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Relaxed Two-Coloring of Cubic Graphs

Robert Berke¹ and Tibor Szabó¹

¹Department of Computer Science, ETH Zürich, 8092 Switzerland

We show that any graph of maximum degree at most 3 has a two-coloring, such that one color-class is an independent set while the other color induces monochromatic components of order at most 189. On the other hand for any constant C we exhibit a 4-regular graph, such that the deletion of any independent set leaves at least one component of order greater than C . Similar results are obtained for coloring graphs of given maximum degree with $k + \ell$ colors such that k parts form an independent set and ℓ parts span components of order bounded by a constant. A lot of interesting questions remain open.

Keywords: Vertex coloring, bounded size components

1 Introduction

In this paper we consider a relaxation of proper coloring by allowing “errors” of certain controlled kind. We say that a coloring of a graph is C -relaxed if all monochromatic components have order at most C . With this definition, 1-relaxed is equivalent to proper coloring. It is easy to see that any graph of maximum degree at most 3 has a 2-relaxed two-coloring. Alon, Ding, Oporowski and Vertigan [1] proved that every graph of maximum degree 4 has a 57-relaxed two-coloring. They also gave a construction of a 6-regular graph for arbitrary C , which does *not* admit a C -relaxed coloring. Haxell, Szabó and Tardos [4] established that even a 6-relaxed two-coloring of graphs of maximum degree 4 is possible and proved that every graph of maximum degree 5 has a C -relaxed two-coloring with some constant C (In fact $C < 20000$).

Earlier work related to relaxed colorings were focusing on special kinds of graphs, like line-graphs of cubic graphs [2, 5]. These works culminated in the result of Thomassen [8], who proved that there exists a two-coloring of the edges of any cubic graph such that not only every monochromatic component is bounded, but is a *path* of length at most five.

2 Two-coloring cubic graphs

In this paper we are concerned about the asymmetric version of the relaxation of proper two-coloring. Namely, we allow larger components in only one of the color classes, the other one has to be an independent set. Obviously, any 2-regular graph has a two-coloring where one of the color-classes is an independent set, and the other induces monochromatic components of order at most 2. Our main theorem claims that a similar statement holds for graphs of maximum degree 3 as well.

Theorem 1 *Let G be a graph of maximum degree at most 3. There exists a partition of the vertex set of G into subsets I and B where I is an independent set and every component of $G[B]$ is of order at most 189.*

We prove Theorem 1 in several steps. Our argument is quite lengthy, here we only give brief synopsis.

2.1 Synopsis of the proof of Theorem 1

After some initial simplification we break the graph G into two pieces: one containing vertices which don't participate in a triangle, the other containing vertices from triangles. We solve our problem separately for each piece, then we finish the proof of Theorem 1 by combining the two-coloring of the two pieces through a series of modifications.

More formally, first we show that Theorem 1 (with a better constant) holds if G is triangle-free.

Theorem 2 *For any triangle-free graph G with $\Delta(G) \leq 3$, there exists a partition of the vertex set into I and B where I is an independent set and no component of $G[B]$ is larger than 6.*

The key point in the proof of this statement is to define an appropriate auxiliary graph and apply the following useful lemma from [4] about *matching transversals*.

Lemma 1 [4, Corollary 4.3] *Let H be a graph with $\Delta(H) \leq 2$. Suppose that $V(H)$ is partitioned into subsets of size two, $V(H) = W_1 \cup \dots \cup W_m$, $|W_i| = 2$ for $i = 1, \dots, m$. Then there exists a “matching transversal”, i.e. a subset $T \subseteq V(H)$ of the vertices such that $|W_i \cap T| = 1$ for every $i = 1, \dots, m$ and $\Delta(G[T]) \leq 1$.*

For graphs whose vertex set is the union of vertex disjoint triangles one can also show that Theorem 1 (with a better constant) holds.

Lemma 2 *For any graph G in which every vertex participates in a triangle, and $\Delta(G) \leq 3$, there exists a partition of the vertex set into I and B where I is an independent set and no component of $G[B]$ is larger than 8.*

The proof utilizes the following theorem of Thomassen about certain edge-two-coloring of cubic graphs.

Theorem 3 [8, Theorem 2.] *Let H be a graph of maximum degree at most 3. Then the edge set of H has a red/blue coloring and an orientation of the edges such that*

- (i) *each monochromatic component is a directed path of length at most 5, and*
- (ii) *each vertex of degree 2 is either an interior vertex of a monochromatic directed path or the endpoint of a monochromatic directed path of length at most 3.*

The third part of the proof of our main theorem, containing the process of combining Theorem 2 and Lemma 2, is quite technical. It starts by taking a “good” two-coloring of the triangle-free part of G (Theorem 2) and the part containing only vertices from triangles. Then we perform a series of small modifications to ensure that each B -component of the “triangle-full” part is joined to at most one B -component of the triangle-free part. In particular we need to use the following strengthened version of Lemma 2, which provides us with the flexibility needed to stick together the two “good” two-colorings. The flexibility is represented by the set X , which can be included in both the “independent” and the “bounded-component” part with some sacrifice in the constant.

Let V_i be the set of vertices of degree i .

Lemma 3 For any G , which is the vertex disjoint union of triangles, and $\Delta(G) \leq 3$, there exists a partition of the vertex set $V(G)$ into three sets I , B and X , such that

- (i) $I \subseteq V_3$, $X \subseteq V_2$, $I \cup X$ is an independent set and no component of $G[B \cup X]$ is larger than 21.
- (ii) every component of $G[B \cup X]$ contains at most three vertices from $B \cap V_2$, all of which are contained in the same triangle. Any component of $G[B \cup X]$ containing exactly one vertex from $B \cap V_2$ is of order at most 8, and any component containing two or three vertices from $B \cap V_2$ is fully contained in a triangle.

3 4-regular graphs

To complement Theorem 1 we prove that a similar statement cannot hold for 4-regular graphs.

Theorem 4 For any constant C there exists a 4-regular graph G such that for any independent set $I \subseteq V(G)$, $G[V(G) \setminus I]$ has a component of order larger than C .

4 More than two colors

We also investigate relaxed colorings of graphs with more than two colors. For this we need the following definition. A graph G is called C -relaxed (k, ℓ) -colorable if there exists a C -relaxed $(k + \ell)$ -coloring of G such that each of the first k color classes are independent sets. A set of graphs \mathcal{G} is called (k, ℓ) -colorable if there exists an absolute constant C , such that every member $G \in \mathcal{G}$ admits a C -relaxed (k, ℓ) -coloring. Obviously, $(k, 0)$ -colorability is the same as the usual k -colorability. The main result of [4] could be formulated as the family of 5-regular graphs is $(0, 2)$ -colorable. Our main results state that cubic graphs are $(1, 1)$ -colorable, while 4-regular graphs are not.

In [4] the maximum degree condition for $(0, k)$ -colorability is investigated. We define $\Delta(k, \ell)$ to be the smallest integer Δ such that the family of graphs with maximum degree Δ is not (k, ℓ) -colorable. In [4] it is shown that there exists a constant $\delta > 0$, such that for large ℓ , $3 + \delta < \Delta(0, \ell)/\ell < 4$,

Here we give bounds on $\Delta(k, \ell)$ and raise several open questions.

Theorem 5 Let $\ell > 0$. For any constant C there exists a graph of maximum degree $\Delta = 2(k + 2\ell - 1)$ which is not C -relaxed (k, ℓ) -colorable. That is $\Delta(k, \ell) \leq 2k + 4\ell - 2$.

Theorem 4 is a special case of Theorem 5 with $k = \ell = 1$. The construction of Alon, Ding, Oporowski and Vertigan [1] is a special case with $k = 0, \ell = 2$.

Theorem 6 Let k, ℓ be nonnegative integers. The family of graphs of maximum degree at most $k + 3\ell - 1$ is (k, ℓ) -colorable. That is $\Delta(k, \ell) > k + 3\ell - 1$.

This statement is a consequence of a theorem of [4], a lemma from [7] and our Theorem 1.

5 Open Problems

One would like to know more about the behavior of the function $\Delta(k, \ell)$ in general, or at least tighten the existing asymptotic gap. The following are two important special cases.

Maximum degree condition for $(0, \ell)$ -colorability. The main theorem of [4] states that $\Delta(0, 2) = 6$. One of the outstanding questions of the topic is to determine the asymptotics of $\Delta(0, \ell)/\ell$. In [4] it is shown that there exists $\delta > 0$, such that for large ℓ , $3 + \delta < \Delta(0, \ell)/\ell < 4$. It would be of great interest to determine asymptotically $\Delta(0, \ell)$.

Maximum degree condition for $(k, 1)$ -colorability. Our main result in this paper states that $\Delta(1, 1) = 4$. The value of $\Delta(2, 1)$ is either 5 or 6. Asymptotically, $\Delta(k, 1)$ is between k and $2k$. We conjecture the lower bounds are (closer to) the truth.

Density version. A natural way to weaken the maximum degree condition is by rather bounding the maximum average degree of the graph, which allows a few very large degree vertices.

Let $\mu(G) = \max\{2|E(G[W])|/|W| : W \subseteq V(G)\}$. For non-negative integers k, ℓ what is the supremum value $\alpha(k, \ell)$, such that every graph G with $\mu(G) < \alpha(k, \ell)$ has a C -relaxed $(k + \ell)$ -coloring with some constant C . Obviously $\alpha(k, \ell) \leq \Delta(k, \ell)$. In [4] the determination of $\alpha(0, 2)$ was raised. The *wheel* graph shows that $\alpha(0, 2) \leq 4$, while Kostochka [6] proved a lower bound of 3. The greedy coloring implies that $\alpha(k, 0) = k$, for any k . We would be very much interested in the value of $\alpha(1, 1)$.

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