterboul hypergraphs?

## References

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