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# Chromatic Turán problems and a new upper bound for the Turán density of $\mathcal{K}_4^-$

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We consider a new type of extremal hypergraph problem: given an  $r$ -graph  $\mathcal{F}$  and an integer  $k \geq 2$  determine the maximum number of edges in an  $\mathcal{F}$ -free,  $k$ -colourable  $r$ -graph on  $n$  vertices.

Our motivation for studying such problems is that it allows us to give a new upper bound for an old problem due to Turán. We show that a 3-graph in which any four vertices span at most two edges has density less than  $\frac{33}{100}$ , improving previous bounds of  $\frac{1}{3}$  due to de Caen [1], and  $\frac{1}{3} - 4.5305 \times 10^{-6}$  due to Mubayi [9].

**Keywords:** Extremal combinatorics, Turán-type problems, Hypergraphs

Given an  $r$ -graph  $\mathcal{F}$  the Turán number  $\text{ex}(n, \mathcal{F})$  is the maximum number of edges in an  $n$ -vertex  $r$ -graph not containing a copy of  $\mathcal{F}$ . The Turán density of  $\mathcal{F}$  is

$$\pi(\mathcal{F}) = \lim_{n \rightarrow \infty} \frac{\text{ex}(n, \mathcal{F})}{\binom{n}{r}}.$$

For 2-graphs the Turán density is determined completely by the chromatic number of  $\mathcal{F}$ , but for  $r \geq 3$  there are very few  $r$ -graphs for which  $\pi(\mathcal{F})$  is known. (Examples of 3-graphs for which  $\pi(\mathcal{F})$  is now known include the Fano plane [2],  $\mathcal{F} = \{abc, abd, abe, cde\}$  [8] and  $\mathcal{F} = \{abc, abd, cde\}$  [7].)

The two most well-known problems in this area are to determine  $\pi(\mathcal{K}_4)$  and  $\pi(\mathcal{K}_4^-)$ , where  $\mathcal{K}_4 = \{abc, abd, acd, bcd\}$  is the complete 3-graph on 4 vertices and  $\mathcal{K}_4^- = \{abc, abd, acd\}$  is the complete 3-graph on 4 vertices with an edge removed. For  $\pi(\mathcal{K}_4)$  we have the following bounds due to Turán and Chung and Lu [3] respectively

$$\frac{5}{9} \leq \pi(\mathcal{K}_4) \leq \frac{3 + \sqrt{17}}{12} = 0.59359 \dots$$

Although the problem of determining  $\pi(\mathcal{K}_4)$  is an extremely natural question in some respects the problem of determining  $\pi(\mathcal{K}_4^-)$  is even more fundamental since  $\mathcal{K}_4^-$  is the smallest 3-graph satisfying  $\pi(\mathcal{F}) \neq 0$ . Note also that this problem may be restated as follows: determine the maximum density of a 3-graph in which any four vertices span at most two edges.

The problem of determining  $\pi(\mathcal{K}_4^-)$  has been considered by many people, including Turán [12], Erdős and Sós [5], Frankl and Füredi [6], de Caen [1] and Mubayi [9]. Previously the best known bounds were

$$\frac{2}{7} \leq \pi(\mathcal{K}_4^-) \leq \frac{1}{3} - (4.5305 \times 10^{-6}).$$

The upper bound was proved by Mubayi [9], improving on the upper bound  $\pi(\mathcal{K}_4^-) \leq 1/3$ , due to de Caen [1]. The lower bound is from a construction due to Frankl and Füredi [6].

Our main result is the following theorem, improving the upper bound for  $\pi(\mathcal{K}_4^-)$ .

**Theorem 1** *The Turán density of  $\mathcal{K}_4^-$  satisfies*

$$\frac{2}{7} \leq \pi(\mathcal{K}_4^-) < \frac{33}{100}.$$

The proof of this result leads naturally to a new general class of extremal problems which we call *chromatic Turán problems*. These are questions of the form: given an  $r$ -graph  $\mathcal{F}$  and an integer  $k \geq 2$  determine the maximum number of edges in an  $\mathcal{F}$ -free,  $k$ -colourable  $r$ -graph on  $n$  vertices. (Recall that an  $r$ -graph is  $k$ -colourable iff its vertices can be partitioned into  $k$  classes, none of which contain an edge.) We denote this quantity by  $\text{ex}_k(n, \mathcal{F})$ . The corresponding  $k$ -chromatic Turán density is then defined to be

$$\pi_k(\mathcal{F}) = \lim_{n \rightarrow \infty} \frac{\text{ex}_k(n, \mathcal{F})}{\binom{n}{r}}.$$

The following lemma shows that any upper bound for the chromatic Turán density  $\pi_3(\mathcal{K}_4^-)$  will yield an upper bound for the ordinary Turán density  $\pi(\mathcal{K}_4^-)$ . (The proof of Theorem 1 actually requires a slightly stronger result but the principle is the same.)

**Lemma 1** *If  $\pi_3 = \pi_3(\mathcal{K}_4^-)$  and  $\pi = \pi(\mathcal{K}_4^-)$  then*

$$\pi \leq \frac{2 + \pi_3/\pi}{9}.$$

To prove Lemma 1 we first note that if  $\mathcal{F}$  is a  $\mathcal{K}_4^-$ -free 3-graph with  $n$  vertices and  $m$  edges then it satisfies

$$mn = \sum_{xy \in V^{(2)}} d_{xy}^2 + q_1, \tag{1}$$

where  $q_1$  is the number of 4-sets in  $\mathcal{F}$  spanning exactly one edge. Lemma 1 is proved by giving a lower bound for  $q_1$ .

This idea of giving a lower bound for  $q_1$  was previously used by Mubayi [9] (he used supersaturation to achieve this). Our approach is quite different in that it involves a chromatic Turán problem.

To be more precise the key idea used in the proof of Lemma 1 is as follows. Let  $\mathcal{F}$  be a  $\mathcal{K}_4^-$ -free 3-graph of order  $n$  and let  $uvw \in \mathcal{F}$ . If  $E_{ij} = \{k : ijk \in \mathcal{F}\}$  then  $E_{uv}, E_{uw}$  and  $E_{vw}$  are pairwise disjoint. We say that an edge  $xyz \in \mathcal{F}$  is *bad* relative to the edge  $uvw$  if  $xyz \cap uvw = \emptyset$  and none of the following

4-sets span exactly one edge:  $uvw x, uvw y, uvw z, xyz u, xyz v, xyz w$ . It is straightforward to check that if  $xyz$  is bad relative to  $uvw$  then  $xyz \subset E_{uv} \cup E_{uw} \cup E_{vw}$  and  $xyz$  is not contained entirely in either  $E_{uv}, E_{uw}$  or  $E_{vw}$ . Hence the collection of bad edges relative to  $uvw$  is 3-colourable. This fact can then be used to give a lower bound for  $q_1$  in terms of the 3-chromatic Turán density  $\pi_3(\mathcal{K}_4^-)$ .

One obvious reason why chromatic Turán problems do not seem to have been considered before is that for 2-graphs they are rather uninteresting. If  $K_t$  is the complete 2-graph of order  $t$  then Turán's theorem determines not only their Turán numbers but also their chromatic Turán numbers.

**Theorem 2 (Turán[11])** *If  $t \geq 3$  then the unique  $K_t$ -free graph with  $n$  vertices and  $ex(n, K_t)$  edges is the complete  $(t - 1)$ -partite graph whose vertex classes are as equal as possible in size.*

**Corollary 1** *If  $t \geq 3$  and  $k \geq 2$  then*

$$ex_k(n, K_t) = \begin{cases} ex(n, K_t), & k \geq t - 1, \\ ex(n, K_{k+1}), & k \leq t - 2. \end{cases}$$

For general  $r$ -graphs with  $r \geq 3$  the problems of determining chromatic and ordinary Turán numbers seem to be genuinely different. While the extremal  $K_t$ -free 2-graph is not only  $K_t$ -free but also  $(t - 1)$ -colourable this does not seem to be the case in general. For example the conjectured extremal  $\mathcal{K}_4^-$ -free 3-graph has rather large chromatic number.

In order to make use of Lemma 1 we need an upper bound on  $\pi_3(\mathcal{K}_4^-)$ . To achieve this we first consider  $\pi_2(\mathcal{K}_4^-)$ .

**Theorem 3** *The 2-chromatic Turán density  $\pi_2(\mathcal{K}_4^-)$  satisfies*

$$\frac{4}{9\sqrt{3}} \leq \pi_2(\mathcal{K}_4^-) \leq \frac{3}{10}.$$

*Proof:* The lower bound follows from a simple construction. For the upper bound let  $\mathcal{F} = ((A, B), E)$  be a bipartite  $\mathcal{K}_4^-$ -free 3-graph with  $n$  vertices and  $m$  edges. Let  $|A| = \alpha n$ , so  $|B| = (1 - \alpha)n$ . Then

$$\sum_{xy \in A \times B} d_{xy} = 2m, \quad \sum_{xy \in A^{(2)} \cup B^{(2)}} d_{xy} = m. \quad (2)$$

From (1) and the fact that  $q_1 \geq 0$  we obtain

$$mn \geq \sum_{xy \in A \times B} d_{xy}^2 + \sum_{xy \in A^{(2)} \cup B^{(2)}} d_{xy}^2$$

Convexity and (2) then imply that

$$m \leq \frac{n^3}{\frac{4}{\alpha(1-\alpha)} + \frac{2}{\alpha^2 + (1-\alpha)^2}} \leq \frac{n^3}{20}.$$

The result then follows directly. □

With some more work we can obtain an upper bound for  $\pi_3(\mathcal{K}_4^-)$ . (Again the lower bound follows from a simple construction.)

**Theorem 4** *The 3-chromatic Turán density  $\pi_3(\mathcal{K}_4^-)$  satisfies*

$$\frac{5}{18} \leq \pi_3(\mathcal{K}_4^-) < \frac{58}{177}.$$

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