

Every 3-connected, essentially 11-connected line graph is hamiltonian

Hong-Jian Lai, Yehong Shao, Ju Zhou, Hehui Wu

► **To cite this version:**

Hong-Jian Lai, Yehong Shao, Ju Zhou, Hehui Wu. Every 3-connected, essentially 11-connected line graph is hamiltonian. 2005 European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), 2005, Berlin, Germany. pp.379-382. hal-01184441

HAL Id: hal-01184441

<https://hal.inria.fr/hal-01184441>

Submitted on 14 Aug 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Every 3-connected, essentially 11-connected line graph is hamiltonian

Hong-Jian Lai, Yehong Shao, Ju Zhou and Hehui Wu

West Virginia University, Morgantown, WV 26506-6310

Thomassen conjectured that every 4-connected line graph is hamiltonian. A vertex cut X of G is essential if $G - X$ has at least two nontrivial components. We prove that every 3-connected, essentially 11-connected line graph is hamiltonian. Using Ryjáček's line graph closure, it follows that every 3-connected, essentially 11-connected claw-free graph is hamiltonian.

Keywords: Line graph, claw-free graph, supereulerian graphs, collapsible graph, hamiltonian graph, dominating Eulerian subgraph, essential connectivity

We use [1] for terminology and notations not defined here, and consider finite graphs without loops. In particular, $\kappa(G)$ and $\kappa'(G)$ represent the *connectivity* and *edge-connectivity* of a graph G . A graph is trivial if it contains no edges. A vertex cut X of G is essential if $G - X$ has at least two nontrivial components. For an integer $k > 0$, a graph G is *essentially k -connected* if G does not have an essential cut X with $|X| < k$. An edge cut Y of G is essential if $G - Y$ has at least two nontrivial components. For an integer $k > 0$, a graph G is *essentially k -edge-connected* if G does not have an essential edge cut Y with $|Y| < k$.

For a graph G , let $O(G)$ denote the set of odd degree vertices of G . A graph G is *Eulerian* if G is connected with $O(G) = \emptyset$, and G is *supereulerian* if G has a spanning Eulerian subgraph. Let $X \subseteq E(G)$ be an edge subset. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. When $X = \{e\}$, we also use G/e for $G/\{e\}$. For an integer $i > 0$, define

$$D_i(G) = \{v \in V(G) : \deg_G(v) = i\}.$$

For any $v \in V(G)$, define

$$E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}.$$

Let H_1, H_2 be subgraphs of a graph G . Then $H_1 \cup H_2$ is a subgraph of G with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$; and $H_1 \cap H_2$ is a subgraph of G with vertex set $V(H_1) \cap V(H_2)$ and edge set $E(H_1) \cap E(H_2)$. If V_1, V_2 are two disjoint subsets of $V(G)$, then $[V_1, V_2]_G$ denotes the set of edges in G with one end in V_1 and the other end in V_2 . When the graph G is understood from the context, we also omit the subscript G and write $[V_1, V_2]$ for $[V_1, V_2]_G$. If H_1, H_2 are two vertex disjoint subgraphs of G , then we also write $[H_1, H_2]$ for $[V(H_1), V(H_2)]$.

The *line graph* of a graph G , denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common. From the definition of a line graph, if $L(G)$ is not a complete graph, then a subset $X \subseteq V(L(G))$ is a vertex cut of $L(G)$ if and only if X is an essential edge cut of G . In 1986, Thomassen proposed the following conjecture.

Conjecture 1 (Thomassen [8]) *Every 4-connected line graph is hamiltonian.*

A graph that does not have an induced subgraph isomorphic to $K_{1,3}$ is called a *claw-free* graph. It is well known that every line graph is a claw-free graph. Matthews and Sumner proposed a seemingly stronger conjecture.

Conjecture 2 (Matthews and Sumner [5]) *Every 4-connected claw-free graph is hamiltonian.*

The best result towards these conjectures so far were obtained by Zhan and Ryjáček. A graph G is *hamiltonian connected* if for every pair of vertices u and v in G , G has a spanning (u, v) -path.

Theorem 3 (Zhan [10]) *Every 7-connected line graph is hamiltonian connected.*

Theorem 4 (Ryjáček [7])

- (i) *Conjecture 1.1 and Conjecture 1.2 are equivalent.*
- (ii) *Every 7-connected claw-free graph is hamiltonian.*

In this paper, we apply Catlin's reduction method ([2], [3]) on contracting collapsible subgraphs to prove the following.

Theorem 5 *Every 3-connected, essentially 11-connected line graph is hamiltonian.*

Ryjáček [7] introduced the line graph closure of a claw-free graph and used it to show that a claw-free graph G is hamiltonian if and only if its closure $cl(G)$ is hamiltonian, where $cl(G)$ is a line graph. With this argument and using the fact that adding edges will not decrease the connectivity of a graph, The following corollary is obtained.

Corollary 6 *Every 3-connected, essentially 11-connected claw-free graph is hamiltonian.*

References

- [1] J. A. Bondy and U. S. R. Murty, "Graph Theory with Applications". American Elsevier, New York (1976).
- [2] P. A. Catlin, A reduction method to find spanning Eulerian subgraphs. J. Graph Theory 12 (1988), 29 - 45.
- [3] P. A. Catlin, Supereulerian graphs, a survey. J. of Graph Theory, 16 (1992), 177 - 196.
- [4] F. Harary and C. St. J. A. Nash-Williams, On Eulerian and Hamiltonian graphs and line graphs. Canad. Math. Bull. 8 (1965), 701 - 710.
- [5] M. M. Matthews and D. P. Sumner, *Hamiltonian results in $K_{1,3}$ -free graphs*, J. Graph Theory, 8(1984) 139-146.

- [6] C. St. J. A. Nash-Williams, *Edge-disjoint spanning trees of finite graphs*, J. London Math. Soc. 36 (1961), 445-450.
- [7] Z. Ryjáček, *On a closure concept in claw-free graphs*, J. Combin. Theory Ser. B, 70 (1997), 217-224.
- [8] C. Thomassen, *Reflections on graph theory*, J. Graph Theory, 10 (1986), 309-324.
- [9] W. Tutte, *On the problem of decomposing a graph into n connected factors*, J. London Math. Soc. 36 (1961), 221-230.
- [10] S. Zhan, *On hamiltonian line graphs and connectivity*. Discrete Math. 89 (1991) 89–95.

