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# Discrepancy of Products of Hypergraphs

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For a hypergraph  $\mathcal{H} = (V, \mathcal{E})$ , its  $d$ -fold symmetric product is  $\Delta^d \mathcal{H} = (V^d, \{E^d \mid E \in \mathcal{E}\})$ . We give several upper and lower bounds for the  $c$ -color discrepancy of such products. In particular, we show that the bound  $\text{disc}(\Delta^d \mathcal{H}, 2) \leq \text{disc}(\mathcal{H}, 2)$  proven for all  $d$  in [B. Doerr, A. Srivastav, and P. Wehr, Discrepancy of Cartesian products of arithmetic progressions, Electron. J. Combin. 11(2004), Research Paper 5, 16 pp.] cannot be extended to more than  $c = 2$  colors. In fact, for any  $c$  and  $d$  such that  $c$  does not divide  $d!$ , there are hypergraphs having arbitrary large discrepancy and  $\text{disc}(\Delta^d \mathcal{H}, c) = \Omega_d(\text{disc}(\mathcal{H}, c)^d)$ . Apart from constant factors (depending on  $c$  and  $d$ ), in these cases the symmetric product behaves no better than the general direct product  $\mathcal{H}^d$ , which satisfies  $\text{disc}(\mathcal{H}^d, c) = O_{c,d}(\text{disc}(\mathcal{H}, c)^d)$ .

**Keywords:** discrepancy, hypergraphs, Ramsey theory

## Introduction

We investigate the discrepancy of certain products of hypergraphs. In [2], Srivastav, Wehr and the first author noted the following. For a hypergraph  $\mathcal{H} = (V, \mathcal{E})$  define the  $d$ -fold direct product and the  $d$ -fold symmetric product by

$$\begin{aligned} \mathcal{H}^d &:= (V^d, \{E_1 \times \dots \times E_d \mid E_i \in \mathcal{E}\}), \\ \Delta^d \mathcal{H} &:= (V^d, \{E^d \mid E \in \mathcal{E}\}). \end{aligned}$$

Then for the (two-color) discrepancy

$$\text{disc}(\mathcal{H}) := \min_{\chi: V \rightarrow \{-1,1\}} \max_{E \in \mathcal{E}} \left| \sum_{v \in E} \chi(v) \right|,$$

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we have

$$\text{disc}(\mathcal{H}^d) \leq \text{disc}(\mathcal{H})^d \quad \text{and} \quad \text{disc}(\Delta^d \mathcal{H}) \leq \text{disc}(\mathcal{H}).$$

In this paper, we show that the situation is more complicated for discrepancies in more than two colors. In particular, it depends highly on the dimension  $d$  and the number of colors, whether the discrepancy of symmetric products is more like the discrepancy of the original hypergraph or the  $d$ -th power thereof. Let us make this precise:

For  $c \in \mathbb{N}_{\geq 2}$ , a  $c$ -coloring of a hypergraph  $\mathcal{H} = (V, \mathcal{E})$  is a mapping  $\chi : V \rightarrow [c]$ , where  $[r] := \{n \in \mathbb{N} \mid n \leq r\}$  for any  $r \in \mathbb{R}$ . The discrepancy problem asks for balanced colorings of hypergraphs in the sense that each hyperedge shall contain the same number of vertices in each color. The *discrepancy of  $\chi$*  and the  *$c$ -color discrepancy of  $\mathcal{H}$*  are defined by

$$\begin{aligned} \text{disc}(\mathcal{H}, \chi) &:= \max_{E \in \mathcal{E}} \max_{i \in [c]} \left| |\chi^{-1}(i) \cap E| - \frac{1}{c}|E| \right|, \\ \text{disc}(\mathcal{H}, c) &:= \min_{\chi: V \rightarrow [c]} \text{disc}(\mathcal{H}, \chi). \end{aligned}$$

These notions were introduced in [1] extending the discrepancy problem for hypergraphs to arbitrary numbers of colors. Note that  $\text{disc}(\mathcal{H}) = 2 \text{disc}(\mathcal{H}, 2)$  holds for all  $\mathcal{H}$ . In this more general setting, the product bound proven in [2] is

$$\text{disc}(\mathcal{H}^d, c) \leq c^{d-1} \text{disc}(\mathcal{H}, c)^d. \quad (1)$$

However, as we show in this paper the relation  $\text{disc}(\Delta^d \mathcal{H}, c) = O(\text{disc}(\mathcal{H}, c))$  does not hold in general. We give a characterization of those values of  $c$  and  $d$ , for which it is fulfilled for every hypergraph  $\mathcal{H}$ . In particular, we present for all  $c, d, k$  such that  $c$  does not divide  $d!$  a hypergraph  $\mathcal{H}$  having  $\text{disc}(\mathcal{H}, c) \geq k$  and  $\text{disc}(\Delta^d \mathcal{H}, c) = \Omega_d(k^d)$ . In the light of (1), this is largest possible apart from factors depending on  $c$  and  $d$  only.

On the other hand, there are further situations where this worst case does not occur. We state some results of this type in the last section.

## Symmetric Direct Products Having Large Discrepancy

Let  $S(d, l)$ ,  $d, l \in \mathbb{N}$ , denote the Stirling numbers of the second kind. For  $c \in \mathbb{N}$  and  $\lambda \in \mathbb{N}_0$  we write  $c \mid \lambda$  if there exists an  $m \in \mathbb{N}_0$  with  $mc = \lambda$ .

**Theorem 1** *Let  $c, d \in \mathbb{N}$ .*

*If  $c \mid k! S(d, k)$  for all  $k \in \{2, \dots, d\}$ , then every hypergraph  $\mathcal{H}$  satisfies*

$$\text{disc}(\Delta^d \mathcal{H}, c) \leq \text{disc}(\mathcal{H}, c). \quad (2)$$

*If  $c \nmid k! S(d, k)$  for some  $k \in \{2, \dots, d\}$ , then there exists a hypergraph  $\mathcal{K}$  such that*

$$\text{disc}(\Delta^d \mathcal{K}, c) \geq \frac{1}{3k!} \text{disc}(\mathcal{K}, c)^k, \quad (3)$$

*and  $\mathcal{K}$  can be chosen to have arbitrary large discrepancy  $\text{disc}(\mathcal{K}, c)$ .*

We state some simple consequences of Theorem 1:

**Corollary 2** (a) Let  $d \geq 3$  be an odd number. Then  $\text{disc}(\Delta^d \mathcal{H}, 3) \leq \text{disc}(\mathcal{H}, 3)$  holds for any hypergraph  $\mathcal{H}$ .

(b) Let  $d \geq 2$  be an even number and  $c = 3l$ ,  $l \in \mathbb{N}$ . There exists a hypergraph  $\mathcal{H}$  with arbitrary large discrepancy that fulfills  $\text{disc}(\Delta^d \mathcal{H}, c) \geq \frac{1}{6} \text{disc}(\mathcal{H}, c)^2$ .

**Proof:** Obviously  $3 \mid k!$  for all  $k \geq 3$ . Since  $S(d, 2) = 2^{d-1} - 1$ , we have  $3 \mid S(d, 2)$  if and only if  $d$  is odd.  $\square$

**Corollary 3** Let  $l \in \mathbb{N}$  and  $c = 4l$ . For all  $d \geq 2$  there exists a hypergraph  $\mathcal{H}$  with arbitrary large discrepancy such that  $\text{disc}(\Delta^d \mathcal{H}, c) \geq \frac{1}{6} \text{disc}(\mathcal{H}, c)^2$ .

**Proof:** As  $S(d, 2) = 2^{d-1} - 1$  is an odd number, we have  $4 \nmid 2! S(d, 2)$ .  $\square$

Our proof of Theorem 1 uses the following lemma.

**Lemma 4** Let  $c, d \in \mathbb{N}$ . For all  $m \in \mathbb{N}$  there exists an  $n \in \mathbb{N}$  having the following property: For each  $c$ -coloring  $\chi : [n]^d \rightarrow [c]$  we find a subset  $T \subseteq [n]$  with  $|T| = m$  such that for all  $l \in [d]$  each  $l$ -dimensional simplex in  $T^d$  is monochromatic with respect to  $\chi$ .

Hereby an  $l$ -dimensional simplex in  $T^d$  is of the following form: Fix a partition  $\{J_1, \dots, J_l\}$  of  $[d]$  and define vectors  $f^{(i)}$  by  $f_k^{(i)} = 1$  if  $k \in J_i$  and 0 else. Then

$$S := \left\{ \sum_{i=1}^l \alpha_i f^{(i)} \mid \alpha_1, \dots, \alpha_l \in T, \alpha_1 < \dots < \alpha_l \right\}$$

is an  $l$ -dimensional simplex in  $T^d$ . The proof of Lemma 4 is based on an argument from Ramsey theory (see, e.g., [3, Section 1.2]).

Related to Lemma 4 is a result of Gravier, Maffray, Renault and Trotignon [4]. They have shown that for any  $m \in \mathbb{N}$  there is an  $n \in \mathbb{N}$  such that any collection of  $n$  different sets contains an induced subsystem on  $m$  points such that one of the following holds: (a) each vertex forms a singleton, (b) for each vertex there is a set containing all  $m$  points except this one, or (c) by sufficiently ordering the points  $p_1, \dots, p_m$  we have that all sets  $\{p_1, \dots, p_\ell\}$ ,  $\ell \in [m]$ , are contained in the system.<sup>§</sup>

In our language, this means that any 0, 1 matrix having  $n$  distinct rows contains a  $m \times m$  submatrix that can be transformed through row and column permutations into a matrix that is (a) a diagonal matrix, (b) the inverse of a diagonal matrix, or (c) a triangular matrix.

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<sup>§</sup> To be precise, the authors also have the empty set contained in cases (a) and (c) and the whole set in case (b). It is obvious that by altering  $m$  by one, one can transform one result into the other.

Hence this result is very close to the assertion of Lemma 4 for dimension  $d = 2$  and  $c = 2$  colors. It is stronger in the sense that not only monochromatic simplices are guaranteed, but also a restriction to 3 of the 8 possible color combinations for the 3 simplices is given. Of course, this stems from the facts that (a) column and row permutations are allowed, (b) not a submatrix with index set  $T^2$  is provided but only one of type  $S \times T$ , and (c) the assumption of having different sets ensures sufficiently many entries in both colors.

## Further Upper Bounds

Besides the first part of Theorem 1, there are more ways to obtain upper bounds.

**Theorem 5** *Let  $\mathcal{H} = (V, \mathcal{E})$  be a hypergraph. Let  $p$  be a prime number,  $q \in \mathbb{N}$  and  $c = p^q$ . Furthermore, let  $d \geq c$  and  $s = d - (p - 1)p^{q-1}$ . Then  $\text{disc}(\Delta^d \mathcal{H}, c) \leq \text{disc}(\Delta^s \mathcal{H}, c)$ .*

As a corollary, we state a less general (but also less technical) version of Theorem 5:

**Corollary 6** *Let  $\mathcal{H} = (V, \mathcal{E})$  be a hypergraph. If  $c$  is a prime number,  $q \in \mathbb{N}$  and  $d = c^q$ , then  $\text{disc}(\Delta^d \mathcal{H}, c) \leq \text{disc}(\mathcal{H}, c)$ .*

**Proof:** Use  $c^q = 1 + (c - 1) \sum_{j=0}^{q-1} c^j$  and Theorem 5 (repeatedly). □

The following result is an extension of the first statement of Theorem 1.

**Theorem 7** *Let  $c, d \in \mathbb{N}$ , and let  $d' \in \{2, \dots, d\}$ . If  $c \mid k! S(d', k)$  for all  $k \in \{2, \dots, d'\}$ , then*

$$\text{disc}(\Delta^d \mathcal{H}, c) \leq \text{disc}(\Delta^{d-d'+1} \mathcal{H}, c) \tag{4}$$

*holds for every hypergraph  $\mathcal{H}$ .*

**Remark 8** *The condition in Theorem 7 is only sufficient but not necessary for the validity of (4), as the following example shows: Let  $c = 4$ ,  $d \geq c$  and  $d' = 3$ . According to Theorem 5, we get for each hypergraph  $\mathcal{H}$  that  $\text{disc}(\Delta^d \mathcal{H}, c) \leq \text{disc}(\Delta^{d-2} \mathcal{H}, c) = \text{disc}(\Delta^{d-d'+1} \mathcal{H}, c)$ . But we have  $2! S(d', 2) = 6 = 3! S(d', 3)$  and  $4 \nmid 6$ . This example indicates also that the proof methods of Theorem 5 and Theorem 7 are different.*

## References

- [1] B. Doerr and A. Srivastav, Multi-Color Discrepancies, *Comb. Probab. Comput.* 12(2003), 365-399.
- [2] B. Doerr, A. Srivastav, and P. Wehr, Discrepancy of Cartesian products of arithmetic progressions, *Electron. J. Combin.* 11(2004), Research Paper 5, 16 pp.

- [3] R. L. Graham, B. L. Rothschild, and J. H. Spencer, *Ramsey Theory*, Second Edition, Wiley, New York, USA, 1990.
- [4] S. Gravier, F. Maffray, J. Renault, and N. Trotignon, Ramsey-type results on singletons, co-singletons and monotone sequences in large collections of sets, *European J. Combin.* 25 (2004), 719-734.

