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Hamiltonian cycles in torical lattices

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We establish sufficient conditions for a toric lattice $T_{m,n}$ to be Hamiltonian. Also, we give some asymptotics for the number of Hamiltonian cycles in $T_{m,n}$.

Keywords: Hamiltonian cycle, toric lattice, Hardy–Littlewood method.

Let $T_{m,n} = J_m \times J_n$ be a toric lattice, i.e., the Cartesian product of two directed cycles lengths m and n respectively.

Erdős problem [1]. When $T_{m,n}$ contains Hamiltonian cycles?

The next theorem was proved by A.A.Evdokimov [2].

Theorem 1 $T_{m,n}$ is Hamiltonian iff there are solutions of the following Diophantine system

$$\begin{aligned} x + y &= \gcd(m, n), \\ \gcd(x, m) &= 1, \quad \gcd(y, n) = 1 \end{aligned} \tag{1}$$

(gcd means the greatest common divisor).

Let $J_{m,n}$ be the number of solutions of the system (1). We obtain estimates for $J_{m,n}$ in two special cases. Let

$$m = \prod_{i=1}^r p_i^{\alpha_i}, \quad n = \prod_{j=1}^s q_j^{\beta_j}$$

are prime decompositions for m, n . We use the following notations

$$P = \prod_{i=1}^r p_i, \quad Q = \prod_{j=1}^s q_j, \quad \lambda(P, Q) = \prod_{r|\text{lcm}(P, Q)} \left(1 - \frac{1}{r}\right)$$

(lcm means the least common multiple).

Theorem 2 $J_{m,n} \geq 1$ if $\gcd(m, n) > \left[\prod_{i=1}^r (1 + p_i) + \prod_{j=1}^s (1 + q_j) \right] (4\lambda(P, Q))^{-1}$.

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The proofs of the theorems 1, 2 are based on the following analytic and combinatorial results.

Let

$$J_N(u) = \sum_{(a,N)=1} u^a, \quad N = p_1^{\alpha_1} \dots p_k^{\alpha_k}.$$

Lemma 1 $J_N(u) = \frac{1}{1-u} - \sum_{i=1}^k \frac{1}{1-u^{p_i}} + \sum_{1 \leq i < j \leq k} \frac{1}{1-u^{p_i p_j}} - \dots$

This formula can be easily proved by inclusion - exclusion principle.

Let $S_r(m, n)$ be the number of solutions of the system

$$\begin{aligned} x + y &= r, \\ \gcd(x, m) &= 1, \quad \gcd(y, n) = 1. \end{aligned} \quad (2)$$

The generating function for $S_r(m, n)$ is related with $J_n(u)$ by the following formula.

Lemma 2

$$\sum_{r=1}^{\infty} S_r(m, n) u^r = J_m(u) J_n(u). \quad (3)$$

Formula (3) implies an expression for the number of solutions of the system (1).

Lemma 3 Let $N = \gcd(m, n) + 1$. Then the following equation holds

$$\begin{aligned} J_{m,n} = \gcd(m, n) \sum_{u|P, v|Q} \frac{\mu(u)\mu(v)}{\text{lcm}(u, v)} + \sum_{u|P, v|Q} \frac{\mu(u)\mu(v)(u+v)}{2\text{lcm}(u, v)} + \\ \sum_{u|P, v|Q} \frac{\mu(u)}{u} \sum_{\alpha^u=1} \frac{1}{\alpha^{N-1}(\alpha^v-1)} + \sum_{u|P, v|Q} \frac{\mu(v)}{v} \sum_{\alpha^v=1} \frac{1}{\alpha^{N-1}(\alpha^u-1)}. \end{aligned} \quad (4)$$

In sums of type

$$\sum_{\alpha^u=1} \frac{1}{\alpha^{N-1}(\alpha^v-1)} \quad (5)$$

the summation is over those roots of equation $\alpha^u = 1$ that are not the roots of equation $\alpha^v = 1$.

Sums (5) are called Dedekind sums. They are well-known in combinatorial analysis (e.g., see [3]).

To simplify (4) we use identities about Möbius function. They are 2-dimensional analogues of the classical formula

$$\sum_{d|n} \frac{\mu(d)}{d} = \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

An example of these identities is given by the following Lemma.

Lemma 4 ([4]) $\sum_{u|m, v|n} \frac{\mu(u)\mu(v)}{\text{lcm}(u, v)} = \prod_{r|\text{lcm}(P, Q)} \left(1 - \frac{1}{r}\right).$

Dealing with Dedekind sums (5) we use the following useful statement. Let

$$S_n(a) = \sum_{\alpha^b=1} \frac{1}{\alpha^n(\alpha^a - 1)}, \quad (6)$$

where summation is over those roots of equation $x^b = 1$ that are not the roots of equation $x^a = 1$. By m_0 we denote the smallest positive solution of equation

$$ax \equiv -(n + a) \pmod{b}.$$

Let $w(a, b) = m_0 - 1$.

Lemma 5

$$S_n(a) = \frac{b}{2} - \frac{\gcd(a, b)}{2\text{lcm}(a, b)} - w(a, b). \quad (7)$$

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