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Kernel perfect and critical kernel imperfect digraphs structure

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A kernel N of a digraph D is an independent set of vertices of D such that for every $w \in V(D) - N$ there exists an arc from w to N . If every induced subdigraph of D has a kernel, D is said to be a kernel perfect digraph. Minimal non-kernel perfect digraph are called critical kernel imperfect digraph. If F is a set of arcs of D , a semikernel modulo F , S of D is an independent set of vertices of D such that for every $z \in V(D) - S$ for which there exists an Sz -arc of $D - F$, there also exists an zS -arc in D . In this talk some structural results concerning critical kernel imperfect and sufficient conditions for a digraph to be a critical kernel imperfect digraph are presented.

Keywords: kernel, semikernel, semikernel modulo F , kernel perfect digraph, critical kernel imperfect digraph

Let D be a digraph; $V(D)$ and $A(D)$ will denote the set of vertices and arcs of D respectively. Let S_1, S_2 be subsets of $V(D)$. The arc u_1u_2 of D will be called an S_1S_2 -arc whenever $u_1 \in S_1$ y $u_2 \in S_2$. Let H be a subdigraph of D . If $wv \in A(D) - A(H)$ then wv is called a *pseudodiagonal* of H . $\Gamma^+(u)$, (resp. $\Gamma^-(u)$) is the exneighbourhood (resp. inneighbourhood) of u in D .

A *kernel* N of D is an independent set of vertices such that for every $w \in V(D) - N$ there exists an arc from w to a vertex in N . The concept of kernel was introduced by Von Neumann and Morgenstern (10) as an abstract generalization of their concept of solution for cooperative games. The problem of the existence of a kernel in a given digraph has been studied by several authors, since it is important in the context of Game Theory and Decision Theory, so the main question is: Which structural properties of a graph imply the existence of a kernel?

The classical results (1) are:

1. A symmetric digraph is kernel perfect;
2. A transitive digraph is kernel perfect, and all kernels have the same cardinality (König);
3. A digraph without cycles is kernel perfect, and its kernel is unique (von Neumann);
4. A graph without cycles of odd length is kernel perfect (Richardson)

Many extensions of Richardson's Theorem have have been found. An easy one is:

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Proposition 1 *Let D a digraph such that every cycle of odd length is symmetrical. Then D is kernel perfect.*

Others theorems have been found, in particular the following:

1. *If every cycle of odd length $(x_1, x_2, \dots, x_{2k+1}, x_1)$ has two pseudodiagonals of the type $(x_i, x_{i+2}), (x_{i+1}, x_{i+3})$ then the digraph is kernel perfect. (3)*
2. *If every cycle of odd length has at least two symmetrical arcs, then the digraph is kernel perfect. (2)*

A directed cycle of length 3 will be called a triangle and a forbidden triangle is a triangle with at most one symmetrical arc. M -oriented digraphs have no forbidden triangles. The covering number of a digraph D , denoted by $\theta(D)$ is the minimum number of complete subdigraphs of D that partition $V(D)$.

The following are sufficient conditions for a M -oriented digraphs with $\theta(D) \leq 3$ is kernel perfect:

- If each directed cycle \mathcal{C} of length 5 contained in D satisfies at least one of the following properties: (a) \mathcal{C} has two diagonals, (b) \mathcal{C} has three symmetrical arcs.
- If every directed cycle of length 5 has three symmetrical arcs.
- If every directed cycle of length 5 has a symmetrical diagonal.
- If every directed cycle of length 5 has two diagonals.

A *semikernel* S of D is an independent set of vertices such that for every $z \in V(D) - S$ for which there exists an arc from a vertex in S to z , there also exists an arc from z to a vertex in S . Notice that a kernel N of D is a semikernel of D . A digraph D is *kernel perfect* if every non-empty induced subdigraph of D has a kernel. We say that D is a *critical kernel imperfect digraph* if D does not have a kernel but each proper induced subdigraph of D does have at least one .

In (9), Neumann-Lara introduced the concept of a semikernel and, considering the kernel perfect digraphs, obtained sufficient conditions for the existence of a kernel in a digraph in terms of semikernels.

Teorema 2 (9) *Let D be a digraph. If every induced subdigraph of D has a non-empty semikernel then D is kernel perfect.*

This result provides another equivalent definition of a kernel perfect digraph: a digraph is kernel perfect if every non-empty induced subdigraph has a non-empty semikernel.

Theorem 2 allows us to prove in a simpler way Richardson's Theorem (7), which originally had a long and complicated proof: any digraph which does not contain directed cycles of odd length has a kernel; its enough to prove that every bipartite digraph has a semikernel. Theorem 2 also provides tools to give some general sufficient conditions for a digraph to be a kernel perfect digraph and some structural properties on critical kernel imperfect digraphs. Therefore, the concept of a semikernel has been very important in the development of Kernel Theory.

In (5), Galeana-Sánchez introduced the following concept: let F be a set of arcs of D . A set $S \subseteq V(D)$ is called a *semikernel of D modulo F* if S is an independent set such that for every $z \in V(D) - S$ for which there exists an arc from a vertex in S to z of $D - F$, there also exists an zS -arc in D . We can observe that a semikernel S is a semikernel modulo F , (for some F).

A digraph D will be called *asymmetrically transitive* whenever $uv, vw \in \text{Asym}(D)$ implies $uw \in \text{Asym}(D)$, where $\text{Asym}(D)$ is the spanning subdigraph of D whose arcs are asymmetrical arcs of D .

In this work the concept of semikernel modulo F is used to obtain new sufficient conditions for the existence of kernels in digraphs; this results are more general than those obtained by using the concept of semikernel and also apply for infinite digraphs.

An infinite sequence (x_1, x_2, \dots) of distinct vertices of D_1 , such that $x_i x_{i+1} \in A(D_1)$ for each i is called *infinite outward path*.

Teorema 3 *Let D be a (possibly infinite) digraph. Let D_1 be an asymmetrically transitive subdigraph of D without infinite outward path, such that every induced subdigraph of D has a non-empty semikernel modulo $A(D_1)$. If D has no induced subdigraph isomorphic to a member of a special family of 14 digraphs, then D is a kernel perfect digraph.*

We will provide an equivalent definition of a kernel perfect digraph for a class of digraphs; If D satisfy:

- There exists $D_1 \subset D$ such that, there is a partial order, \leq , in the set of non-empty semikernels of D modulo $A(D_1)$, with a maximal element.
- If S is a non-empty semikernel of D modulo $A(D_1)$, such that $B_S = \{v \in D - S \mid \nexists vS - \text{arc in } D\} \neq \emptyset$ and, if S' is a non-empty semikernel of $D[B_S]$ modulo $A(D_1)$, then $T_S \cup S'$ is non-empty semikernel of D modulo $A(D_1)$ and $T_S \cup S' > S$, where $T_S = \{v \in S \mid \nexists vS' - \text{arc in } D_1\}$.
- If S_0 is maximal with respect to \leq , then $S \subset S_0 \cup \{x \in V(D) \mid \exists xS_0 - \text{arc in } D\}$, for each $S < S_0$

we say that D holds the property $P(\alpha_{D_1}, \leq)$. We say that D satisfy *hereditarily* $P(\alpha_{D_1}, \leq)$ if D holds the property $P(\alpha_{D_1}, \leq)$ and every $H \subset^* D$ holds $P(\alpha_{D_1[V(H)]}, \leq)$, with \leq restricted to $\alpha_{D_1[V(H)]}$. Note that the independent sets of H are also independent in D .

Teorema 4 *Let D be a digraph that satisfy hereditarily $P(\alpha_{D_1}, \leq)$. D is kernel perfect if every non-empty induced subdigraph has a non-empty semikernel modulo $A(D_1)$.*

Notice that Theorem 3 implies Theorem 2, if we have that D_1 is $\text{Sym}(D)$ (the spanning subdigraph of D whose arcs are symmetrical arcs of D). As a consequence of Theorem 3, we obtain a generalization of the following result due to B. Sands, N. Sauer and R. Woodrow (8): *Let D be a digraph whose arcs are colored with two colors. If D contains no monochromatic infinite outward path, then there exists a set S of vertices of D such that no two vertices of S are connected by a monochromatic directed path and for every vertex not in S there is a monochromatic directed path from x to a vertex in S .*

In (6), Galeana-Sánchez and V. Neumann-Lara, using the notions of semikernels, gave sufficient conditions for a digraph to be a kernel perfect digraph. Those conditions generalized those studied by, e.g. Duchet (2). As an example, we have:

Teorema 5 *If every directed cycle C of odd length in D has two pseudodiagonals with consecutive terminal endpoints then D is kernel perfect.*

Galeana-Sánchez and Neumann-Lara also gave some structural properties of critical kernel imperfect digraphs. In particular they proved that every vertex (resp. arc) in a critical kernel imperfect digraph D , is contained in an odd directed cycle containing some "special pseudodiagonals".

In this work, we generalize the results of Galeana-Sánchez and Neumann-Lara, using the notions of semikernels modulo $A(D_1)$, where $D_1 \subset D$ and asking for D to hold the property $P(\alpha_{D_1}, \leq)$, (the results of them are obtained if $D_1 = \text{Sym}(D)$).

The following theorems let us know some structures of the critical kernel imperfect digraphs:

We say that a cycle $C = (u_0, u_1, \dots, u_n)$ in D alternate arcs, (resp. vertex), in $A \subset A(D)$, (resp. $B \subset V(D)$), if $u_0 u_1, u_2 u_3, \dots$ in A , (resp. $u_0, u_2, \dots \in B$).

Teorema 6 Every arc in a critical kernel imperfect digraph D (possibly infinite) holding $P(\alpha_{D_1}, \leq)$ is contained in an odd directed cycle that alternate arcs in $A(D) - A(D_1)$ not containing special pseudo-diagonals.

Remark: Up to now, it is not known if an infinite critical kernel imperfect digraph exists.

Teorema 7 Every vertex in a critical kernel imperfect digraph D (possibly infinite), holding $P(\alpha_{D_1}, \leq)$, which is not a directed cycle of odd length, belongs to at least $\Delta_D(u) + 1$ directed cycle of odd length that alternate arcs in $A(D) - A(D_1)$. ($\Delta_D(u) = \max\{|\Gamma^-(u)|, |\Gamma^+(u)|\}$).

In particular, we provide sufficient conditions, as in the following theorems, to assure when a digraph is kernel perfect:

Teorema 8 Any finite digraph holding $P(\alpha_{D_1}, \leq)$ in which every odd directed cycle that alternate arcs in $A(D) - A(D_1)$, has two pseudodiagonals with consecutive terminal endpoints, is kernel perfect.

Denote by \mathcal{V}_{D_1} , (resp. \mathcal{F}_{D_1}), the set of vertices (resp. arcs) of D which do not belong to a directed cycle of odd length that alternate arcs in $A(D) - A(D_1)$.

Teorema 9 D is kernel perfect digraph iff $D - \mathcal{V}_{D_1}$, (resp. every induced subdigraph H of D such that $A(H) \cap \mathcal{F}_{D_1} = \emptyset$), is a kernel perfect digraph.

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