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► **To cite this version:**

Philippe Nadeau. Enumeration of walks reaching a line. Stefan Felsner. 2005 European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), 2005, Berlin, Germany. Discrete Mathematics and Theoretical Computer Science, DMTCS Proceedings vol. AE, European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), pp.401-406, 2005, DMTCS Proceedings. <hal-01184438>

HAL Id: hal-01184438

<https://hal.inria.fr/hal-01184438>

Submitted on 17 Aug 2015

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Enumeration of walks reaching a line

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We enumerate walks in the plane \mathbb{R}^2 , with steps East and North, that stop as soon as they reach a given line; these walks are counted according to the distance of the line to the origin, and we study the asymptotic behavior when the line has a fixed slope and moves away from the origin. When the line has a rational slope, we study a more general class of walks, and give exact as well as asymptotic enumerative results ; for this, we define a nice bijection from our walks to words of a rational language. For a general slope, asymptotic results are obtained; in this case, the method employed leads us to find asymptotic results for a wider class of walks in \mathbb{R}^m .

Keywords: walk, generating function, rational language, singularity analysis

1 Introduction

In this work we consider primarily two classes of walks in the plane \mathbb{R}^2 , noted $\mathcal{W}_{a,\delta}^+$ and $\mathcal{W}_{a,\delta}^-$, defined in the following manner :

Definition 1 Let $a \in [0, 1[$ and $\delta \geq 0$ be real numbers. We denote by $\mathcal{D}_{a,\delta}$ the line of \mathbb{R}^2 with slope $-a$, going through the point $(\delta, 0)$. An equation of $\mathcal{D}_{a,\delta}$ is $y = -a(x - \delta)$. We denote by $\mathcal{W}_{a,\delta}^+$ (resp. $\mathcal{W}_{a,\delta}^-$) the set of walks in the plane \mathbb{R}^2 starting at the origin $O = (0, 0)$ with steps East or North, which end as soon as they reach the open (resp. closed) half plane above $\mathcal{D}_{a,\delta}$. The cardinalities of the sets $\mathcal{W}_{a,\delta}^+$ and $\mathcal{W}_{a,\delta}^-$ are denoted respectively by $W_{a,\delta}^+$ and $W_{a,\delta}^-$.

These definitions are illustrated on Figure 1.

These walks stop as soon as they cross the line $\mathcal{D}_{a,\delta}$, those in $\mathcal{W}_{a,\delta}^+$ having to go strictly beyond the line, whereas those in $\mathcal{W}_{a,\delta}^-$ stop on it if they happen to touch it. We are interested in the enumeration of these walks according to the parameter δ ; that is, we fix the slope $-a$ of the line $\mathcal{D}_{a,\delta}$, and study the numbers $W_{a,\delta}^+$ and $W_{a,\delta}^-$ in function of δ . Note that, up to a constant factor a , δ represents the distance of the line $\mathcal{D}_{a,\delta}$ to the origin.

We can now state our first theorem which gives all asymptotic results for $W_{a,\delta}^+$ and $W_{a,\delta}^-$ when δ goes to infinity.

Theorem 1 Let $a \in]0, 1]$, and let λ be the unique positive solution to the equation $\lambda^{-1} + \lambda^{-1/a} = 1$.

If $a = p/q > 0$ is a fixed rational number, where p and q are relatively prime positive integers, then the asymptotic approximations

$$W_{a,\delta}^+ \sim \frac{a}{p(1 - \lambda^{-1/p})} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}} \lambda^{\lfloor p\delta \rfloor / p} \quad \text{and} \quad W_{a,\delta}^- \sim \frac{a}{p(\lambda^{1/p} - 1)} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}} \lambda^{\lfloor p\delta \rfloor / p}$$

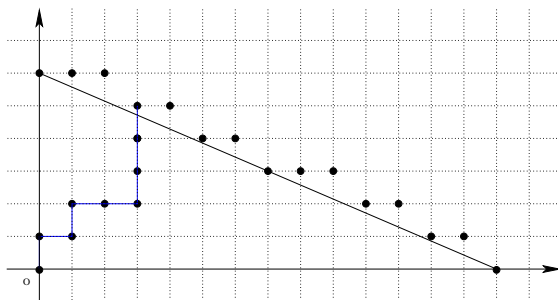


Fig. 1: An example of walk in $\mathcal{W}_{a,\delta}^+$ with $a = \frac{3}{7}$ and $n = 14$.

hold when δ goes to infinity. If a is irrational, then the asymptotic approximations

$$W_{a,\delta}^+ \sim \frac{a}{\ln \lambda} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}} \lambda^\delta \quad \text{and} \quad W_{a,\delta}^- \sim \frac{a}{\ln \lambda} \cdot \frac{1}{1 - (1 - a)\lambda^{-1}} \lambda^\delta$$

hold when δ goes to infinity.

As this theorem shows, the behavior of $W_{a,\delta}^+$ and $W_{a,\delta}^-$ depends on the rationality of the number a ; if a is rational, then we will find the generating function of the numbers $W_{a,n}^+$ and $W_{a,n}^-$. In this case, we will actually introduce another class of walks that includes $\mathcal{W}_{a,n}^+$ and $\mathcal{W}_{a,n}^-$ and find a bijection that sends walks to words of a rational language; various enumerative and asymptotic results derive from there. In the case of a general a , we will proceed differently, and start from an easily obtained functional equation to obtain asymptotic results. Our method is close to Erdős et al. (EHO⁺87), method that is also applicable to a wider class of walks defined in \mathbb{R}^n .

2 Walks reaching a set of points

As announced in the introduction, we now introduce a new class of walks that will include our original walks when the slope of $\mathcal{D}_{a,\delta}$ is rational. The reader is advised to look at Figure 2 while reading the following definition.

Definition 2 ($V_{d,n}$ and $\mathcal{W}_{d,n}$) Let $d = (d_i)_{i \geq 1}$ be an infinite sequence of positive integers, and let $e = (e_i)_{i \in \mathbb{N}}$ be the corresponding sequence of partial sums, defined by $e_0 = 0$ and $e_k = d_1 + d_2 + \dots + d_k$, for $k \geq 1$. We associate to d a set of points V_d in the plane, with integer coordinates: the set $V_d \subset \mathbb{Z} \times \mathbb{N}$ consists in the origin O together with, for every $k \geq 1$, the d_k points with y -coordinate equal to k and x -coordinate in $\llbracket -e_k, -e_{k-1} - 1 \rrbracket$.

For any integer n , $V_{d,n}$ is defined as the translated of V_d by the vector $(n, 0)$. That is, $V_{d,n} = V_d + (n, 0)$. The generalized set of walks $\mathcal{W}_{d,n}$ consists of the walks that start at the origin O , make steps East or North, and have their last points, and no other one, in $V_{d,n}$.

These walks are a generalization of our walks $\mathcal{W}_{a,n}^+$ and $\mathcal{W}_{a,n}^-$. Indeed, let d_a^+ and d_a^- be the sequences whose k th terms are given respectively by $\lceil \frac{k}{a} \rceil - \lceil \frac{k-1}{a} \rceil$ and $\lfloor \frac{k}{a} \rfloor - \lfloor \frac{k-1}{a} \rfloor$. Then we have the following proposition :

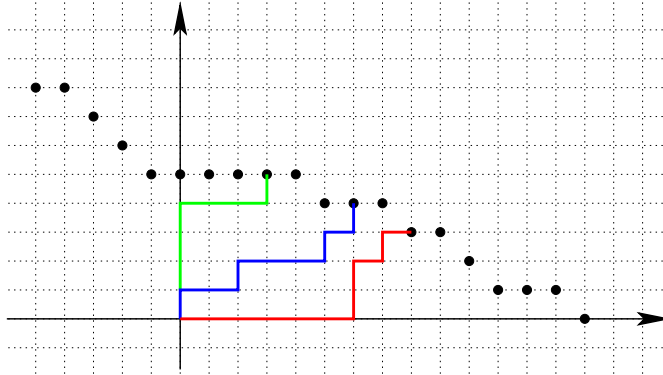


Fig. 2: A set $V_{d,n}$ with examples of walks in $W_{d,n}$. Here $n = 14$ and $d = (3, 1, 2, 3, 6, 1, 1, \dots)$

Proposition 1 For every $n \in \mathbb{N}$ and $a \in]0, 1]$, we have the equalities

$$\mathcal{W}_{a,n}^+ = \mathcal{W}_{d_a^+,n+1} \text{ and } \mathcal{W}_{a,n}^- = \mathcal{W}_{d_a^-,n}.$$

An interesting case happens when the sequence d is periodic. It is easy to see that d_a^+ and d_a^- are periodic exactly when a is a rational number. If $d = (d_1, \dots, d_p)$ is a finite sequence, we will note $V_{d,n} = V_{\bar{d},n}$ and $\mathcal{W}_{d,n} = \mathcal{W}_{\bar{d},n}$, where \bar{d} is the periodic infinite sequence $(d_1, \dots, d_p, d_1, \dots, d_p, \dots)$.

From now on d will stand for a finite sequence $d = (d_1, \dots, d_p)$ of positive integers. We define $q = d_1 + \dots + d_p$, and $a = p/q$. To such a sequence we attach the following language on a finite alphabet (recall that a run in a finite word is a maximal factor composed of identical letters)

Definition 3 (Language \mathcal{L}_d) The language \mathcal{L}_d is the set of words w on the alphabet $\Sigma = \{a_0, a_1, \dots, a_{p-1}\}$ that satisfy the following conditions (where we set by convention $d_0 = d_p$ and $a_p = a_0$):

- C1. w is the empty word, or its initial letter belongs to $\{a_0, a_1\}$
- C2. for all i , a run of a_i in w is terminal or is followed by a run of a_{i+1} ;
- C3. for all i , the runs of a_i in w are of length at least d_i ; this constraint does not apply to the last run, and, if w begins with a_0 , it does not apply to the first run either.

We can finally state the theorem announced in the introduction:

Theorem 2 Let $n \geq 0$ be an integer. There exists an explicit bijection between walks in $\mathcal{W}_{d,n}$ and words of \mathcal{L}_d of length n .

The language \mathcal{L}_d is rational, and we give an unambiguous rational expression that represents it. Then the existence of a bijection as stated in Theorem 2 allows us to explicit the generating function $W_d(x) = \sum_{k=0}^{\infty} W_{d,k} x^k$ of the sequence $(W_{d,n})_{n \in \mathbb{N}}$:

Theorem 3 The generating function $W_d(x)$ has the following expression:

$$W_d(x) = \frac{N(x)}{(1-x)^p - x^q}, \text{ with } N(x) = (1-x)^{p-2} + \sum_{i=1}^{p-2} x^{e_i+1} (1-x)^{p-2-i} + \sum_{k=e_{p-1}+1}^{e_p-1} x^k.$$

Given a rational function, we can easily have access to asymptotic approximations of the coefficients of its series expansion, and we show that the first part of Theorem 1 can thus be obtained as a consequence of Theorem 3.

In fact, thanks to the bijection of Theorem 2, we can even find the bivariate generating function of the numbers $(W_{d,n,k})_{n,k}$ which enumerate walks in $\mathcal{W}_{d,n}$ of length k . By the techniques of singularity analysis exposed in chapter 8 of (FS), we can then prove that the average length of a walk in $\mathcal{W}_{d,n}$ is asymptotically $C_a \cdot n$ when n goes to infinity, where C_a is positive constant depending only on a .

3 Asymptotic results in the general case

Let W_a^+ be the function defined on \mathbb{R} by $W_a^+(\delta) = 1$ if $\delta < 0$, and by $W_a^+(\delta) = W_{a,\delta}^+$ if $\delta \geq 0$. Then, by decomposing walks according to their first step, one shows that W_a^+ satisfies the following functional equation :

$$\forall \delta \geq 0, \quad W_a^+(\delta) = W_a^+(\delta - 1/a) + W_a^+(\delta - 1). \quad (1)$$

This equation and related ones have appeared in various contexts, and have been studied in numerous works, including (CG01; FK74; Pip93). Here we use a method inspired by the paper (EHO⁺87). This consists in interpreting Equation 1 as a “renewal equation”, so that its asymptotic behavior is given by the celebrated *Renewal Limit Theorem* (RLT) of probability theory; see Feller (Fel71) for all necessary background. Application of the RLT immediately leads to a proof of Theorem 1 as far as $W_{a,\delta}^+$ is concerned. It is then extended to $W_{\bar{a},\delta}^-$ by finding simple relations between the two numbers.

Our walks have a natural generalization in any dimension. Let $\bar{a} = (a_1, \dots, a_m)$ be a vector in \mathbb{R}^m , with all coordinates being positive, and \mathcal{H}_δ be the hyperplane of equation $\mathcal{H}_\delta : a_1x_1 + \dots + a_{m-1}x_{m-1} + a_mx_m - \delta = 0$. Then define $W_{\bar{a},\delta}^+$ (resp. $W_{\bar{a},\delta}^-$) to be the numbers of walks in \mathbb{R}^m from the origin with steps in $\{e_i\}_{1 \leq i \leq m}$ defined by the fact that their last points, and no other one, are “above \mathcal{H}_δ ” (resp. “above or on \mathcal{H}_δ ”).

Assume $1 = a_m \leq a_1 \leq a_2 \leq \dots \leq a_{m-1}$, and let λ designate the unique positive solution to $\sum_{i=1}^m \lambda^{-a_i} = 1$. If all a_i are rational numbers and we write $a_i = p_i/q_i$ in reduced form for each i , we define $q = \text{lcm}(q_i)$. Then the proof of the following theorem is proved along the same lines as described above :

Theorem 4 *Let λ and q be defined as above. Then we have the following asymptotics when δ tends to ∞ :*

(i) *if at least one a_i is irrational, then*

$$W_{\bar{a},\delta}^+ \sim \frac{m-1}{\ln \lambda \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^\delta \quad \text{and} \quad W_{\bar{a},\delta}^- \sim \frac{m-1}{\ln \lambda \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^\delta,$$

(ii) *and if all a_i are rational, then*

$$W_{\bar{a},\delta}^+ \sim \frac{m-1}{q(1 - \lambda^{-1/q}) \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^{\lfloor q\delta \rfloor / q} \quad \text{and} \quad W_{\bar{a},\delta}^- \sim \frac{m-1}{q(\lambda^{1/q} - 1) \cdot \sum_{i=1}^m a_i \lambda^{-a_i}} \cdot \lambda^{\lfloor q\delta \rfloor / q}.$$

In fact, the same reasoning shows that similar approximations hold in the irrational case when the steps are allowed to be any finite number of non zero vectors with nonnegative coordinates.

Acknowledgements

I would like to thank Yves Verhoeven who is at the origin of this work and helped me on many occasions during my researches.

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