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Maximal sets of integers not containing $k + 1$ pairwise coprimes and having divisors from a specified set of primes

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We find the formula for the cardinality of maximal set of integers from $[1, \dots, n]$ which does not contain $k + 1$ pairwise coprimes and has divisors from a specified set of primes. This formula is defined by the set of multiples of the generating set, which does not depend on n .

Keywords: greatest common divisor, coprimes, squarefree numbers

1 Formulation of the result

Let $\mathbb{P} = \{p_1 < p_2, \dots\}$ be the set of primes and \mathbb{N} be the set of natural numbers. Write $\mathbb{N}(n) = \{1, \dots, n\}$, $\mathbb{P}(n) = \mathbb{P} \cap \mathbb{N}(n)$. For $a, b \in \mathbb{N}$ denote the greatest common divisor of a and b by (a, b) . Let $S(n, k)$ be the family of sets $A \subset \mathbb{N}(n)$ of positive integers which does not contain $k + 1$ coprimes. Define

$$f(n, k) = \max_{A \in S(n, k)} |A|.$$

In the paper [1] the following was proved.

Theorem 1 For all sufficiently large

$$f(n, k) = |\mathbb{E}(n, k)|,$$

where

$$\mathbb{E}(n, k) = \{a \in \mathbb{N}(n) : a = up_i, \text{ for some } i = 1, \dots, k\}. \quad (1)$$

Let now $\mathbb{Q} = \{q_1 < q_2 < \dots < q_r\} \subset \mathbb{P}$ be finite set of primes and $R(n, \mathbb{Q}) \subset S(n, 1)$ is such family of sets of positive integers that for the arbitrary $a \in A \in R(n, \mathbb{Q})$, $(a, \prod_{j=1}^r q_j) > 1$. In [2] was proved the following

Theorem 2 Let $n \geq \prod_{j=1}^r q_j$, then

$$f(n, \mathbb{Q}) \triangleq \max_{A \in R(n, \mathbb{Q})} |A| = \max_{1 \leq t \leq r} |M(2q_1, \dots, 2q_t, q_1 \dots q_t) \cap \mathbb{N}(n)|, \quad (2)$$

where $M(B)$ is the set of multiples of the set of integers B .

In [2] the problem was stated of finding the maximal set of positive integers from $\mathbb{N}(n)$ which satisfies the conditions of Theorems 1 and 2 simultaneously i.e. which is a set A without $k + 1$ coprimes and such that each element of this set has a divisor from \mathbb{Q} . This paper is devoted to the solution of this problem. In our work we use the methods from the paper [1].

Denote $R(n, k, \mathbb{Q}) \subset S(n, k)$ the family of sets of positive integers with the property that an arbitrary $a \in A \in R(n, k, \mathbb{Q})$ has divisor from \mathbb{Q} . For given s and $\mathbb{T} = \{r_1 < r_2 < \dots\} = \mathbb{P} - \mathbb{Q}$ let $F(n, k, s, \mathbb{Q}) \subset R(n, k, \mathbb{Q})$ is the family of sets of squarefree positive numbers such that for the arbitrary $a \in A \in F(n, k, s, \mathbb{Q})$ we have $(r_i, a) = 1, i > s$. For given s, r cardinality of the family $F(n, k, s, \mathbb{Q})$ and cardinalities of each $A \in F(n, k, s, \mathbb{Q})$ are bounded from above as $n \rightarrow \infty$.

Next we formulate our main result which extent the result of the Theorems 1, 2 and in some sense include both of them.

Theorem 3 *If $\mathbb{Q} \neq \emptyset$, then for sufficiently large n the following relation is valid*

$$\varphi(n, k, \mathbb{Q}) \triangleq \max_{A \in R(n, k, \mathbb{Q})} |A| = \max_{F \in F(n, k, s-1, \mathbb{Q})} |M(F) \cap \mathbb{N}(n)|, \tag{3}$$

where s is the minimal integer which satisfies the inequality $r_s > r$.

2 Proof of the Theorem 3

Let's remind the definition of the left pushing which the reader can find in [2]. For the arbitrary

$$a = up_j^\alpha, p_i < p_j, (p_i p_j, u) = 1, \alpha > 0 \text{ and } p_j \notin \mathbb{Q} \text{ or } p_i, p_j \in \mathbb{Q} \tag{4}$$

define

$$L_{i,j}(a, \mathbb{Q}) = p_i^\alpha u.$$

For a not of the form (4) we set $L_{i,j}(a, \mathbb{Q}) = a$. For $A \subset \mathbb{N}$ denote

$$L_{i,j}(a, A, \mathbb{Q}) = \begin{cases} L_{i,j}(a, \mathbb{Q}), & L_{i,j}(a, \mathbb{Q}) \notin A, \\ a, & L_{i,j}(a, \mathbb{Q}) \in A. \end{cases}$$

At last set

$$L_{i,j}(A, \mathbb{Q}) = \{L_{i,j}(a, A, \mathbb{Q}); a \in A\}.$$

We say that A is left compressed if for the arbitrary $i < j$

$$L_{i,j}(A, \mathbb{Q}) = A.$$

It can be easily seen that every finite $A \subset \mathbb{N}$ after finite number of left pushing operations can be made left compressed,

$$|L_{i,j}(A, \mathbb{Q})| > |A|$$

and if $A \in R(n, k, \mathbb{Q})$, then $L_{i,j}(A, \mathbb{Q}) \in R(n, k, \mathbb{Q})$.

If we denote $O(n, k, \mathbb{Q}) \subset R(n, k, \mathbb{Q})$ the families of sets on which achieved max in (3) and $C(n, k, \mathbb{Q}) \subset R(n, k, \mathbb{Q})$ is the family of left compressed sets from $R(n, k, \mathbb{Q})$, then it follows that $O(n, k, \mathbb{Q}) \cap C(n, k, \mathbb{Q}) \neq \emptyset$. Next we assume that $A \in C(n, k, \mathbb{Q}) \cap O(n, k, \mathbb{Q})$.

For the arbitrary $a \in A$ we have the decomposition $a = a^1 a^2$, where $a^1 = r_{i_1}^{\alpha_1} \dots r_{i_f}^{\alpha_f}$, $r_i < r_j$, $i < j$, $a^2 = q_{j_1}^{\beta_1} \dots q_{j_\ell}^{\beta_\ell}$; $q_{j_m} < q_{j_s}$, $m < s$; $\alpha_j, \beta_j > 0$. If $a = r_{i_1}^{\alpha_1} \dots r_{i_f}^{\alpha_f} q_{j_1}^{\beta_1} \dots q_{j_\ell}^{\beta_\ell} \in A$, $\alpha_j, \beta_j > 0$, then $\bar{a} = r_{i_1} \dots r_{i_f} q_{j_1} \dots q_{j_\ell} \in A$ as well and also $\hat{a} = ua \in A$ for all $u \in \mathbb{N} : ua \leq n$. Consider all squarefree numbers $A^* \subset A$ and for given a^2 the set of all a^1 such that $a^1 a^2 \in A^*$. This set is the ideal generated by the division. The set of minimal elements from this ideal denote by $P(a^2, A^*)$. It follows that $(A \in O(n, k, \mathbb{N}))$,

$$A = M(\{a^1 a^2; a^1 \in P(a^2, A^*)\}) \cap \mathbb{N}(n),$$

For each a^2 we order $\{a_1^1 < a_2^1 < \dots\} = P(a^2, A^*)$ lexicographically according to their decomposition $a_i^1 = r_{i_1} \dots r_{i_f}$. Let ρ is the maximal over the choice of a^2 positive integer such that r_ρ divide some a_i^1 for which $a_i^1 a^2 \in A^*$. From the left compressedness of the set A it follows that $a' = a_j^1 a^2$, $j < i$ also belongs to A . Then the set B of elements $b = b^1 b^2 \leq n$, $(b^1, \prod_{j=1}^r q_j) = 1$ such that $b^2 = q_{j_1}^{\beta_1} \dots q_{j_\ell}^{\beta_\ell}$, $\beta_j > 0$ and $a_i^1 | b^1$, $a_j^1 \nmid b^1$, $j < i$ is exactly the set

$$B(a) = \left\{ u \leq n : u = r_{i_1}^{\alpha_1} \dots r_{i_f}^{\alpha_f} r_\rho^{\alpha_\rho} q_{j_1}^{\beta_1} \dots q_{j_\ell}^{\beta_\ell} F; \alpha_i, \beta_i > 0, \left(F, \prod_{j=1}^{\rho} r_j \prod_{j=1}^r q_j \right) = 1 \right\}.$$

Denote

$$P^\rho(a^2, A^*) = \{a \in P(a^2, A^*) : (a, r_\rho) = r_\rho\},$$

$$P_s^\rho(A^*) = \left\{ a \in P^\rho(a^2, A^*) \text{ for some } a^2, \text{ such that } \left(a^2, \prod_{j=1}^s q_j \right) = q_s \right\}$$

and

$$L^\rho(a^2) = \bigcup_{a \in P^\rho(a^2, A^*)} B(a).$$

Then the set $\bigcup_{s=1}^r P_s^\rho(A^*)$ is exactly the set $\bigcup_{a^2} P^\rho(a^2, A^*)$ of numbers which are divisible by r_ρ . Because each $a \in P(a^2, A^*)$ for all a^2 has divisor from \mathbb{Q} it follows that for some $1 \leq s \leq r$

$$\left| \bigcup_{a \in P_s^\rho(A^*)} B(a) \right| \geq \frac{1}{r} \left| \bigcup_{a^2} L^\rho(a^2) \right|. \tag{5}$$

Next for this s we define the transformation

$$\bar{P}(a^2, A^*) = (P(a^2, A^*) - P^\rho(a^2, A^*)) \bigcup R_s^\rho(a^2, A^*),$$

where

$$R_s^\rho(a^2, A^*) = \{v \in \mathbb{N}; vr_\rho \in P_s^\rho(a^2, A)\},$$

$$P_s^\rho(a^2, A^*) = \{a = a^1 a^2 \in P_s^\rho(A^*)\}.$$

It is easy to see that

$$\bigcup_{a^2} \bar{P}(a^2, A^*) \subset S(n, k, \mathbb{Q}).$$

Next we prove that if $r_\rho > r$, then

$$\left| M \left(\bigcup_{a^2} \bar{P}(a^2, A^*) \right) \cap \mathbb{N}(n) \right| > |A| \tag{6}$$

which gives the contradiction to the maximality of A .

For $a \in R_s^\rho(a^2, A^*)$, $a = r_{i_1} \dots r_{i_f} q_{j_1} \dots q_{j_\ell}$, $r_{i_1} < \dots < r_{i_f} < r_\rho$, $q_{j_1} \dots q_{j_\ell} = a^2$ denote

$$D(a) = \left\{ v \in \mathbb{N}(n) : v = r_{i_1}^{\alpha_1} \dots r_{i_f}^{\alpha_f} q_{j_1}^{\beta_1} \dots q_{j_\ell}^{\beta_\ell} T, \alpha_j, \beta_j \geq 1, \left(T, \prod_{j=1}^{\rho-1} r_j \prod_{j=1}^r q_j \right) = 1 \right\}.$$

It can be easily seen that

$$D(a) \cap D(a') = \emptyset, a \neq a'$$

and

$$M \left(\bigcup_{a^2} (P(a^2, A^*) - P^\rho(a^2, A^*)) \right) \cap D(a) = \emptyset.$$

Thus to prove (6) it is sufficient to show, that for large $n > n_0$

$$|D(a) > r|B(ar_\rho)|. \tag{7}$$

To prove (7) we consider three cases.

First case when $n/(ar_\rho) \geq 2$ and $\rho > \rho_0$. It follows that

$$\begin{aligned} |B(ar_\rho)| &\leq c_2 \sum_{\alpha_i, \alpha, \beta_i \geq 1} \frac{n}{r_{i_1}^{\alpha_1} \dots r_{i_f}^{\alpha_f} r_\rho^\alpha q_{j_1}^{\beta_1} \dots q_{j_\ell}^{\beta_\ell}} \prod_{j=1}^\rho \left(1 - \frac{1}{r_j}\right) \prod_{j=1}^r \left(1 - \frac{1}{q_j}\right) \\ &= c_2 \frac{n}{(r_{i_1} - 1) \dots (r_{i_f} - 1) (r_\rho - 1) (q_{j_1} - 1) \dots (q_{j_\ell} - 1)} \prod_{j=1}^\rho \left(1 - \frac{1}{r_j}\right) \prod_{j=1}^r \left(1 - \frac{1}{q_j}\right). \end{aligned} \tag{8}$$

At the same time

$$\bar{D}(a) \triangleq \left\{ v \in \mathbb{N}(n); v = r_{i_1} \dots r_{i_f} q_{j_1} \dots q_{j_\ell} F_1, \left(F_1, \prod_{j=1}^{\rho-1} r_j \prod_{j=1}^r q_j \right) = 1 \right\} \subset D(a)$$

and we obtain the inequalities

$$|D(a)| \geq |\bar{D}(a)| \geq c_1 \frac{n}{r_{i_1} \dots r_{i_f} q_{j_1} \dots q_{j_\ell}} \prod_{j=1}^{\rho-1} \left(1 - \frac{1}{r_j}\right) \prod_{j=1}^r \left(1 - \frac{1}{q_j}\right). \tag{9}$$

Thus from (8), (9) it follows that

$$\begin{aligned} \frac{|D(a)|}{|B(ar_\rho)|} &\geq \frac{c_1}{c_2} r_\rho \frac{(r_{i_1} - 1) \dots (r_{i_f} - 1)}{r_{i_1} \dots r_{i_f}} \prod_{j \in [r] - \{j_1, \dots, j_\ell\}} \left(1 - \frac{1}{q_j}\right) \\ &\geq \frac{c_1}{c_2} \prod_{j=1}^f \left(1 - \frac{1}{r_j}\right) r_\rho \prod_{j=1}^r \left(1 - \frac{1}{q_j}\right) > r. \end{aligned}$$

Now let's $n/(ar_\rho) \geq 2$, $\rho < \rho_0$. Then we obtain the inequalities

$$|B(ar_\rho)| < (1 + \epsilon) \frac{n}{(r_{i_1} - 1) \dots (r_{i_f} - 1)(r_\rho - 1)(q_{j_1} - 1) \dots (q_{j_\ell} - 1)} \prod_{j=1}^{\rho} \left(1 - \frac{1}{r_j}\right) \prod_{j=1}^r \left(1 - \frac{1}{q_j}\right),$$

$$|D(a)| > (1 - \epsilon) \frac{n}{(r_{i_1} - 1) \dots (r_{i_f} - 1)(q_{j_1} - 1) \dots (q_{j_\ell} - 1)} \prod_{j=1}^{\rho-1} \left(1 - \frac{1}{r_j}\right) \prod_{j=1}^r \left(1 - \frac{1}{q_j}\right).$$

From these inequalities it follows that

$$\frac{|D(a)|}{|B(ar_\rho)|} > \frac{1 - \epsilon}{1 + \epsilon} r_\rho > r.$$

Here the last inequality is valid for sufficiently small ϵ because $r_\rho > r$.

The last case is when $1 \leq n/(ar_\rho) < 2$. In this case $|B(ar_\rho)| = 1$. Let's $r_{i_1} \dots r_{i_f} r_\rho q_{j_1} \dots q_{j_\ell} = B(ar_\rho)$. Then we choose $r_g > (q_{j_1})^r$ and $n > \prod_{j=1}^g r_j \prod_{j=1}^r q_j$. We have $r_\rho > r_g$. Indeed, otherwise

$$n > \prod_{j=1}^g r_j \prod_{j=1}^r q_j > 2 \prod_{j=1}^{\rho} \prod_{j=1}^r q_j > 2ar_\rho$$

which is the contradiction to our case.

Hence

$$\{r_{i_1} \dots r_{i_f} q_{j_1} \dots q_{j_\ell}, r_{i_1} \dots r_{i_f} q_{j_1}^2 \dots q_{j_\ell}, \dots, r_{i_1} \dots r_{i_f} q_{j_1}^r \dots q_{j_\ell}, r_{i_1} \dots r_{i_f} q_{j_1} \dots q_{j_\ell} r_\rho\} \subset D(a).$$

Thus in this case also $|D(a)| > r = r|B(ar_\rho)|$.

From the above follows that for sufficiently large $n > n_0(\mathbb{Q})$ for all $a \in R_g^{\rho}(a^2, A^*)$ inequality (7) is valid and taking into account (5) we obtain (6). This gives the contradiction to the maximality of A . Hence the maximal $r_\rho \in \mathbb{P} - \mathbb{Q}$ which appear as the divisor of some $a \in \bigcup_{a^2} P(a^2, A^*)$ such that $M(A^*) \cap \mathbb{N}(n) \in O(n, k, \mathbb{Q})$ satisfies the condition $r_\rho \leq r$. This inequality gives the statement of Theorem.

This is joint work with R.Ahlswede.

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