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# On the $L(p, 1)$ -labelling of graphs

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In this paper we improve the best known bound for the  $L(p, 1)$ -labelling of graphs with given maximal degree.

**Keywords:** lambda-labelling

## 1 Introduction.

In all the paper we work on a graph  $G$  with maximal degree  $\Delta$ . For a set of vertices  $S \subset V(G)$ , the graph  $G \setminus S$  is the graph induced by  $V(G) \setminus S$ . The distance  $d(u, v)$  between two vertices  $u$  and  $v$  is the number of edges in the shortest path from  $u$  to  $v$ . We say that  $v$  is a  $d$ -neighbor of  $u$  if  $d(u, v) = d$ . Let  $N_d(v)$  be the set of  $d$ -neighbors of  $v$ . We will generally use the common term *neighbor* instead of 1-neighbor. A  $L(\alpha_1, \alpha_2, \dots, \alpha_k)$ -labelling of a graph  $G$  is a function  $l : V(G) \rightarrow [0, \lambda]$  such that for any pair of vertices  $u$  and  $v$  if  $d(u, v) = d \leq k$  then  $|l(u) - l(v)| \geq \alpha_d$ . The problem is to find an  $L(\alpha_1, \alpha_2, \dots, \alpha_k)$ -labelling of  $G$  that minimizes  $\lambda$ . We denote  $\lambda_{\alpha_1, \alpha_2, \dots, \alpha_k}(G)$  such minimal  $\lambda$ . For a sequence of non-negative integers  $S = (\alpha_1, \alpha_2, \dots, \alpha_k)$ , we will use the notation  $\lambda_S(G)$  instead of  $\lambda_{\alpha_1, \alpha_2, \dots, \alpha_k}(G)$ .

This problem arises from the *channel assignment problem*. The channel assignment problem is to assign a channel to each radio transmitter so that close transmitters do not interfere and such that we use the minimum span of frequency. Roberts proposed to assign channels such that “close” transmitters receive different channels and “very close” transmitters receive channels that are at least two channels apart. This is a  $L(2, 1)$ -labelling of a graph  $G$  where the vertices are the transmitters, the “very close” transmitters are adjacent vertices and the “close” transmitters are vertices at distance two in  $G$ . Since the constraints between transmitters diminish with the distance, the  $L(\alpha_1, \alpha_2, \dots, \alpha_k)$ -labelling of graph is interesting for this problem when the sequence  $\alpha_1, \alpha_2, \dots, \alpha_k$  is decreasing. Many work has been done on  $L(2, 1)$ -labeling since the first paper of J.R.Griggs and R.K.Yeh [7]. Many papers deal with bounding  $\lambda_{\alpha_1, \alpha_2}$  for some family of graphs or given some graphs invariants such as  $\chi(G)$  and  $\Delta$  (See for example [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14]). In their paper [7], Griggs and Yeh proved that  $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta$  and made the following conjecture.

**Conjecture 1** For any graph  $G$  with maximal degree  $\Delta \geq 2$ ,  $\lambda_{2,1}(G) \leq \Delta^2$ .

Actually they proved it for  $\Delta = 2$  and for graphs of diameter at most two. They also proved that determining  $\lambda_{2,1}(G)$  is NP-complete. In this paper we focus on bounding  $\lambda_{p,1}$  according to  $\Delta$ . In [3] the authors gave an algorithm for the  $L(2, 1)$ -labeling and improved the upper bound of  $\lambda_{2,1}$  to  $\Delta^2 + \Delta$ . In [4], with the same algorithm they obtained that  $\lambda_{p,1}(G) \leq \Delta^2 + (p - 1)\Delta$ . Let  $\sigma(S, \Delta)$  be the function

defined for any sequence  $S = (\alpha_1, \dots, \alpha_k)$  by  $\sigma(S, \Delta) = \sum_{i=1}^k \alpha_i \Delta (\Delta - 1)^{i-1}$ . With the algorithm used in [3, 4], we generalise their result as follow.

**Proposition 1** *For any sequence of non-negative integers  $S = (\alpha_1, \alpha_2, \dots, \alpha_k)$ , with  $k \geq 1$ , and any graph  $G$  with maximum degree  $\Delta$ , we have that  $\lambda_S(G) \leq \sigma(S, \Delta)$ .*

But this is not the best known bound. In [9], Král and Škrekovski had a result on the list channel assignment problem. As a corollary of their result we have that :

**Theorem 1** *For any sequence of non-negative integers  $S = (\alpha_1, \alpha_2, \dots, \alpha_k)$ , such that  $k \geq 2$  and  $\alpha_1 > \alpha_2$ , and any graph  $G$  with maximum degree  $\Delta \geq 3$ , we have that  $\lambda_S(G) \leq \sigma(S, \Delta) - 1$ .*

In this paper, we improve this last bound by two different ways for some specific sequences  $S$ .

**Theorem 2** *For any sequence  $S = (\alpha_1, \dots, \alpha_k)$  with  $k \geq 2$  and such that  $\alpha_1 > \alpha_2 \geq \alpha_3 \geq \dots \geq \alpha_k = 1$ , and any connected graph  $G$  with maximum degree  $\Delta \geq 3$ , there is an ordering of the vertices,  $v_0, v_1, \dots, v_n$  and a  $L(\alpha_1, \dots, \alpha_k)$ -labelling  $l$  of  $G$  such that :*

- $l(v_0) \leq \sigma(S, \Delta) - 1$
- $l(v_j) \leq \sigma(S, \Delta) - j$  for  $1 \leq j < k$
- $l(v_j) \leq \sigma(S, \Delta) - k$  for  $k \leq j$

This implies that just a constant number of vertices,  $k$ , are labelled more than  $\sigma(S, \Delta) - k$ .

**Theorem 3** *For any sequence  $S = (p, 1)$  with  $p \geq 2$  and any graph  $G$  with maximum degree  $\Delta \geq 3$ , we have that  $\lambda_{p,1}(G) \leq \sigma(S, \Delta) - 2 = \Delta^2 + (p - 1)\Delta - 2$ .*

So, for the  $L(2,1)$ -labelling we obtain that  $\lambda_{2,1}(G) \leq \Delta^2 + \Delta - 2$  and we get a little closer to Conjecture 1. To prove Theorem 3 we need the following structural lemma.

**Lemma 1** *Every graph  $G$  with maximal degree  $\Delta \geq 3$  has either :*

- (i) *a vertex  $v$  with degree less than  $\Delta$ .*
- (ii) *a cycle of length three.*
- (iii) *two cycle of length four passing through the same vertex  $v$ .*
- (iv) *a vertex  $v$  with three neighbors  $u, x$  and  $y$ , such that there is a cycle of length four passing through the edge  $wv$  and such that the graph  $G \setminus \{x, y\}$  is connected.*
- (v) *a vertex  $u$  with two adjacent vertices  $v$  and  $w$  such that the graph  $G \setminus X$  is connected, where  $X$  is the set  $(N(v) \cup N(u)) \setminus \{w\}$ .*

For proving Theorem 2, the following corollary of Lemma 1 is sufficient.

**Corollary 1** *Every graph  $G$  with maximal degree  $\Delta \geq 3$  has either :*

- (i) *a vertex  $v$  with degree less than  $\Delta$ .*
- (ii) *a cycle of length  $\leq 4$ .*
- (iii) *a vertex  $v$  with two neighbors  $x$  and  $y$  such that the graph  $G \setminus \{x, y\}$  is connected.*

In this abstract we do not prove Lemma 1 and Theorem 3, but most of the arguments used in the proof of Theorem 3 are in the proof of Theorem 2. In section 2 we generalise the labeling algorithm presented in [3] and we obtain Proposition 1. In section 3 we modify it to prove Theorem 2.

## 2 The basic algorithm.

In [3] the authors present an algorithm that  $L(2, 1)$ -label graphs and establish that for a graph  $G$  of maximal degree  $\Delta$  we have  $\lambda_{2,1}(G) \leq \Delta^2 + \Delta$ . Here we present an extended version of this algorithm that  $L(\alpha_1, \dots, \alpha_k)$ -label graphs and establishes Proposition 1.

```

i = 0 ;
while there are unlabelled vertices do
  for  $v_j = v_n$  to  $v_0$  do
    if  $v_j$  is unlabelled and  $v_j$  can be labelled  $i$  then
      | let  $v_j$  be labelled  $i$ ;
    end
  end
   $i = i + 1$ ;
end

```

In this algorithm a vertex  $v_j$  can be labelled  $i$  if it has no  $d$ -neighbor already labelled  $x$  with  $i - \alpha_d < x < i + \alpha_d$ .

Let us denote  $l(v)$  the value the algorithm assigns to the vertex  $v$ . Observe that if the vertex  $v$  is not labelled  $i$  it cannot be because its  $d$ -neighbor  $u$  is labelled  $l(u)$ , with  $i < l(u) < i + \alpha_d$ . Indeed, when the algorithm “proposed”  $v$  to be labelled  $i$ , the vertex  $u$  was still unlabelled. So, a vertex  $u$  which has been labelled  $l(u)$  could only “forbid” its  $d$ -neighbor  $v$  to be labelled  $l(u)$ ,  $l(u) + 1, \dots$ , and  $l(u) + \alpha_d - 1$ . Let us denote  $F(u, v)$ , the set of values which have been forbidden by  $u$  to  $v$  during the execution of the algorithm, we have that  $F(u, v) = \{l(u), l(u) + 1, \dots, l(u) + \alpha_d - 1\}$ , if  $d(u, v) = d$ . The set  $F(v)$  of all the values that have been forbidden to  $v$  is the union on all the vertices  $u$  of  $F(u, v)$ ,  $F(v) = \bigcup_{u \in V(G)} F(u, v)$ . Note that the algorithm labels  $v$  with the smallest value which is not in  $F(v)$ . So  $l(v) \leq |F(v)|$ , since there are  $|F(v)| + 1$  values in the interval  $[0, |F(v)|]$ . The set  $F(v)$  being a union of possibly disjoint sets we have  $|F(v)| \leq \sum_{u \in V(G)} |F(u, v)|$ . In a graph of maximal degree  $\Delta$ , one can easily see by induction on  $i$  that there are at most  $\Delta(\Delta - 1)^{i-1}$  vertices in  $N_i(v)$ . Since if  $u$  is a  $i$ -neighbor of  $v$  we have  $|F(u, v)| = \alpha_i$ , we obtain that  $l(v) \leq \sum_{i=1}^k \alpha_i \Delta (\Delta - 1)^{i-1}$ .

## 3 The improved algorithm and proof of Theorem 2.

To improve the bound we have in Proposition 1, we have to be more carefull on the order the algorithm considers the vertices. If we have two vertices  $v_p$  and  $v_q$ , with  $p < q$  and  $d(v_p, v_q) = d \leq k$ , the vertex  $v_p$  only forbids  $\alpha_d - 1$  values to  $v_q$ . Indeed, the vertex  $v_p$  does not forbid to  $v_q$  the value  $l(v_p)$ , when the algorithm considered the possibility to label the vertex  $v_q$  with the value  $l(v_p)$  the vertex  $v_p$ , being considered after  $v_q$  by the algorithm, was still unlabelled. This observation reduces the size of  $F(v_p, v_q)$  by one and so the bound on  $l(v_q)$ . So, if for a vertex  $v_q$  there are  $x$  vertices  $v_p$ , with  $p < q$  and  $d(v_p, v_q) = d \leq k$ , then  $l(v_q) \leq |F(v_q)| = \sigma(S, \Delta) - x$ . It would be interesting to have an order

such that many vertices have some  $d$ -neighbors, with  $d \leq k$ , posterior to them. It is not possible for all the vertices, the vertex  $v_0$  being the last vertex considered by the algorithm, we cannot reduce the size of  $F(v_0)$  with this observation.

Given a spanning tree  $T$  of  $G$  rooted in  $r$ , numbering the vertices of  $G$  according to a preorder traversal of  $T$  we obtain that the vertices of  $G$  numbered  $v_0, \dots, v_n$  are such that :

- $v_0 = r$
- If  $i < j < k$  then  $d(v_i, v_j) \leq k$ .
- If  $k \leq j$ , there are at least  $k$  vertices  $v_i$  such that  $d(v_i, v_j) \leq k$ .

With such numbering of the vertices, by the previous observation, we clearly prove the two last points of Theorem 2. Now we are going to show how to choose  $T$  and  $r$  in order to obtain the first point. To do that, consider the case of Corollary 1 we are in.

**Case (i)** If there is a vertex of degree less than  $\Delta$ , let  $r$  be this vertex and consider any spanning tree  $T$  of  $G$ . In this case, since there are at most  $\Delta - 1$  vertices in  $N_1(v_0)$ , we easily bound  $|F(v_0)|$  by  $\sigma(S, \Delta) - \alpha_1$ .

**Case (ii)** If there is a cycle of length  $\leq 4$ , let  $r$  be a vertex of this cycle and consider any spanning tree  $T$  of  $G$ . In this case, since there are at most  $\Delta(\Delta - 1) - 1$  vertices in  $N_2(v_0)$ , we easily bound  $|F(v_0)|$  by  $\sigma(S, \Delta) - \alpha_2$ .

**Case (iii)** If there is a vertex with two neighbors  $x$  and  $y$  such that the graph  $G \setminus \{x, y\}$  is connected, let  $r$  be this vertex. We construct  $T$  from any spanning tree of  $G \setminus \{x, y\}$  by adding the edges  $rx$  and  $ry$ . We then number the vertices by a preorder traversal of  $T$  such that  $x$  and  $y$  are the two last numbered vertices. It is possible since  $x$  and  $y$  are leaves in  $T$ . So we have that  $v_0 = r$ ,  $v_{n-1} = x$  and  $v_n = y$ . Since  $v_n$  is the first vertex considered by the algorithm, it clearly labels it 0. Since  $d(v_n, v_{n-1}) = 2$  (else, see the previous case), the algorithm cannot label  $v_{n-1}$  less than  $\alpha_2$ . We consider two cases according to the label of  $v_{n-1}$ .

- If  $v_{n-1}$  is labelled  $\alpha_2$ , since  $\alpha_1 > \alpha_2$ , the value  $\alpha_2$  is in both  $F(v_{n-1}, v_0)$  and  $F(v_n, v_0)$ . This implies that  $|F(v_0)| \leq \sigma(S, \Delta) - 1$ .
- If  $v_{n-1}$  is not labelled  $\alpha_2$ , since there was no vertex labelled  $\alpha_2$  when the algorithm considered this value for  $v_{n-1}$ , there is a vertex  $v_k$  labelled  $l$  such that  $d(v_k, v_{n-1}) = d$  and  $l + \alpha_d > \alpha_2$ . Since  $l < \alpha_2$  and  $\alpha_k = 1$  we have that  $d < k$ . This implies that  $d(v_k, v_0) \leq k$  and that the value  $l$  is in both  $F(v_k, v_0)$  and  $F(v_n, v_0)$ . This implies that  $|F(v_0)| \leq \sigma(S, \Delta) - 1$ .

## References

- [1] P. Bella, D. Král, B. Mohar and K. Quittnerová, Labeling planar graphs with a condition at distance two. In these proceedings.
- [2] A.A. Bertossi, C.M. Pinotti and R.B. Tan, Channel assignment with separation for special classes of wireless networks : Girds and rings. In Proc. IPDPS'02 (International Parallel and Distributed Processing Symposium), pp. 28-33. IEEE Computer Society, 2002.

- [3] G.J. Chang and D. Kuo, The  $L(2,1)$ -labeling problem on graphs, *SIAM J. Discrete Math.*, 9 (1996), pp. 309-316.
- [4] G.J. Chang, W.-T. Ke, D. Kuo, D.D.-F. Liu, and R.K. Yeh. On  $L(d,1)$ -labelings of graphs, *Discrete Mathematics*, 220(2002), pp. 57-66.
- [5] G. Fertin and A. Raspaud,  $L(p,q)$  Labeling of  $d$ -Dimensional Grids, talk presented at EURO-COMB'03, Charles University, Prague, Czech Republic, September 2003.
- [6] J. Fiala, A.V. Fishkin and F.V. Fomin, On distance constrained labeling of disk graphs, *Theoretical Computer Science*, 326(2004), pp. 261-292.
- [7] J.R. Griggs and R.K. Yeh, Labelling graphs with a condition at distance 2, *SIAM J. Discrete Math.*, 5 (1992), pp. 586-595.
- [8] J.-H Kang,  $L(2,1)$ -labelling of 3-regular Hamiltonian cubic graphs, submitted.
- [9] D. Král and R. Škrekovski, A theorem about the channel assignment problem, *SIAM J. Discrete Math.*, 16(3) (2003), pp. 426-437.
- [10] D. Král, Coloring powers of chordal graphs, *SIAM J. Discrete Math.*, 18(3) (2004), pp. 451-461.
- [11] C. McDiarmid, On the span in channel assignment problem: bounds, computing and counting, *Discrete Math.*, 266(2003), pp. 387-397.
- [12] M. Molloy and M.R. Salavatipour. Frequency channel assignment on planar networks, In Proc. 10th Annual European Symposium (ESA 2002), Rome, Italy, September 2002, volume 2461, pages 736-747. Lecture Notes Computer Science, Springer-Verlag Berlin, 2002.
- [13] D. Sakai, Labeling chordal graphs: Distance two condition, *SIAM J. Discrete Math.*, 7 (1994), pp. 133-140.
- [14] M. Whittlesey, J. Georges and D.W. Mauro, On the  $\lambda$ -number of  $Q_n$  and related graphs. *SIAM J. Discrete Math.*, 8 (1995), pp. 499-506.

