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# Every 3-connected, essentially 11-connected line graph is hamiltonian

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Thomassen conjectured that every 4-connected line graph is hamiltonian. A vertex cut  $X$  of  $G$  is essential if  $G - X$  has at least two nontrivial components. We prove that every 3-connected, essentially 11-connected line graph is hamiltonian. Using Ryjáček's line graph closure, it follows that every 3-connected, essentially 11-connected claw-free graph is hamiltonian.

**Keywords:** Line graph, claw-free graph, supereulerian graphs, collapsible graph, hamiltonian graph, dominating Eulerian subgraph, essential connectivity

We use [1] for terminology and notations not defined here, and consider finite graphs without loops. In particular,  $\kappa(G)$  and  $\kappa'(G)$  represent the *connectivity* and *edge-connectivity* of a graph  $G$ . A graph is trivial if it contains no edges. A vertex cut  $X$  of  $G$  is essential if  $G - X$  has at least two nontrivial components. For an integer  $k > 0$ , a graph  $G$  is *essentially  $k$ -connected* if  $G$  does not have an essential cut  $X$  with  $|X| < k$ . An edge cut  $Y$  of  $G$  is essential if  $G - Y$  has at least two nontrivial components. For an integer  $k > 0$ , a graph  $G$  is *essentially  $k$ -edge-connected* if  $G$  does not have an essential edge cut  $Y$  with  $|Y| < k$ .

For a graph  $G$ , let  $O(G)$  denote the set of odd degree vertices of  $G$ . A graph  $G$  is *Eulerian* if  $G$  is connected with  $O(G) = \emptyset$ , and  $G$  is *supereulerian* if  $G$  has a spanning Eulerian subgraph. Let  $X \subseteq E(G)$  be an edge subset. The *contraction*  $G/X$  is the graph obtained from  $G$  by identifying the two ends of each edge in  $X$  and then deleting the resulting loops. When  $X = \{e\}$ , we also use  $G/e$  for  $G/\{e\}$ . For an integer  $i > 0$ , define

$$D_i(G) = \{v \in V(G) : \deg_G(v) = i\}.$$

For any  $v \in V(G)$ , define

$$E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}.$$

Let  $H_1, H_2$  be subgraphs of a graph  $G$ . Then  $H_1 \cup H_2$  is a subgraph of  $G$  with vertex set  $V(H_1) \cup V(H_2)$  and edge set  $E(H_1) \cup E(H_2)$ ; and  $H_1 \cap H_2$  is a subgraph of  $G$  with vertex set  $V(H_1) \cap V(H_2)$  and edge set  $E(H_1) \cap E(H_2)$ . If  $V_1, V_2$  are two disjoint subsets of  $V(G)$ , then  $[V_1, V_2]_G$  denotes the set of edges in  $G$  with one end in  $V_1$  and the other end in  $V_2$ . When the graph  $G$  is understood from the context, we also omit the subscript  $G$  and write  $[V_1, V_2]$  for  $[V_1, V_2]_G$ . If  $H_1, H_2$  are two vertex disjoint subgraphs of  $G$ , then we also write  $[H_1, H_2]$  for  $[V(H_1), V(H_2)]$ .

The *line graph* of a graph  $G$ , denoted by  $L(G)$ , has  $E(G)$  as its vertex set, where two vertices in  $L(G)$  are adjacent if and only if the corresponding edges in  $G$  have at least one vertex in common. From the definition of a line graph, if  $L(G)$  is not a complete graph, then a subset  $X \subseteq V(L(G))$  is a vertex cut of  $L(G)$  if and only if  $X$  is an essential edge cut of  $G$ . In 1986, Thomassen proposed the following conjecture.

**Conjecture 1** (Thomassen [8]) *Every 4-connected line graph is hamiltonian.*

A graph that does not have an induced subgraph isomorphic to  $K_{1,3}$  is called a *claw-free* graph. It is well known that every line graph is a claw-free graph. Matthews and Sumner proposed a seemingly stronger conjecture.

**Conjecture 2** (Matthews and Sumner [5]) *Every 4-connected claw-free graph is hamiltonian.*

The best result towards these conjectures so far were obtained by Zhan and Ryjáček. A graph  $G$  is *hamiltonian connected* if for every pair of vertices  $u$  and  $v$  in  $G$ ,  $G$  has a spanning  $(u, v)$ -path.

**Theorem 3** (Zhan [10]) *Every 7-connected line graph is hamiltonian connected.*

**Theorem 4** (Ryjáček [7])

- (i) *Conjecture 1.1 and Conjecture 1.2 are equivalent.*
- (ii) *Every 7-connected claw-free graph is hamiltonian.*

In this paper, we apply Catlin's reduction method ([2], [3]) on contracting collapsible subgraphs to prove the following.

**Theorem 5** *Every 3-connected, essentially 11-connected line graph is hamiltonian.*

Ryjáček [7] introduced the line graph closure of a claw-free graph and used it to show that a claw-free graph  $G$  is hamiltonian if and only if its closure  $cl(G)$  is hamiltonian, where  $cl(G)$  is a line graph. With this argument and using the fact that adding edges will not decrease the connectivity of a graph, The following corollary is obtained.

**Corollary 6** *Every 3-connected, essentially 11-connected claw-free graph is hamiltonian.*

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