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► **To cite this version:**

Vladimir K. Leontiev. Hamiltonian cycles in torical lattices. Stefan Felsner. 2005 European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), 2005, Berlin, Germany. Discrete Mathematics and Theoretical Computer Science, DMTCS Proceedings vol. AE, European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), pp.397-400, 2005, DMTCS Proceedings. <hal-01184437>

HAL Id: hal-01184437

<https://hal.inria.fr/hal-01184437>

Submitted on 14 Aug 2015

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Hamiltonian cycles in torical lattices

Vladimir K. Leontiev^{1†}

¹*Dorodnitsyn Computing Center of RAS, & Vavilova, 40, Moscow, 117967, Russia*

We establish sufficient conditions for a toric lattice $T_{m,n}$ to be Hamiltonian. Also, we give some asymptotics for the number of Hamiltonian cycles in $T_{m,n}$.

Keywords: Hamiltonian cycle, toric lattice, Hardy–Littlewood method.

Let $T_{m,n} = J_m \times J_n$ be a toric lattice, i.e., the Cartesian product of two directed cycles lengths m and n respectively.

Erdős problem [1]. When $T_{m,n}$ contains Hamiltonian cycles?

The next theorem was proved by A.A.Evdokimov [2].

Theorem 1 $T_{m,n}$ is Hamiltonian iff there are solutions of the following Diophantine system

$$\begin{aligned} x + y &= \gcd(m, n), \\ \gcd(x, m) &= 1, \quad \gcd(y, n) = 1 \end{aligned} \tag{1}$$

(gcd means the greatest common divisor).

Let $J_{m,n}$ be the number of solutions of the system (1). We obtain estimates for $J_{m,n}$ in two special cases. Let

$$m = \prod_{i=1}^r p_i^{\alpha_i}, \quad n = \prod_{j=1}^s q_j^{\beta_j}$$

are prime decompositions for m, n . We use the following notations

$$P = \prod_{i=1}^r p_i, \quad Q = \prod_{j=1}^s q_j, \quad \lambda(P, Q) = \prod_{r|\text{lcm}(P, Q)} \left(1 - \frac{1}{r}\right)$$

(lcm means the least common multiple).

Theorem 2 $J_{m,n} \geq 1$ if $\gcd(m, n) > \left[\prod_{i=1}^r (1 + p_i) + \prod_{j=1}^s (1 + q_j) \right] (4\lambda(P, Q))^{-1}$.

[†]This work was supported by grants RFBR 05–01–01019 and NSH 1721.2003.1

The proofs of the theorems 1, 2 are based on the following analytic and combinatorial results.

Let

$$J_N(u) = \sum_{(a,N)=1} u^a, \quad N = p_1^{\alpha_1} \dots p_k^{\alpha_k}.$$

Lemma 1 $J_N(u) = \frac{1}{1-u} - \sum_{i=1}^k \frac{1}{1-u^{p_i}} + \sum_{1 \leq i < j \leq k} \frac{1}{1-u^{p_i p_j}} - \dots$

This formula can be easily proved by inclusion - exclusion principle.

Let $S_r(m, n)$ be the number of solutions of the system

$$\begin{aligned} x + y &= r, \\ \gcd(x, m) &= 1, \quad \gcd(y, n) = 1. \end{aligned} \tag{2}$$

The generating function for $S_r(m, n)$ is related with $J_n(u)$ by the following formula.

Lemma 2

$$\sum_{r=1}^{\infty} S_r(m, n) u^r = J_m(u) J_n(u). \tag{3}$$

Formula (3) implies an expression for the number of solutions of the system (1).

Lemma 3 Let $N = \gcd(m, n) + 1$. Then the following equation holds

$$\begin{aligned} J_{m,n} = \gcd(m, n) \sum_{u|P, v|Q} \frac{\mu(u)\mu(v)}{\text{lcm}(u, v)} + \sum_{u|P, v|Q} \frac{\mu(u)\mu(v)(u+v)}{2\text{lcm}(u, v)} + \\ \sum_{u|P, v|Q} \frac{\mu(u)}{u} \sum_{\alpha^u=1} \frac{1}{\alpha^{N-1}(\alpha^v-1)} + \sum_{u|P, v|Q} \frac{\mu(v)}{v} \sum_{\alpha^v=1} \frac{1}{\alpha^{N-1}(\alpha^u-1)}. \end{aligned} \tag{4}$$

In sums of type

$$\sum_{\alpha^u=1} \frac{1}{\alpha^{N-1}(\alpha^v-1)} \tag{5}$$

the summation is over those roots of equation $\alpha^u = 1$ that are not the roots of equation $\alpha^v = 1$.

Sums (5) are called Dedekind sums. They are well-known in combinatorial analysis (e.g., see [3]).

To simplify (4) we use identities about Möbius function. They are 2-dimensional analogues of the classical formula

$$\sum_{d|n} \frac{\mu(d)}{d} = \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

An example of these identities is given by the following Lemma.

Lemma 4 ([4]) $\sum_{u|m, v|n} \frac{\mu(u)\mu(v)}{\text{lcm}(u, v)} = \prod_{r|\text{lcm}(P, Q)} \left(1 - \frac{1}{r}\right).$

Dealing with Dedekind sums (5) we use the following useful statement. Let

$$S_n(a) = \sum_{\alpha^b=1} \frac{1}{\alpha^n(\alpha^a - 1)}, \quad (6)$$

where summation is over those roots of equation $x^b = 1$ that are not the roots of equation $x^a = 1$. By m_0 we denote the smallest positive solution of equation

$$ax \equiv -(n + a) \pmod{b}.$$

Let $w(a, b) = m_0 - 1$.

Lemma 5

$$S_n(a) = \frac{b}{2} - \frac{\gcd(a, b)}{2\text{lcm}(a, b)} - w(a, b). \quad (7)$$

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