

# Survival probability of a critical multi-type branching process in random environment

Elena Dyakonova

► **To cite this version:**

Elena Dyakonova. Survival probability of a critical multi-type branching process in random environment. Fourth Colloquium on Mathematics and Computer Science Algorithms, Trees, Combinatorics and Probabilities, 2006, Nancy, France. pp.375-380. hal-01184679

**HAL Id: hal-01184679**

**<https://hal.inria.fr/hal-01184679>**

Submitted on 17 Aug 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Survival probability of a critical multi-type branching process in random environment

Elena Dyakonova<sup>1†</sup>

<sup>1</sup>Department of Discrete Mathematics, Steklov Mathematical Institute, Gubkin St. 8, 119991 Moscow, Russia

We study a multi-type branching process in i.i.d. random environment. Assuming that the associated random walk satisfies the Doney-Spitzer condition, we find the asymptotics of the survival probability at time  $n$  as  $n \rightarrow \infty$ .

**Keywords:** branching processes in random environment, Doney-Spitzer condition, survival probability

## Introduction

Branching processes in random environment constitute an important part of the theory of branching processes (see, for example, (1), (2), (4)-(7), (9)-(14)). A branching process in random environment was first considered by Smith and Wilkinson (10). The subsequent papers (2), (7), (11) investigated single- and multi-type Galton-Watson processes in random environment. The asymptotics of the survival probability of the critical branching processes in a random environment generated by a sequence of independent identically distributed random variables under the condition  $\mathbf{E}X^2 < \infty$  for the increment  $X$  of the associated random walk was found in (6), (9) for single-type processes, and in (4) for multi-type processes. Recent papers (1), (5), (12)-(14) study the survival probability for an extended class of the critical single-type branching processes in random environment where the case  $\mathbf{E}X^2 = \infty$  is not excluded and, moreover,  $\mathbf{E}X$  may not exist. The present paper investigates an extended class of multi-type critical branching processes in random environment whose associated random walks satisfy the Doney-Spitzer condition. In particular, we generalize some results established in (1) and (4) concerning the asymptotic behavior of survival probability.

Let  $Z(n) = (Z_1(n), \dots, Z_p(n))$ ,  $n = 0, 1, \dots$ , be a  $p$ -type Galton-Watson branching process in a random environment. This process can be described as follows.

Let  $\mathbf{N}_0 = \{0, 1, 2, \dots\}$  and  $\mathbf{N}_0^p$  be the set of all vectors  $t = (t_1, \dots, t_p)$  with non-negative integer coordinates. Denote by  $(\Delta_1, \mathcal{B}(\Delta_1))$  a set of probability measures on  $\mathbf{N}_0^p$  with  $\sigma$ -algebra  $\mathcal{B}(\Delta_1)$  of Borel sets endowed with the metric of total variation, and by  $(\Delta, \mathcal{B}(\Delta))$  the  $p$ -times product of the space  $(\Delta_1, \mathcal{B}(\Delta_1))$  on itself. Let  $\mathbf{F} = (\mathbf{F}^{(1)}, \dots, \mathbf{F}^{(p)})$  be a random variable (random measure) taking values in  $(\Delta, \mathcal{B}(\Delta))$ . An infinite sequence  $\Pi = (\mathbf{F}_0, \mathbf{F}_1, \mathbf{F}_2, \dots)$  of independent identically distributed copies of  $\mathbf{F}$  is said to form a random environment and we will say that  $\mathbf{F}$  generates  $\Pi$ . A sequence of random  $p$ -dimensional vectors  $Z(0), Z(1), Z(2), \dots$  with non-negative integer coordinates is called a  $p$ -type branching process in random environment  $\Pi$ , if  $Z(0)$  is independent of  $\Pi$  and for all  $n \geq 0, z = (z_1, \dots, z_p) \in \mathbf{N}_0^p$  and  $f_0, f_1, \dots \in \Delta$

$$\begin{aligned} \mathcal{L}(Z(n+1) \mid Z(n) = (z_1, \dots, z_p), \Pi = (f_1, f_2, \dots)) \\ &= \mathcal{L}(Z(n+1) \mid Z(n) = (z_1, \dots, z_p), \mathbf{F}_n = f_n) \\ &= \mathcal{L}((\xi_{n,1}^{(1)} + \dots + \xi_{n,z_1}^{(1)}) + (\xi_{n,1}^{(2)} + \dots + \xi_{n,z_2}^{(2)}) + \dots + (\xi_{n,1}^{(p)} + \dots + \xi_{n,z_p}^{(p)})), \end{aligned} \quad (1)$$

where  $f_n = (f_n^{(1)}, f_n^{(2)}, \dots, f_n^{(p)}) \in \Delta$ ,  $\xi_{n,i}^{(i)}, \xi_{n,2}^{(i)}, \dots, \xi_{n,z_i}^{(i)}, i = 1, \dots, p$ , are independent  $p$ -dimensional random vectors, and for each  $i = 1, \dots, p$  the random vectors  $\xi_{n,1}^{(i)}, \xi_{n,2}^{(i)}, \dots, \xi_{n,z_i}^{(i)}$  are identically distributed according to the measure  $f_n^{(i)}$ . Relation (1) defines a branching Galton-Watson process  $Z(n)$  in random environment which describes the evolution of a particle population  $Z(n) = (Z_1(n), \dots, Z_p(n))$ ,  $n = 0, 1, \dots$ , where  $Z_i(n), i = 1, \dots, p$ , is the number of type  $i$  particles in the  $n$ -th generation.

<sup>†</sup>Supported by RFBR grant 05-01-00035, grant Scientific School-4129.2006.1 and by the program "Contemporary Problems of Theoretical Mathematics" of the Russian Academy of Sciences.

This population evolves as follows. If  $\mathbf{F}_n = f_n$  then each of the  $Z_i(n)$  particles of type  $i$  existing at the time  $n$ , produces offspring in accordance with the  $p$ -dimensional probability measure  $f_n^{(i)}$  independently of the reproduction of other particles. Thus, the  $i$ -th component of the vector  $Z(n+1) = (Z_1(n+1), \dots, Z_p(n+1))$  is equal to the number of type  $i$  particles among all direct descendants of the particles of the  $n$ -th generation. The distribution of  $Z(0)$  will be specified later.

### The main results

Let  $J^p$  be the set of all column vectors  $s = (s_1, \dots, s_p)^T, 0 \leq s_i \leq 1, i = 1, \dots, p$ . For  $s \in J^p$  and  $t \in \mathbf{N}_0^p$  set  $s^t = \prod_{i=1}^p s_i^{t_i}$ . Taking into account existence of a one-to-one correspondence between probability measures and generating functions we associate with  $\mathbf{F} = (\mathbf{F}^{(1)}, \dots, \mathbf{F}^{(p)})$  generating  $\Pi$  a random  $p$ -dimensional column vector  $F(s) = (F^{(1)}(s), \dots, F^{(p)}(s))^T, s \in J^p$ , whose components are  $p$ -dimensional (random) generating functions  $F^{(i)}(s)$  corresponding to  $\mathbf{F}^{(i)}, 1 \leq i \leq p$ :

$$F^{(i)}(s) = \sum_{t \in \mathbf{N}_0^p} \mathbf{F}^{(i)}(\{t\})s^t, s \in J^p.$$

In a similar way we associate with the component  $\mathbf{F}_n = (\mathbf{F}_n^{(1)}, \dots, \mathbf{F}_n^{(p)}), n \geq 0$ , of the random environment  $\Pi = (\mathbf{F}_0, \mathbf{F}_1, \mathbf{F}_2, \dots)$  a random vector  $F_n(s) = (F_n^{(1)}(s), \dots, F_n^{(p)}(s))^T, s \in J^p$ , the components of which are multidimensional (random) generating functions  $F_n^{(i)}(s)$ , corresponding to  $\mathbf{F}_n^{(i)}, 1 \leq i \leq p$ ,

$$F_n^{(i)}(s) = \sum_{t \in \mathbf{N}_0^p} \mathbf{F}_n^{(i)}(\{t\})s^t.$$

Let  $e_j, j = 1, \dots, p$ , be the  $p$ -dimensional row vector whose  $j$ -th component is equal to 1 and the others are zeros,  $\bar{0} = (0, \dots, 0)$  be the  $p$ -dimensional row vector all whose components are zeros, and let  $\bar{1} = (1, \dots, 1)^T$  be the  $p$ -dimensional column vector all whose components are equal to 1. For  $x = (x_1, \dots, x_p)$  and  $y = (y_1, \dots, y_p)^T$  we set  $|x| = \sum_{i=1}^p |x_i|, |y| = \sum_{i=1}^p |y_i|, (x, y) = \sum_{i=1}^p x_i y_i$ . Let  $A = \|A(i, j)\|_{i,j=1}^p$  be an arbitrary positive  $p \times p$  matrix. Denote by  $\rho(A)$  the Perron root of  $A$  and by  $u(A) = (u_1(A), \dots, u_p(A))^T$  and  $v(A) = (v_1(A), \dots, v_p(A))$  the right and left eigenvectors of  $A$  corresponding to the eigenvalue  $\rho(A)$  and such that

$$|v(A)| = 1, (v(A), u(A)) = 1.$$

For vector-valued generating functions  $F(s)$  and  $F_n(s)$  we introduce the mean matrices

$$M = M(\mathbf{F}) = \|M(i, j)\|_{i,j=1}^p = \left\| \frac{\partial F^{(i)}(\bar{1})}{\partial s_j} \right\|_{i,j=1}^p$$

and

$$M_n = M_n(\mathbf{F}_n) = \|M_n(i, j)\|_{i,j=1}^p = \left\| \frac{\partial F_n^{(i)}(\bar{1})}{\partial s_j} \right\|_{i,j=1}^p.$$

Let  $\mathcal{C}_\alpha, 0 < \alpha < 1$ , be the class of all matrices  $A = \|A(i, j)\|_{i,j=1}^p$  such that

$$\alpha \leq \frac{A(i_1, j_1)}{A(i_2, j_2)} \leq \alpha^{-1}, 1 \leq i_1, i_2, j_1, j_2 \leq p.$$

One of our basic hypotheses is the following condition.

**Assumption A0.** There exist a number  $0 < \alpha < 1$  and a positive row vector  $v = (v_1, \dots, v_p), |v| = 1$ , such that, with probability 1

$$M = M(\mathbf{F}) \in \mathcal{C}_\alpha,$$

and

$$vM = \rho(M)v. \tag{2}$$

Set  $\rho = \rho(M), \rho_n = \rho(M_n), n \geq 0$ . It is not difficult to see that in our settings  $X := \ln \rho, X_i := \ln \rho_{i-1}, i \geq 1$ , are independent and identically distributed random variables. Our next hypothesis imposes a restriction on the so-called associated random walk  $S = (S_0, S_1, \dots)$ , where

$$S_n = X_1 + \dots + X_n, n \geq 1, S_0 = 0.$$

**Assumption A1.** There exists a number  $0 < a < 1$  such that

$$\mathbf{P}(S_n > 0) \rightarrow a, n \rightarrow \infty. \tag{3}$$

Extending the known classification of single-type branching processes in random environment (see (1), (12)), we call a  $p$ -type branching process  $Z(n), n \geq 0$ , in random environment  $\Pi$  critical if its associated random walk is of the oscillating type, i.e.,  $\limsup_{n \rightarrow \infty} S_n = +\infty$  a.s. and  $\liminf_{n \rightarrow \infty} S_n = -\infty$  a.s. It is known that any random walk satisfying Assumption A1 oscillates. From now on we consider only critical  $p$ -type branching processes in random environment.

Let  $0 =: \gamma_0 < \gamma_1 < \dots$  be the strict descending ladder epochs of  $S$ . Put

$$V(x) := \sum_{i=0}^{\infty} \mathbf{P}(S_{\gamma_i} \geq -x), x \geq 0; V(x) = 0, x < 0.$$

Since  $S$  is oscillating, the following relation holds (3):

$$\mathbf{E}V(x + X) = V(x), x \geq 0. \tag{4}$$

For  $d \in \mathbf{N}_0$  set

$$O_d = \{t = (t_1, \dots, t_p) \in \mathbf{N}_0^p \mid t_i < d, i = 1, \dots, p\}, U_d = \mathbf{N}_0^p \setminus O_d.$$

Introduce the random variable

$$\kappa(d) = \sum_{t \in U_d} \sum_{i=1}^p v_i \sum_{j,k=1}^p \mathbf{F}^{(i)}(\{t\}) t_j t_k / \rho^2, d \in \mathbf{N}_0,$$

where  $v = (v_1, \dots, v_p)$  is from (2). Our next condition is connected with the random variable  $\kappa(d)$ , which is a generalization of the standardized truncated second moment of the reproduction law to the multi-type case.

**Assumption A2.** There exist  $\varepsilon > 0$  and  $d \in \mathbf{N}_0$  such that

$$\mathbf{E}(\ln^+ \kappa(d))^{1/a+\varepsilon} < \infty, \mathbf{E} \left( V(X)(\ln^+ \kappa(d))^{1/a+\varepsilon} \right) < \infty.$$

Let  $T = \min\{n \geq 0 : Z(n) = \bar{0}\}$  be the extinction moment for  $Z(n)$ . Introduce the random variables

$$Q^{(i)}(n) = \mathbf{P}(T > n \mid Z(0) = e_i, \Pi), Q(n) = (Q^{(1)}(n), \dots, Q^{(p)}(n)),$$

and let

$$q_i(k) = \mathbf{P}(T > k \mid Z(0) = e_i) = \mathbf{E}Q^{(i)}(k).$$

Note that under Assumptions A0 and A1  $Q^{(i)}(n) \rightarrow 0$   $\mathbf{P}$ -a.s. as  $n \rightarrow \infty$  for all  $1 \leq i \leq p$ , since  $\mathbf{P}$ -a.s.

$$(v, Q(n)) \leq \min_{0 \leq k \leq n-1} |vM_0 \cdots M_k| \leq \exp\left\{ \min_{0 \leq k \leq n-1} S_k \right\} \rightarrow 0$$

as  $n \rightarrow \infty$ . Denote by  $u(n) = (u_1(n), \dots, u_p(n))^T := u(M_0 \cdots M_n), n \geq 0$ , the right eigenvector of the product  $M_0 \cdots M_n$ , corresponding to the Perron root  $\rho(M_0 \cdots M_n) = \rho_0 \cdots \rho_n$ . To investigate the asymptotic behavior of  $q_i(n)$  and  $Q^{(i)}(n)$  as  $n \rightarrow \infty$  we need the following statement describing the behavior of  $u(n)$ .

**Theorem 1** *If Assumption A0 is valid, then there exist a random vector  $u = (u_1, \dots, u_p)^T$  and a function  $g(n) \geq 0, g(n) \rightarrow 0, n \rightarrow \infty$ , such that with probability 1*

$$|u_i(n) - u_i| \leq g(n), i = 1, \dots, p.$$

In addition,

$$(v, u) = 1, \alpha \leq u_i \leq 1/v^*,$$

where  $v^* = \min(v_1, \dots, v_p)$  and  $v = (v_1, \dots, v_p)$  is from (2).

The following statement describes the behavior of  $Q(n)$  as  $n \rightarrow \infty$ .

**Theorem 2** Assume Assumptions A0 and A1. Then  $\mathbf{P}$ -a.s., as  $n \rightarrow \infty$ ,

$$\frac{Q_i(n)}{(v, Q(n))} \rightarrow u_i, \quad i = 1, \dots, p,$$

where  $u = (u_1, \dots, u_p)$  is from Theorem 1.

Now we are ready to formulate the main result of the paper.

**Theorem 3** Assume Assumptions A0, A1, and A2. Then, as  $n \rightarrow \infty$ ,

$$q_i(n) \sim c_i n^{-(1-a)} l(n), \quad c_i > 0, \quad i = 1, \dots, p,$$

where  $l(n)$  is a function slowly varying at infinity.

Note that under our approach one of the key facts to prove Theorems 1, 2, 3 is convergence in distribution, as  $n \rightarrow \infty$ , of the products  $\prod_{i=0}^n M_i \rho_i^{-1}$  of random matrices to a limit matrix whose distribution is not concentrated at zero matrix. It is known (8) that for  $p = 2$  the products  $\prod_{i=0}^n M_i \rho_i^{-1}$  of the positive bounded independent identically distributed  $2 \times 2$  matrices  $A_i = M_i \rho_i^{-1}$  converges in distribution, as  $n \rightarrow \infty$ , to a limit matrix whose distribution is not concentrated at zero matrix if and only if all the matrices  $A_i$  have a common positive right or left eigenvector. Hence, for the 2-type process  $Z(n)$  our assumption on existence of a common positive left eigenvector of the matrices  $M$  is essential indeed.

Observe also that Assumption A1 covers non-degenerate random walks with zero mean and finite variance of there increments, as well as all non-degenerate symmetric random walks. In these cases  $a = 1/2$ . Another example when Assumption A1 is valid gives the random walk, whose increments have distribution belonging to the domain of attraction of a stable law.

In conclusion we give an example where Assumption A2 is fulfilled (given that Assumption A0 is valid as well). Clearly, if the measure  $\mathbf{F}$  generating our random environment has a bounded support, i.e., if there exists a  $p$ -dimensional cube  $B = [0, b]^p, b > 0$ , such that  $\mathbf{P}(\mathbf{F}(B) = 1) = 1$ , then Assumption A2 holds since  $\kappa(d) = 0$   $\mathbf{P}$ -a.s. for  $d > b$ .

One can show that if  $\mathbf{F}$  satisfies Assumption A0 and

$$F(s) = \bar{1} - \frac{M(\bar{1} - s)}{1 + \gamma(\bar{1} - s)}, \quad s \in J^p,$$

where the  $p$ -dimensional random row vector  $\gamma$  with positive components and the random matrix  $M$  are such that the components of the vector  $y = (M\bar{1})/|\gamma|$  are uniformly bounded from below then Assumption A2 holds true.

Note that if the distribution of  $X = \ln \rho$  has a regular varying tail then Assumptions A1 and A2 can be replaced by the following hypotheses (see (13) or (1)):

**Assumption A1'**. There exist constants  $c_n, n \geq 0$ , such that as  $n \rightarrow \infty$  the scaled sums  $c_n S_n$  converge weakly to a stable distribution  $\mu$  with parameter  $\beta \in (0, 2]$ . The limit law  $\mu$  is not one-side, i.e.,  $0 < \mu(\mathbb{R}^+) < 1$ .

**Assumption A2'**. There exist  $\varepsilon > 0$  and  $d \in \mathbb{N}_0$  such that

$$\mathbf{E}(\ln^+ \kappa(d))^{\beta+\varepsilon} < \infty,$$

where  $\beta$  is from Assumption A1'.

Note that Assumption A1' implies the validity of Assumption A1 with  $a = \mu(\mathbb{R}^+)$ , and Assumption A2 is stronger than Assumption A2' since  $a\beta \leq 1$ .

**Corollary 1** Assume Assumptions A0 and A1'. Then the statement of Theorem 2 remains true.

Using the proof of Theorem 3 and results of paper (1), one can obtain also the following statement.

**Theorem 4** Assume Assumptions A0, A1', and A2'. Then the statement of Theorem 3 remains true.

## References

- [1] Afanasyev V.I., Geiger J., Kersting G., and Vatutin V.A. *Criticality for branching processes in random environment*. Ann. Prob., 33(2005), 2, p. 645-673.
- [2] Athreya K.B. and Karlin S. *On branching processes with random environments: I,II*. Ann. Math. Statist., 42 (1971), 5, p. 1499-1520; 42(1971), 6, p. 1843-1858.

- [3] Bertoin J. and Doney R.A. *On conditioning a random walk to stay nonnegative*. Ann. Prob., 22 (1994), p. 2152-2167.
- [4] Dyakonova E.E. *Asymptotic behavior of the probability of non-extinction for a multitype branching process in random environment*. Discrete Math. Appl., 9 (1999), 2, p. 119-136.
- [5] Dyakonova E.E., Geiger J., and Vatutin V.A. *On the survival probability and a functional limit theorem for branching processes in random environment*. Markov Process. Related Fields, 10 (2004), 2, p. 289-306.
- [6] Geiger J. and Kersting G. *The survival probability of a critical branching process in random environment*. Theory Probab. Appl., 45(2000), 3, p. 607-615.
- [7] Kaplan N. *Some results about multidimensional branching processes with random environments*. Ann. Prob., 2 (1974), 3, p. 441-455.
- [8] Kesten H. and Spitzer F. *Convergence in distribution of products of random matrices*. Z. Wahrsch. verw. Geb., 67 (1984), 4, p. 363-386.
- [9] Kozlov M.V. *On the asymptotic behavior of the probability of non-extinction for critical branching processes in a random environment*. Theory Probab. Appl., 21 (1976), 4, p. 791-804.
- [10] Smith W.L. and Wilkinson W. *On branching processes in random environment*. Ann. Math. Statist., 40 (1969), 3, p. 814-827.
- [11] Tanny D. *On multitype branching processes in a random environment*. Adv. Appl. Prob., 13 (1981), 3, p. 464-497.
- [12] Vatutin V.A. and Dyakonova E.E. *Galton-Watson branching processes in random environments, I: limit theorems*. Theory Probab. Appl., 48(2003), 2, p. 314-336.
- [13] Vatutin V.A. and Dyakonova E.E. *Galton-Watson branching processes in random environments, II: joint distributions*. Theory Probab. Appl., 49 (2004), 2, p. 275-308.
- [14] Vatutin V.A. and Dyakonova E.E. *Branching processes in random environment and bottlenecks in the evolution of populations*. Theory Probab. Appl., 51(2006), 1.

