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Thomassen conjectured that every 4-connected line graph is hamiltonian. A vertex cut X of G is essential if $G - X$ has at least two nontrivial components. We prove that every 3-connected, essentially 11-connected line graph is hamiltonian. Using Ryjáček's line graph closure, it follows that every 3-connected, essentially 11-connected claw-free graph is hamiltonian.

Keywords: Line graph, claw-free graph, supereulerian graphs, collapsible graph, hamiltonian graph, dominating Eulerian subgraph, essential connectivity

We use [1] for terminology and notations not defined here, and consider finite graphs without loops. In particular, $\kappa(G)$ and $\kappa'(G)$ represent the *connectivity* and *edge-connectivity* of a graph G. A graph is trivial if it contains no edges. A vertex cut X of G is essential if $G - X$ has at least two nontrivial components. For an integer $k > 0$, a graph G is *essentially k-connected* if G does not have an essential cut X with $|X| < k$. An edge cut Y of G is essential if $G - Y$ has at least two nontrivial components. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G does not have an essential edge cut Y with $|Y| < k$.

For a graph G, let $O(G)$ denote the set of odd degree vertices of G. A graph G is *Eulerian* if G is connected with $O(G) = \emptyset$, and G is *supereulerian* if G has a spanning Eulerian subgraph. Let $X \subseteq E(G)$ be an edge subset. The *contraction* G/X is the graph obtained from G be identifying the two ends of each edge in X and then deleting the resulting loops. When $X = \{e\}$, we also use G/e for $G/\{e\}$. For an integer $i > 0$, define

$$D_i(G) = \{v \in V(G) : deg_G(v) = i\}.$$

For any $v \in V(G)$, define

$$E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}.$$

Let H_1, H_2 be subgraphs of a graph G. Then $H_1 \cup H_2$ is a subgraph of G with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$; and $H_1 \cap H_2$ is a subgraph of G with vertex set $V(H_1) \cap V(H_2)$ and edge set $E(H_1) \cap E(H_2)$. If V_1, V_2 are two disjoint subsets of $V(G)$, then $[V_1, V_2]_G$ denotes the set of edges in G with one end in V_1 and the other end in V_2. When the graph G is understood from the context, we also omit the subscript G and write $[V_1, V_2]$ for $[V_1, V_2]_G$. If H_1, H_2 are two vertex disjoint subgraphs of G, then we also write $[H_1, H_2]$ for $[V(H_1), V(H_2)]$.

The *line graph* of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common. From the definition of a line graph, if $L(G)$ is not a complete graph, then a subset $X \subseteq V(L(G))$ is a vertex cut of $L(G)$ if and only if X is an essential edge cut of G. In 1986, Thomassen proposed the following conjecture.

Conjecture 1 *(Thomassen [8]) Every 4-connected line graph is hamiltonian.*

A graph that does not have an induced subgraph isomorphic to $K_{1,3}$ is called a *claw-free* graph. It is well known that every line graph is a claw-free graph. Matthews and Sumner proposed a seemingly stronger conjecture.

Conjecture 2 *(Matthews and Sumner [5]) Every 4-connected claw-free graph is hamiltonian.*

The best result towards these conjectures so far were obtained by Zhan and Ryjáček. A graph G is *hamiltonian connected* if for every pair of vertices u and v in G, G has a spanning (u, v)-path.

Theorem 3 *(Zhan [10]) Every 7-connected line graph is hamiltonian connected.*

Theorem 4 *(Ryjáček [7])*
(i) Conjecture 1.1 and Conjecture 1.2 are equivalent.
(ii) Every 7-connected claw-free graph is hamiltonian.

In this paper, we apply Catlin's reduction method ([2], [3]) on contracting collapsible subgraphs to prove the following.

Theorem 5 *Every 3-connected, essentially 11-connected line graph is hamiltonian.*

Ryjáček [7] introduced the line graph closure of a claw-free graph and used it to show that a claw-free graph G is hamiltonian if and only if it closure $cl(G)$ is hamiltonian, where $cl(G)$ is a line graph. With this argument and using the fact that adding edges will not decrease the connectivity of a graph, The following corollary is obtained.

Corollary 6 *Every 3-connected, essentially 11-connected claw-free graph is hamiltonian.*

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