# On the Minimum Number of Completely 3-Scrambling Permutations

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# *On the Minimum Number of Completely 3-Scrambling Permutations*

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A family $\mathcal{P} = \{\pi_1, \ldots, \pi_q\}$ of permutations of $[n] = \{1, \ldots, n\}$ is *completely $k$-scrambling* [Spencer, 1972; Füredi, 1996] if for any distinct $k$ points $x_1, \ldots, x_k \in [n]$, permutations $\pi_i$'s in $\mathcal{P}$ produce all $k!$ possible orders on $\pi_i(x_1), \ldots, \pi_i(x_k)$. Let $N^*(n,k)$ be the minimum size of such a family. This paper focuses on the case $k = 3$. By a simple explicit construction, we show the following upper bound, which we express together with the lower bound due to Füredi for comparison.

$$\frac{2}{\log_2 e}\log_2 n \le N^*(n,3) \le 2\log_2 n + (1 + o(1))\log_2\log_2 n.$$

We also prove the existence of $\lim_{n\to\infty} N^*(n,3)/\log_2 n = c_3$. Determining the value $c_3$ and proving the existence of $\lim_{n\to\infty} N^*(n,k)/\log_2 n = c_k$ for $k \ge 4$ remain open.

## 1 Introduction and Summary

Following Spencer [Sp72] and Füredi [Fü96], call a family $\mathcal{P} = \{\pi_1, \ldots, \pi_q\}$ of permutations of $[n]$ *completely $k$-scrambling* if for any distinct $x_1, x_2, \ldots, x_k \in [n]$, there exists a permutation $\pi_i \in \mathcal{P}$ such that $\pi_i(x_1) < \pi_i(x_2) < \cdots < \pi_i(x_k)$; or equivalently, $\pi_i$'s applied to $x_1, x_2, \ldots, x_k$ produce all $k!$ orders. This paper focuses on the case $k = 3$. Following Füredi [Fü96], say that a family $\mathcal{P}$ is 3-*mixing* if for any distinct $x, y, z \in [n]$, there is a permutation $\pi_i \in \mathcal{P}$ that places $x$ between $y$ and $z$, i.e., there is a permutation $\pi_i$ such that either $\pi_i(y) < \pi_i(x) < \pi_i(z)$ or $\pi_i(z) < \pi_i(x) < \pi_i(y)$.

Let $N^*(n,k)$ be the minimum $q$ such that completely $k$-scrambling $q$ permutations exist for $[n]$. The best known bounds for $N^*(n,k)$ can be expressed as follows. For arbitrary fixed $k \ge 3$, as $n \to \infty$,

$$\left(\frac{1}{\log_2 e}(k-1)! + o(1)\right)\log_2 n \le N^*(n,k) \le \frac{k}{\log_2(k!/(k!-1))}\log_2 n. \tag{1}$$

The coefficient of the upper bound in (1) is $\Theta(k \cdot k!)$; thus the gap between the coefficients of the lower and upper bounds in (1) is $\Theta(k^2)$. The upper bound in (1) was shown by Spencer [Sp72] by a probabilistic argument, where one considers the probability that some order among some $x_1, \ldots, x_k$ is never produced by $q$ independent random permutations. The lower bound in (1) was first proved by Füredi [Fü96] for $k = 3$, and was proved for $k \ge 3$ by Radhakrishnan [Ra03]; entropy arguments are used in both work;

the factor $\log_2 e$ in the lower bound comes from the fact that $\int_0^1 H(x)dx = (\log_2 e)/2$, where $H(x)$ is the binary entropy function.

As for the case $k = 3$, Füredi [Fü96] has shown that

$$\frac{2}{\log_2 e} \log_2 n \le N^*(n,3) \le \left(\frac{10}{\log_2 7}\right) \log_2 n + O(1), \tag{2}$$

where the coefficients of $\log_2 n$ are $1.38\ldots$ and $3.56\ldots$ in (2). The lower bound in (2) is in fact a lower bound for the case where we only require a family to be 3-mixing. No better lower bound for completely 3-scrambling families is known. If a family $\mathcal{P} = \{\pi_1, \ldots, \pi_q\}$ is 3-mixing, by adding to $\mathcal{P}$ the $q$ reverse permutations of $\pi_i$'s mapping $x \mapsto n+1-\pi_i(x)$, we can obtain completely 3-scrambling $2q$ permutations. Ishigami [Is95] has given an efficient recursive construction of 3-mixing families starting with a 3-mixing family of five permutations of $\{1, \ldots, 7\}$. Füredi [Fü96] gave the upper bound in (2) by making these observations and doubling the size of Ishigami's 3-mixing family.

In this paper, we first give an improved upper bound for $N^*(n,3)$ by a simple construction. Let $f(q)$ be the maximum $n$ such that completely 3-scrambling $q$ permutations exist for $[n]$.

**Theorem 1**

$$f(q) \ge \binom{\lfloor q/2 \rfloor}{\lfloor q/4 \rfloor}.$$

The following upper bound on $N^*(n,3)$ readily follows.

**Corollary 1**

$$N^*(n,3) \le 2\log_2 n + (1 + o(1)) \log_2 \log_2 n.$$

It seems natural to conjecture that for every fixed $k \ge 3$, as $n \to \infty$, $N^*(n,k) = (c_k + o(1)) \log_2 n$ for some $c_k$. We show the existence of limit for the case $k = 3$:

**Theorem 2**

$$\lim_{q \to \infty} \frac{\log_2 f(q)}{q} = C \ \text{ exists.}$$

The following immediately follows.

**Corollary 2**

$$\lim_{n \to \infty} \frac{N^*(n,3)}{\log_2 n} = 1/C = c_3 \ \text{ exists.}$$

# 2 Proofs

We can identify in a natural way a total order $\phi$ on $[n]$ and the permutation of $[n]$ induced by $\phi$; thus we speak interchangeably in terms of permutations and total orders. In fact for an arbitrary finite set $U$ with $n$ elements, we can assume for our purposes that $U$ is identified with $[n]$ in an arbitrary fixed way, and speak about permutations of $U$ in terms of total orders on $U$.

**Proof of Theorem 1.** Put $r = \lfloor q/2 \rfloor$ and let $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ be a family of subsets of $\{1, \ldots, r\}$ such that $A_i \not\subseteq A_j$ for all $i \neq j$; i.e., $\mathcal{F}$ is an antichain.

For each point $x \in \{1, \ldots, r\}$, define two orders $\phi_x$ and $\psi_x$ on $\mathcal{F}$. In both orders $\phi_x$ and $\psi_x$, the sets $A_i$ containing the point $x$ are smaller than all the sets $A_k$ not containing $x$. Among the sets containing $x$ and among the sets not containing $x$: in the order $\phi_x$, $A_i < A_j$ precisely when $i < j$; in the order $\psi_x$, this is reversed, and $A_i < A_j$ precisely when $i > j$.

We claim that for arbitrary distinct $i, j, k \in [m]$, there exists an order $\theta \in \{\phi_1, \psi_1, \phi_2, \psi_2, \ldots, \phi_r, \psi_r\}$ such that $A_i < A_j < A_k$ in the order $\theta$. To see the claim fix a point $x \in (A_i - A_k) \neq \emptyset$, i.e., $x \in A_i$ and $x \notin A_k$. Depending on whether $x \in A_j$ or $x \notin A_j$, we specify an order $\theta$ that produces the ordering $A_i < A_j < A_k$.
Case $x \in A_j$ : Let $\theta = \phi_x$ if $i < j$ and let $\theta = \psi_x$ if $i > j$.
Case $x \notin A_j$ : Let $\theta = \phi_x$ if $j < k$ and let $\theta = \psi_x$ if $j > k$.

Clearly under the order $\theta$, $A_i < A_j < A_k$. Hence the $2r$ orders thus defined on $[m]$ are completely 3-scrambling. We obtain the theorem by taking $\mathcal{F}$ to be the family of all subsets of $\{1, \ldots, r\}$ with cardinality $\lfloor r/2 \rfloor = \lfloor q/4 \rfloor$. □

**Proof of Theorem 2.** Our proof of Theorem 2 will be basically similar to Füredi's proof [Fü96] of the existence of $\lim_{q\to\infty} (\log_2 g(q))/q$, where $g(q)$ is the maximum $n$ such that 3-mixing $q$ permutations exist for $[n]$. To make a recursive construction go through for scrambling permutations, we introduce and use red-blue colored doubly reversing permutations: Call a family $\mathcal{P} = \{\pi_1, \ldots, \pi_q\}$ of permutations of $[n]$ *2-reversing* if there is a coloring $\chi : \{\pi_1, \ldots, \pi_q\} \to \{\text{red}, \text{blue}\}$ such that for every distinct $i, j \in [n]$, there are red $\pi_\kappa$, red $\pi_\lambda$, blue $\pi_\mu$, and blue $\pi_\nu$ satisfying

$$\pi_\kappa(i) < \pi_\kappa(j), \;\; \pi_\lambda(i) > \pi_\lambda(j); \quad \pi_\mu(i) < \pi_\mu(j), \;\; \pi_\nu(i) > \pi_\nu(j).$$

For a permutation $\pi$ of $[n]$, let $\mathrm{reverse}(\pi)$ be the permutation of $[n]$ mapping $x \mapsto n + 1 - \pi(x)$. Let $\mathcal{P}$ be a family of permutations of $[n]$ with $|\mathcal{P}| \geq 3$. We can easily transform $\mathcal{P}$ to a 2-reversing family by adding at most two permutations as follows. Arbitrarily fix two distinct permutations $\sigma, \tau \in \mathcal{P}$ such that $\tau \neq \mathrm{reverse}(\sigma)$; such $\sigma$ and $\tau$ exist since $|\mathcal{P}| \geq 3$; add $\mathrm{reverse}(\sigma)$ and $\mathrm{reverse}(\tau)$ to $\mathcal{P}$; color $\sigma$ and $\mathrm{reverse}(\sigma)$ red; color $\tau$ and $\mathrm{reverse}(\tau)$ blue; color the remaining permutations arbitrarily.

Let $f^*(q)$ be the maximum $n$ such that completely 3-scrambling *and* 2-reversing $q$ permutations exist for $[n]$. By definition and from the discussion above we have

$$f^*(q) \leq f(q) \leq f^*(q + 2). \tag{3}$$

**Claim 1**

$$f^*(q + r) \geq f^*(q) f^*(r).$$

For the moment we assume that Claim 1 holds and go on to derive Theorem 2.

The sequence $(1/q)\log_2 f^*(q)$ is bounded above. ¿From this and Claim 1 it follows by classical calculus (Fekete's theorem) that

$$\lim_{q\to\infty} \frac{1}{q}\log_2 f^*(q) = \limsup_{q\to\infty} \frac{1}{q}\log_2 f^*(q).$$

From (3) it now follows that

$$\lim_{q\to\infty} \frac{1}{q}\log_2 f(q) = \lim_{q\to\infty} \frac{1}{q}\log_2 f^*(q).$$

Thus we are left to prove Claim 1.

Let $\mathcal{S} = \{\sigma_1, \dots, \sigma_q\}$ and $\mathcal{T} = \{\tau_1, \dots, \tau_r\}$ be completely 3-scrambling and 2-reversing families of permutations of $[l]$ and $[m]$ respectively. Assume that both families are validly red-blue colored. Let $U = \{(i,j) : 1 \le i \le l, 1 \le j \le m\}$; think of $U$ as a matrix with $l$ rows and $m$ columns. We will show that we can define $q + r$ orders on $U$ that are completely 3-scrambling and 2-reversing. Note that from this Claim 1 follows.

Let $x = (i,j)$ and $y = (i',j')$ be distinct elements of $U$. For $k = 1, \dots, q$, define the order $\tilde{\sigma}_k$ using $\sigma_k$ in a row-major form as follows: if $i \ne i'$, order $x$ and $y$ according to the order of $\sigma_k(i)$ and $\sigma_k(i')$. When $i = i'$: if $\sigma_k$ is red, $(i,j) < (i,j') \iff j < j'$; if $\sigma_k$ is blue, $(i,j) < (i,j') \iff j > j'$. Similarly for $k = 1, \dots, r$, define the order $\tilde{\tau}_k$ on $U$ in a column-major form: when $j \ne j'$, $x < y \iff \tau_k(j) < \tau_k(j')$; when $j = j'$: if $\tau_k$ is red, $(i,j) < (i',j) \iff i < i'$; if $\tau_k$ is blue, $(i,j) < (i',j) \iff i > i'$. As for colors, let $\tilde{\sigma}_k$ and $\tilde{\tau}_k$ inherit the colors of $\sigma_k$ and $\tau_k$.

**Claim 2** The family $\mathcal{F} = \{\tilde{\sigma}_1, \dots, \tilde{\sigma}_q, \tilde{\tau}_1, \dots, \tilde{\tau}_r\}$ is completely 3-scrambling and 2-reversing.

To see Claim 2, let $x_1 = (i_1, j_1), x_2 = (i_2, j_2), x_3 = (i_3, j_3)$ be distinct elements of $U$. If $i_1, i_2, i_3$ are all distinct, $\sigma_k$'s produce all six orderings of $i_1, i_2, i_3$, and hence $\tilde{\sigma}_k$'s produce all six orderings of $x_1, x_2, x_3$. Similar arguments with $\tau_k$'s and $\tilde{\tau}_k$'s apply for the case when $j_1, j_2, j_3$ are all distinct.

The remaining case is when $|\{i_1, i_2, i_3\}| = |\{j_1, j_2, j_3\}| = 2$. We write, e.g., 231 to express the ordering $x_2 < x_3 < x_1$. Assume that

$$x_1 = (i,j),\ x_2 = (i,j'),\ x_3 = (i',j),\ \ i \ne i', j \ne j'.$$

We will see that all six orderings of $x_1, x_2, x_3$ are produced by checking that (1) all the four orders in which $x_3$ is smallest or largest, i.e., 312, 321, 123, 213 are produced and that (2) all the four orders in which $x_2$ is smallest or largest are produced.

A red $\tilde{\sigma}_\kappa$ and a blue $\tilde{\sigma}_\mu$ satisfying $\sigma_\kappa(i) < \sigma_\kappa(i')$ and $\sigma_\mu(i) < \sigma_\mu(i')$ produce 123 and 213 respectively. Similarly, a red $\tilde{\sigma}_\lambda$ and a blue $\tilde{\sigma}_\nu$ satisfying $\sigma_\lambda(i) > \sigma_\lambda(i')$ and $\sigma_\nu(i) > \sigma_\nu(i')$ produce 312 and 321 respectively. Thus all the four orders in which $x_3$ is smallest or largest are produced. Similarly, two red $\tilde{\tau}$'s and two blue $\tilde{\tau}$'s ordering $j$ and $j'$ in both directions produce the four orders in which $x_2$ is smallest or largest.

Finally, if $x = (i,j)$ and $y = (i',j')$ are distinct points in $U$, either (i) $i \ne i'$ or (ii) $j \ne j'$. The 2-reversing condition is satisfied by $\tilde{\sigma}_k$'s in case (i) and by $\tilde{\tau}_k$'s in case (ii). □

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