# Kernel perfect and critical kernel imperfect digraphs structure

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# *Kernel perfect and critical kernel imperfect digraphs structure*

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A kernel $N$ of a digraph $D$ is an independent set of vertices of $D$ such that for every $w \in V(D) - N$ there exists an arc from $w$ to $N$. If every induced subdigraph of $D$ has a kernel, $D$ is said to be a kernel perfect digraph. Minimal non-kernel perfect digraph are called critical kernel imperfect digraph. If $F$ is a set of arcs of $D$, a semikernel modulo $F$, $S$ of $D$ is an independent set of vertices of $D$ such that for every $z \in V(D) - S$ for which there exists an $Sz-$arc of $D - F$, there also exists an $zS-$arc in $D$. In this talk some structural results concerning critical kernel imperfect and sufficient conditions for a digraph to be a critical kernel imperfect digraph are presented.

**Keywords:** kernel, semikernel, semikernel modulo F, kernel perfect digraph, critical kernel imperfect digraph

Let $D$ be a digraph; $V(D)$ and $A(D)$ will denote the set of vertices and arcs of $D$ respectively. Let $S_1, S_2$ be subsets of $V(D)$. The arc $u_1u_2$ of $D$ will be called an $S_1S_2-$arc whenever $u_1 \in S_1$ y $u_2 \in S_2$. Let $H$ be a subdigraph of $D$. If $uv \in A(D) - A(H)$ then $uv$ is called a *pseudodiagonal of* $H$. $\Gamma^+(u)$, (resp. $\Gamma^-(u)$) is the exneighbourhood (resp. inneighbourhood) of $u$ in $D$.

A *kernel* $N$ of $D$ is an independent set of vertices such that for every $w \in V(D) - N$ there exists an arc from $w$ to a vertex in $N$. The concept of kernel was introduced by Von Neumann and Morgenstern (10) as an abstract generalization of their concept of solution for cooperative games. The problem of the existence of a kernel in a given digraph has been studied by several authors, since it is important in the context of Game Theory and Decision Theory, so the main question is: Which structural properties of a graph imply the existence of a kernel?

The classical results (1) are:

1. A symmetric digraph is kernel perfect;
2. A transitive digraph is kernel perfect, and all kernels have the same cardinality (König);
3. A digraph without cycles is kernel perfect, and its kernel is unique (von Neumann);
4. A graph without cycles of odd length is kernel perfect (Richardson)

Many extensions of Richardson's Theorem have have been found. An easy one is:

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**Proposition 1** *Let $D$ a digraph such that every cycle of odd length is symmetrical. Then $D$ is kernel perfect.*

Others theorems have been found, in particular the following:

1. *If every cycle of odd length $(x_1, x_2, \dots x_{2k+1}, x_1)$ has two pseudodiagonals of the type $(x_i, x_{i+2})$, $(x_{i+1}, x_{i+3})$ then the digraph is kernel perfect.* (3)

2. *If every cycle of odd length has at least two symmetrical arcs, then the digraph is kernel perfect.* (2)

A directed cycle of length 3 will be called a triangle and a forbidden triangle is a triangle with at most one symmetrical arc. *$M$-oriented digraphs* have no forbidden triangles. The *covering number* of a digraph $D$, denoted by $\theta(D)$ is the minimum number of complete subdigraphs of $D$ that partition $V(D)$.

The following are sufficient conditions for a $M$-oriented digraphs with $\theta(D) \leq 3$ is kernel perfect:

- If each directed cycle $\mathcal{C}$ of length 5 contained in $D$ satisfies at least one of the following properties: (a) $\mathcal{C}$ has two diagonals, (b) $\mathcal{C}$ has three symmetrical arcs.
- If every directed cycle of length 5 has three symmetrical arcs.
- If every directed cycle of length 5 has a symmetrical diagonal.
- If every directed cycle of length 5 has two diagonals.

A *semikernel* $S$ of $D$ is an independent set of vertices such that for every $z \in V(D) - S$ for which there exists an arc from a vertex in $S$ to $z$, there also exists an arc from $z$ to a vertex in $S$. Notice that a kernel $N$ of $D$ is a semikernel of $D$. A digraph $D$ is *kernel perfect* if every non-empty induced subdigraph of $D$ has a kernel. We say that $D$ is a *critical kernel imperfect digraph* if $D$ does not have a kernel but each proper induced subdigraph of $D$ does have at least one .

In (9), Neumann-Lara introduced the concept of a semikernel and, considering the kernel perfect digraphs, obtained sufficient conditions for the existence of a kernel in a digraph in terms of semikernels.

**Teorema 2** *(9) Let $D$ be a digraph. If every induced subdigraph of $D$ has a non-empty semikernel then $D$ is kernel perfect.*

This result provides another equivalent definition of a kernel perfect digraph: a digraph is kernel perfect if every non-empty induced subdigraph has a non-empty semikernel.

Theorem 2 allows us to prove in a simpler way Richardson's Theorem (7), which originally had a long and complicated proof: any digraph which does not contain directed cycles of odd length has a kernel; its enough to prove that every bipartite digraph has a semikernel. Theorem 2 also provides tools to give some general sufficient conditions for a digraph to be a kernel perfect digraph and some structural properties on critical kernel imperfect digraphs. Therefore, the concept of a semikernel has been very important in the development of Kernel Theory.

In (5), Galeana-Sánchez introduced the following concept: let $F$ be a set of arcs of $D$. A set $S \subseteq V(D)$ is called a *semikernel of $D$ modulo $F$* if $S$ is an independent set such that for every $z \in V(D) - S$ for which there exists an arc from a vertex in $S$ to $z$ of $D - F$, there also exists an $zS-$arc in $D$. We can observe that a semikernel $S$ is a semikernel modulo $F$, (for some $F$).

A digraph $D$ will be called *asymmetrically transitive* whenever $uv, vw \in Asym(D)$ implies $uw \in Asym(D)$, where $Asym(D)$ is the spanning subdigraph of $D$ whose arcs are asymmetrical arcs of $D$.

In this work the concept of semikernel modulo $F$ is used to obtain new sufficient conditions for the existence of kernels in digraphs; this results are more general than those obtained by using the concept of semikernel and also apply for infinite digraphs.

An infinite sequence $(x_1, x_2, \ldots)$ of distinct vertices of $D_1$, such that $x_i x_{i+1} \in A(D_1)$ for each $i$ is called *infinite outward path*.

**Teorema 3** *Let $D$ be a (possibly infinite) digraph. Let $D_1$ be an asymetrically transitive subdigraph of $D$ without infinite outward path, such that every induced subdigraph of $D$ has a non-empty semikernel modulo $A(D_1)$. If $D$ has no induced subdigraph isomorphic to a member of a special family of 14 digraphs, then $D$ is a kernel perfect digraph.*

We will provide an equivalent definition of a kernel perfect digraph for a class of digraphs; If $D$ satisfy:

- There exists $D_1 \subset D$ such that, there is a partial order, $\leq$, in the set of non-empty semikernels of $D$ modulo $A(D_1)$, with a maximal element.
- If $S$ is a non-empty semikernel of $D$ modulo $A(D_1)$, such that $B_S = \{v \in D - S \mid \nexists\, vS-$arc in $D\} \neq \emptyset$ and, if $S'$ is a non-empty semikernel of $D[B_S]$ modulo $A(D_1)$, then $T_S \cup S'$ is non-empty semikernel of $D$ modulo $A(D_1)$ and $T_S \cup S' > S$, where $T_S = \{v \in S \mid \nexists\, vS'-$arc in $D_1\}$.
- If $S_0$ is maximal with respect to $\leq$, then $S \subset S_0 \cup \{x \in V(D) \mid \exists\, xS_0 -$ arc in $D\}$, for each $S < S_0$

we say that $D$ holds the property $P(\alpha_{D_1}, \leq)$. We say that $D$ satisfy *hereditarily* $P(\alpha_{D_1}, \leq)$ if $D$ holds the property $P(\alpha_{D_1}, \leq)$ and every $H \subset^* D$ holds $P(\alpha_{D_1[V(H)]}, \leq)$, with $\leq$ restricted to $\alpha_{D_1[V(H)]}$. Note that the independent sets of $H$ are also independent in $D$.

**Teorema 4** *Let $D$ be a digraph that satisfy hereditarilly $P(\alpha_{D_1}, \leq)$. $D$ is kernel perfect if every non-empty induced subdigraph has a non-empty semikernel modulo $A(D_1)$.*

Notice that Theorem 3 implies Theorem 2, if we have that $D_1$ is $Sym(D)$ (the spanning subdigraph of $D$ whose arcs are symmetrical arcs of $D$). As a consequence of Theorem 3, we obtain a generalization of the following result due to B. Sands, N. Sauer and R. Woodrow (8): *Let $D$ be a digraph whose arcs are colored with two colors. If $D$ contains no monochromatic infinite outward path, then there exists a set $S$ of vertices of $D$ such that no two vertices of $S$ are connected by a monochromatic directed path and for every vertex not in $S$ there is a monochromatic directed path from $x$ to a vertex in $S$.*

In (6), Galeana-Sánchez and V. Neumann-Lara, using the notions of semikernels, gave sufficient conditions for a digraph to be a kernel perfect digraph. Those conditions generalized those studied by, e.g. Duchet (2). As a example, we have:

**Teorema 5** *If every directed cycle $C$ of odd length in $D$ has two pseudodiagonals with consecutive terminal endpoints then $D$ is kernel perfect.*

Galeana-Sánchez and Neumann-Lara also gave some structural properties of critical kernel imperfect digraphs. In particular they proved that every vertex (resp. arc) in a critical kernel imperfect digraph $D$, is contained in an odd directed cycle containing some ”special pseudodiagonals”.

In this work, we generalize the results of Galeana-Sánchez and Neumann-Lara, using the notions of semikernels modulo $A(D_1)$, where $D_1 \subset D$ and asking for $D$ to hold the property $P(\alpha_{D_1}, \leq)$, (the results of them are obtained if $D_1 = Sym(D)$).

The following theorems let us know some structures of the critical kernel imperfect digraphs:

We say that a cycle $C = (u_0, u_1, \ldots u_n)$ in $D$ *alternate* arcs, (resp. vertex), in $A \subset A(D)$, (resp. $B \subset V(D)$), if $u_0u_1, u_2u_3, \ldots inA$, (resp. $u_0, u_2, \ldots \in B$).

**Teorema 6** *Every arc in a critical kernel imperfect digraph $D$ (possibly infinite) holding $P(\alpha_{D_1}, \leq)$ is contained in an odd directed cycle that alternate arcs in $A(D) - A(D_1)$ not containing special pseudo-diagonals.*

**Remark:** Up to now, it is not known if an infinite critical kernel imperfect digraph exists.

**Teorema 7** *Every vertex in a critical kernel imperfect digraph $D$ (possibly infinite), holding $P(\alpha_{D_1},\leq)$, which is not a directed cycle of odd length, belongs to at least $\Delta_D(u) + 1$ directed cycle of odd length that alternate arcs in $A(D) - A(D_1)$. ($\Delta_D(u)$=max$\{|\Gamma^-(u)|, |\Gamma^+(u)|\}$).*

In particular, we provide sufficient conditions, as in the following theorems, to assure when a digraph is kernel perfect:

**Teorema 8** *Any finite digraph holding $P(\alpha_{D_1}, \leq)$ in which every odd directed cycle that alternate arcs in $A(D) - A(D_1)$, has two pseudodiagonals with consecutive terminal endpoints, is kernel perfect.*

Denote by $\mathcal{V}_{D_1}$, (resp. $\mathcal{F}_{D_1}$), the set of vertices (resp. arcs) of $D$ which do not belong to a directed cycle of odd length that alternate arcs in $A(D) - A(D_1)$.

**Teorema 9** *$D$ is kernel perfect digraph iff $D - \mathcal{V}_{D_1}$, (resp. every induced subdigraph $H$ of $D$ such that $A(H) \cap \mathcal{F}_{D_1} = \emptyset$), is a kernel perfect digraph.*

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