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Enumeration of walks reaching a line

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We enumerate walks in the plane \mathbb{R}^2 , with steps East and North, that stop as soon as they reach a given line; these walks are counted according to the distance of the line to the origin, and we study the asymptotic behavior when the line has a fixed slope and moves away from the origin. When the line has a rational slope, we study a more general class of walks, and give exact as well as asymptotic enumerative results ; for this, we define a nice bijection from our walks to words of a rational language. For a general slope, asymptotic results are obtained; in this case, the method employed leads us to find asymptotic results for a wider class of walks in \mathbb{R}^m .

Keywords: walk, generating function, rational language, singularity analysis

1 Introduction

In this work we consider primarily two classes of walks in the plane \mathbb{R}^2 , noted $\mathcal{W}_{a,\delta}^+$ and $\mathcal{W}_{a,\delta}^-$, defined in the following manner :

Definition 1 Let $a \in [0, 1[$ and $\delta \geq 0$ be real numbers. We denote by $\mathcal{D}_{a,\delta}$ the line of \mathbb{R}^2 with slope $-a$, going through the point $(\delta, 0)$. An equation of $\mathcal{D}_{a,\delta}$ is $y = -a(x - \delta)$. We denote by $\mathcal{W}_{a,\delta}^+$ (resp. $\mathcal{W}_{a,\delta}^-$) the set of walks in the plane \mathbb{R}^2 starting at the origin $O = (0, 0)$ with steps East or North, which end as soon as they reach the open (resp. closed) half plane above $\mathcal{D}_{a,\delta}$. The cardinalities of the sets $\mathcal{W}_{a,\delta}^+$ and $\mathcal{W}_{a,\delta}^-$ are denoted respectively by $W_{a,\delta}^+$ and $W_{a,\delta}^-$.

These definitions are illustrated on Figure 1.

These walks stop as soon as they cross the line $\mathcal{D}_{a,\delta}$, those in $\mathcal{W}_{a,\delta}^+$ having to go strictly beyond the line, whereas those in $\mathcal{W}_{a,\delta}^-$ stop on it if they happen to touch it. We are interested in the enumeration of these walks according to the parameter δ ; that is, we fix the slope $-a$ of the line $\mathcal{D}_{a,\delta}$, and study the numbers $W_{a,\delta}^+$ and $W_{a,\delta}^-$ in function of δ . Note that, up to a constant factor a , δ represents the distance of the line $\mathcal{D}_{a,\delta}$ to the origin.

We can now state our first theorem which gives all asymptotic results for $W_{a,\delta}^+$ and $W_{a,\delta}^-$ when δ goes to infinity.

Theorem 1 Let $a \in]0, 1]$, and let λ be the unique positive solution to the equation $\lambda^{-1} + \lambda^{-1/a} = 1$.

If $a = p/q > 0$ is a fixed rational number, where p and q are relatively prime positive integers, then the asymptotic approximations

$$W_{a,\delta}^+ \sim \frac{a}{p(1 - \lambda^{-1/p})} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}} \lambda^{\lfloor p\delta \rfloor / p} \quad \text{and} \quad W_{a,\delta}^- \sim \frac{a}{p(\lambda^{1/p} - 1)} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}} \lambda^{\lfloor p\delta \rfloor / p}$$

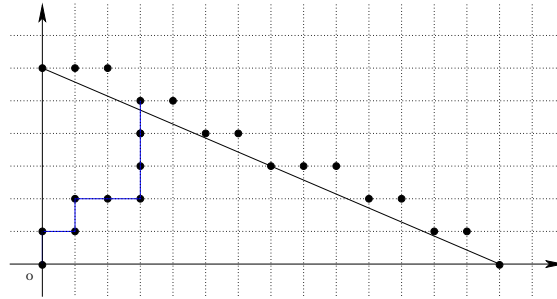


Fig. 1: An example of walk in $\mathcal{W}_{a,\delta}^+$ with $a = \frac{3}{7}$ and $n = 14$.

hold when δ goes to infinity. If a is irrational, then the asymptotic approximations

$$W_{a,\delta}^+ \sim \frac{a}{\ln \lambda} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}} \lambda^\delta \quad \text{and} \quad W_{a,\delta}^- \sim \frac{a}{\ln \lambda} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}} \lambda^\delta$$

hold when δ goes to infinity.

As this theorem shows, the behavior of $W_{a,\delta}^+$ and $W_{a,\delta}^-$ depends on the rationality of the number a ; if a is rational, then we will find the generating function of the numbers $W_{a,n}^+$ and $W_{a,n}^-$. In this case, we will actually introduce another class of walks that includes $\mathcal{W}_{a,n}^+$ and $\mathcal{W}_{a,n}^-$ and find a bijection that sends walks to words of a rational language; various enumerative and asymptotic results derive from there. In the case of a general a , we will proceed differently, and start from an easily obtained functional equation to obtain asymptotic results. Our method is close to Erdős et al. (EHO⁺87), method that is also applicable to a wider class of walks defined in \mathbb{R}^n .

2 Walks reaching a set of points

As announced in the introduction, we now introduce a new class of walks that will include our original walks when the slope of $\mathcal{D}_{a,\delta}$ is rational. The reader is advised to look at Figure 2 while reading the following definition.

Definition 2 ($V_{d,n}$ and $\mathcal{W}_{d,n}$) Let $d = (d_i)_{i \geq 1}$ be an infinite sequence of positive integers, and let $e = (e_i)_{i \in \mathbb{N}}$ be the corresponding sequence of partial sums, defined by $e_0 = 0$ and $e_k = d_1 + d_2 + \dots + d_k$, for $k \geq 1$. We associate to d a set of points V_d in the plane, with integer coordinates: the set $V_d \subset \mathbb{Z} \times \mathbb{N}$ consists in the origin O together with, for every $k \geq 1$, the d_k points with y -coordinate equal to k and x -coordinate in $\llbracket -e_k, -e_{k-1} - 1 \rrbracket$.

For any integer n , $V_{d,n}$ is defined as the translated of V_d by the vector $(n, 0)$. That is, $V_{d,n} = V_d + (n, 0)$. The generalized set of walks $\mathcal{W}_{d,n}$ consists of the walks that start at the origin O , make steps East or North, and have their last points, and no other one, in $V_{d,n}$.

These walks are a generalization of our walks $\mathcal{W}_{a,n}^+$ and $\mathcal{W}_{a,n}^-$. Indeed, let d_a^+ and d_a^- be the sequences whose k th terms are given respectively by $\lceil \frac{k}{a} \rceil - \lceil \frac{k-1}{a} \rceil$ and $\lfloor \frac{k}{a} \rfloor - \lfloor \frac{k-1}{a} \rfloor$. Then we have the following proposition :

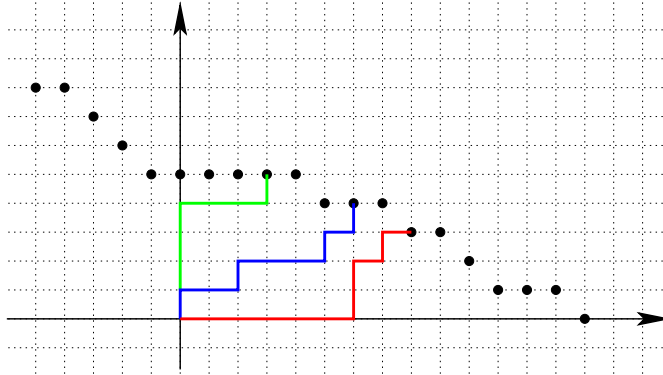


Fig. 2: A set $V_{d,n}$ with examples of walks in $W_{d,n}$. Here $n = 14$ and $d = (3, 1, 2, 3, 6, 1, 1, \dots)$

Proposition 1 For every $n \in \mathbb{N}$ and $a \in]0, 1]$, we have the equalities

$$\mathcal{W}_{a,n}^+ = \mathcal{W}_{d_a^+,n+1} \text{ and } \mathcal{W}_{a,n}^- = \mathcal{W}_{d_a^-,n}.$$

An interesting case happens when the sequence d is periodic. It is easy to see that d_a^+ and d_a^- are periodic exactly when a is a rational number. If $d = (d_1, \dots, d_p)$ is a finite sequence, we will note $V_{d,n} = V_{\bar{d},n}$ and $\mathcal{W}_{d,n} = \mathcal{W}_{\bar{d},n}$, where \bar{d} is the periodic infinite sequence $(d_1, \dots, d_p, d_1, \dots, d_p, \dots)$.

From now on d will stand for a finite sequence $d = (d_1, \dots, d_p)$ of positive integers. We define $q = d_1 + \dots + d_p$, and $a = p/q$. To such a sequence we attach the following language on a finite alphabet (recall that a run in a finite word is a maximal factor composed of identical letters)

Definition 3 (Language \mathcal{L}_d) The language \mathcal{L}_d is the set of words w on the alphabet $\Sigma = \{a_0, a_1, \dots, a_{p-1}\}$ that satisfy the following conditions (where we set by convention $d_0 = d_p$ and $a_p = a_0$):

- C1. w is the empty word, or its initial letter belongs to $\{a_0, a_1\}$
- C2. for all i , a run of a_i in w is terminal or is followed by a run of a_{i+1} ;
- C3. for all i , the runs of a_i in w are of length at least d_i ; this constraint does not apply to the last run, and, if w begins with a_0 , it does not apply to the first run either.

We can finally state the theorem announced in the introduction:

Theorem 2 Let $n \geq 0$ be an integer. There exists an explicit bijection between walks in $\mathcal{W}_{d,n}$ and words of \mathcal{L}_d of length n .

The language \mathcal{L}_d is rational, and we give an unambiguous rational expression that represents it. Then the existence of a bijection as stated in Theorem 2 allows us to explicit the generating function $W_d(x) = \sum_{k=0}^{\infty} W_{d,k} x^k$ of the sequence $(W_{d,n})_{n \in \mathbb{N}}$:

Theorem 3 The generating function $W_d(x)$ has the following expression:

$$W_d(x) = \frac{N(x)}{(1-x)^p - x^q}, \text{ with } N(x) = (1-x)^{p-2} + \sum_{i=1}^{p-2} x^{e_i+1} (1-x)^{p-2-i} + \sum_{k=e_{p-1}+1}^{e_p-1} x^k.$$

Given a rational function, we can easily have access to asymptotic approximations of the coefficients of its series expansion, and we show that the first part of Theorem 1 can thus be obtained as a consequence of Theorem 3.

In fact, thanks to the bijection of Theorem 2, we can even find the bivariate generating function of the numbers $(W_{d,n,k})_{n,k}$ which enumerate walks in $\mathcal{W}_{d,n}$ of length k . By the techniques of singularity analysis exposed in chapter 8 of (FS), we can then prove that the average length of a walk in $\mathcal{W}_{d,n}$ is asymptotically $C_a \cdot n$ when n goes to infinity, where C_a is positive constant depending only on a .

3 Asymptotic results in the general case

Let W_a^+ be the function defined on \mathbb{R} by $W_a^+(\delta) = 1$ if $\delta < 0$, and by $W_a^+(\delta) = W_{a,\delta}^+$ if $\delta \geq 0$. Then, by decomposing walks according to their first step, one shows that W_a^+ satisfies the following functional equation :

$$\forall \delta \geq 0, \quad W_a^+(\delta) = W_a^+(\delta - 1/a) + W_a^+(\delta - 1). \quad (1)$$

This equation and related ones have appeared in various contexts, and have been studied in numerous works, including (CG01; FK74; Pip93). Here we use a method inspired by the paper (EHO⁺87). This consists in interpreting Equation 1 as a “renewal equation”, so that its asymptotic behavior is given by the celebrated *Renewal Limit Theorem* (RLT) of probability theory; see Feller (Fel71) for all necessary background. Application of the RLT immediately leads to a proof of Theorem 1 as far as $W_{a,\delta}^+$ is concerned. It is then extended to $W_{a,\delta}^-$ by finding simple relations between the two numbers.

Our walks have a natural generalization in any dimension. Let $\bar{a} = (a_1, \dots, a_m)$ be a vector in \mathbb{R}^m , with all coordinates being positive, and \mathcal{H}_δ be the hyperplane of equation $\mathcal{H}_\delta : a_1x_1 + \dots + a_{m-1}x_{m-1} + a_mx_m - \delta = 0$. Then define $W_{\bar{a},\delta}^+$ (resp. $W_{\bar{a},\delta}^-$) to be the numbers of walks in \mathbb{R}^m from the origin with steps in $\{e_i\}_{1 \leq i \leq m}$ defined by the fact that their last points, and no other one, are “above \mathcal{H}_δ ” (resp. “above or on \mathcal{H}_δ ”).

Assume $1 = a_m \leq a_1 \leq a_2 \leq \dots \leq a_{m-1}$, and let λ designate the unique positive solution to $\sum_{i=1}^m \lambda^{-a_i} = 1$. If all a_i are rational numbers and we write $a_i = p_i/q_i$ in reduced form for each i , we define $q = \text{lcm}(q_i)$. Then the proof of the following theorem is proved along the same lines as described above :

Theorem 4 *Let λ and q be defined as above. Then we have the following asymptotics when δ tends to ∞ :*

(i) *if at least one a_i is irrational, then*

$$W_{\bar{a},\delta}^+ \sim \frac{m-1}{\ln \lambda \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^\delta \quad \text{and} \quad W_{\bar{a},\delta}^- \sim \frac{m-1}{\ln \lambda \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^\delta,$$

(ii) *and if all a_i are rational, then*

$$W_{\bar{a},\delta}^+ \sim \frac{m-1}{q(1 - \lambda^{-1/q}) \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^{\lfloor q\delta \rfloor / q} \quad \text{and} \quad W_{\bar{a},\delta}^- \sim \frac{m-1}{q(\lambda^{1/q} - 1) \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^{\lfloor q\delta \rfloor / q}.$$

In fact, the same reasoning shows that similar approximations hold in the irrational case when the steps are allowed to be any finite number of non zero vectors with nonnegative coordinates.

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References

- [CG01] V. Choi and M. J. Golin. Lopsided trees. I. Analyses. *Algorithmica*, 31(3):240–290, 2001. Mathematical analysis of algorithms.
- [EHO⁺87] P. Erdős, A. Hildebrand, A. Odlyzko, P. Pudaite, and B. Reznick. The asymptotic behavior of a family of sequences. *Pacific J. Math.*, 126(2):227–241, 1987.
- [Fel71] William Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons Inc., New York, 1971.
- [FK74] Michael L. Fredman and Donald E. Knuth. Recurrence relations based on minimization. *J. Math. Anal. Appl.*, 48:534–559, 1974.
- [FS] P. Flajolet and R. Sedgewick. Analytic combinatorics. Book in preparation. Preliminary version available at <http://algo.inria.fr/flajolet/Publications/books.html>.
- [Pip93] Nicholas Pippenger. An elementary approach to some analytic asymptotics. *SIAM J. Math. Anal.*, 24(5):1361–1377, 1993.

