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S-constrained random matrices

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Let S be a set of d -dimensional row vectors with entries in a q -ary alphabet. A matrix M with entries in the same q -ary alphabet is S -constrained if every set of d columns of M contains as a submatrix a copy of the vectors in S , up to permutation. For a given set S of d -dimensional vectors, we compute the asymptotic probability for a random matrix M to be S -constrained, as the numbers of rows and columns both tend to infinity. If n is the number of columns and $m = m_n$ the number of rows, then the threshold is at $m_n = \alpha_d \log(n)$, where α_d only depends on the dimension d of vectors and not on the particular set S . Applications to superimposed codes, shattering classes of functions, and Sidon families of sets are proposed. For $d = 2$, an explicit construction of a S -constrained matrix is given.

Keywords: random matrix, Poisson approximation, superimposed code, shattering, VC dimensions, Sidon families

1 Introduction

We propose S -constrained matrices as a unifying framework for two seemingly remote notions, one in the field of cryptography (superimposed codes), the other one in information theory (shattering classes of functions).

Definition 1 Let q , d and s be three fixed integers. Let $S = \{\eta_1, \dots, \eta_s\}$ be a set of d -dimensional row vectors, with entries in the q -ary alphabet $\{0, \dots, q-1\}$.

Let $m \geq d$ and $n \geq s$ be two integers. Let $M = (M_{i,j})$ be a $m \times n$ matrix with entries in the same q -ary alphabet.

The matrix M is said to be S -constrained if for every subset \mathbf{j} of $[n] = \{1, \dots, n\}$, with $|\mathbf{j}| = d$ elements, there exist s indices i_1, \dots, i_s in $[m]$ such that for all $h = 1, \dots, s$ and for all $j \in \mathbf{j}$,

$$M_{i_h,j} = \eta_h(j).$$

In other words, every subset of d columns of the matrix M must contain as a submatrix a copy of all row vectors in S , up to permutation.

The matrices considered here can be interpreted either as q -ary codes (see Cohen and Schaathun's review (5)) or as classes of functions from $[n]$ into $\{0, \dots, q-1\}$ (see for instance Anthony and Bartlett (2)). In the first interpretation, the *columns* of the matrix are the words of the code, in the second one, the *rows* are the functions of the class.

To motivate Definition 1, recall ((16) p. 276) that a binary code is called (w, r) -superimposed if for every subsets of words W, R (sets of columns of the matrix) with respective cardinalities w, r , there exists a position (row index) on which every word of W is 1 and every word of R is 0. Let S be the set of all binary vectors with w ones and r zeros. The code is (w, r) -superimposed if and only if the corresponding matrix is S -constrained.

Consider now the interpretation of a q -ary matrix as a class of functions (rows of the matrix). A set \mathbf{j} of d coordinates (column indices), is *shattered* by the class, if the restrictions of the functions to those coordinates contain all possible q^d functions. The *Vapnik-Chervonenkis dimension* of the class is the size of the largest shattered set, its *testing dimension* is the maximal d such that all sets of size d are shattered (see (3)). Let S be the set of all q^d vectors of size d . The testing dimension of the class is $\geq d$ if and only if the corresponding matrix is S -constrained.

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The aim of this paper is to obtain asymptotic bounds on the size of S -constrained matrices, as the number of columns n tends to infinity, and the number of rows $m = m_n$ increases as a function of n . The problem of finding bounds on matrices satisfying certain constraints, has given rise to an extensive literature: see *e.g.* Kim *et al.* (16) for (w, r) -superimposed codes, Cohen and Schaathun (5; 6) for separating codes, Li *et al.* (17) for hashing codes. The dual problem of finding matrices *not containing* any copy of S has also been considered by a number of authors, following Sauer (20): see Steele (21) for matrices not shattering any sets of size d , and more recently Anstee *et al.* (1) on matrices with forbidden configurations.

We are mainly concerned with probabilistic bounds, derived from examining the threshold of the desired property for random matrices: see (6; 16; 17) for comparisons between probabilistic and deterministic bounds. See also (22; 23), for probabilistic bounds on classes of binary functions under different random models.

Our main result gives the asymptotic probability for a random matrix to be S -constrained. By *random matrix*, we mean a matrix whose entries are mutually independent and uniformly distributed on the alphabet $\{0, \dots, q-1\}$. Its distribution is denoted by \mathbb{P} .

Theorem 1 *Let S be a set of s q -ary vectors of size d . Denote by α_d the following (positive) real.*

$$\alpha_d = -\frac{d}{\log(1 - q^{-d})}. \quad (1)$$

Assume $m = m_n$ is such that:

$$\lim_{n \rightarrow \infty} m_n - \alpha_d \log n = c, \quad (2)$$

where c is a real constant. Let M be q -ary random matrix with m_n rows and n columns. Then:

$$\lim_{n \rightarrow \infty} \mathbb{P}(M \text{ is } S\text{-constrained}) = \exp\left(-\frac{s}{d!}(1 - q^{-d})^c\right). \quad (3)$$

As a consequence of Theorem 1, $m_n = \alpha_d \log n$ is a probabilistic bound for S -constrained matrices: if the number of rows is such that that $m_n - \alpha_d \log n$ tends to $-\infty$, then with high probability (*w.h.p.*), a random matrix is not S -constrained. If $m_n - \alpha_d \log n$ tends to $+\infty$, it is S -constrained *w.h.p.* Observe that the threshold $\alpha_d \log n$ only depends on the dimension of the vectors in S , and not on any particular set S . Thus if $m_n - \alpha_d \log n$ tends to $+\infty$ a random matrix, interpreted as a code, is (w, r) -superimposed for any w, r such that $w + r = d$. Interpreted as a class of functions, it has testing dimension at least d . We shall see in Section 3 another application to Sidon families of sets.

The question naturally arises of whether $m_n = \alpha_d \log n$ is optimal. We will answer it by the negative in section 4. In the case $d = 2$, we construct a S -constrained matrix having $m_n \leq \beta_2 \log n$ rows, with

$$\beta_2 = \frac{q^2}{\log 2} < \alpha_2 = -\frac{2}{\log(1 - q^{-2})}.$$

The problem of finding bounds for the size of S -constrained matrices has been investigated in the different contexts of combinatorics, cryptography and information theory. Deterministic and probabilistic bounds for q -ary superimposed codes are given by (16). D'yachkov and Rykov (8) used a probabilistic approach to construct superimposed codes in the binary case. Cohen and Schaathun (5) propose a very thorough review on bounds for separating codes, completed by (6). Other results concerning Sidon families, cover-free families and superimposed codes can be found in (10; 8; 13; 19; 12; 9; 16).

The rest of the paper is organised as follows. Section 2 is dedicated to the proof of Theorem 1, based on a classical Poisson approximation technique. Section 3 details the application of S -constrained binary matrices to Sidon families of sets. In Section 4, we construct an explicit S -constrained matrix in the case $d = 2$.

2 Proof of Theorem 1

Before proving Theorem 1, we will comment a few of its consequences.

The convergence in (3) expresses a rather sharp concentration result. Consider a random matrix with $m_n = \alpha \log n$ rows, where $\alpha \neq \alpha_d$ for all d , and let d be the largest integer such that $\alpha > \alpha_d$. Then if S is any set of d -dimensional vectors, the matrix will be S -constrained *w.h.p.* But for any set of $(d+1)$ -dimensional vectors, it will not be S -constrained *w.h.p.*

For $m_n = \alpha_d \log n + c$, the probability to be S -constrained converges to a non trivial value, that does depend on the set S , but only as a decreasing function of its cardinality s : the more constraints are imposed,

the less likely it is to satisfy them all. Table 1 below gives numerical values of the asymptotic probability (3), for $q = 2$, $c = 5$, $d = 1, \dots, 10$, and for the two extreme values of $s = 1$ and $s = 2^d$. It turns out that there is relatively little difference between satisfying all 2^d possible constraints, or just a single one.

d	1	2	3	4	5	6	7	8	9	10
$s = 1$	0.9692	0.8881	0.9181	0.9703	0.9929	0.9987	0.9998	1.0000	1.0000	1.0000
$s = 2^d$	0.9394	0.6221	0.5047	0.6171	0.7965	0.9211	0.9759	0.9938	0.9986	0.9997

Tab. 1: Asymptotic probability for a random matrix to be *S*-constrained for a set of d -dimensional vectors, of size $s = 1$ or $s = 2^d$, for $q = 2$, $c = 5$ and $d = 1, \dots, 10$.

Theorem 1 provides an existence result for *S*-constrained matrices: if $m_n = \alpha_d \log n + c$, then for n large enough, the probability for a random matrix to be *S*-constrained is strictly positive, hence there must exist *S*-constrained matrices of size $m_n \times n$. But it also provides an algorithmic way to construct such a matrix. Assume the parameters are such that the probability for a random matrix to be *S*-constrained is $1/2$, then drawing random matrices until one which is *S*-constrained is found, will output the desired matrix after 2 random drawings on average.

Theorem 1 is a Poisson approximation result (see Barbour et al. (4) for a general reference). The technique of proof, based on the Stein-Chen method, is quite standard: we shall use the results stated in Janson (14).

Proof: Let $\mathbf{j} \subset [n]$ be a set of d column indices. For any d -dimensional q -ary row vector η , we denote by $C_{\mathbf{j},\eta}$ the set of all $m \times n$ q -ary matrices such that there exists a row of M whose restriction to \mathbf{j} is η . We denote by $C_{\mathbf{j}}$ the intersection of the $C_{\mathbf{j},\eta}$'s over all vectors $\eta \in S$

$$C_{\mathbf{j}} = \bigcap_{\eta \in S} C_{\mathbf{j},\eta} .$$

The set $C_{\mathbf{j}}$ is made of those matrices whose columns indexed by \mathbf{j} contain a copy of all vectors in S . The set of *S*-constrained matrices is the intersection over all possible subsets of d indices \mathbf{j} , of the events $C_{\mathbf{j}}$.

$$C = \bigcap_{|\mathbf{j}|=d} C_{\mathbf{j}} .$$

Our aim is to compute the probability of C under the uniform distribution on all $m \times n$ q -ary matrices, denoted by \mathbb{P} . We begin with the probability of $C_{\mathbf{j}}$, for $\mathbf{j} \subset [n]$ such that $|\mathbf{j}| = d$. We denote by \overline{B} the complement of an event B . Recall that $C_{\mathbf{j}} = \bigcap C_{\mathbf{j},\eta}$, hence $\overline{C_{\mathbf{j}}} = \bigcup \overline{C_{\mathbf{j},\eta}}$, where the union extends over all elements η of S . If $h \leq s$ and η_1, \dots, η_h are distinct elements of S , then the probability that among m random rows none of them coincides with one of the η_i 's on \mathbf{j} is:

$$\mathbb{P}(\overline{C_{\mathbf{j},\eta_1}} \cap \dots \cap \overline{C_{\mathbf{j},\eta_h}}) = (1 - hq^{-d})^m . \tag{4}$$

By the formula giving the probability of a union (formula (1.5) p. 99 of Feller (11)), one gets

$$\mathbb{P}(\overline{C_{\mathbf{j}}}) = \sum_{h=1}^s (-1)^{h-1} \binom{s}{h} (1 - hq^{-d})^m .$$

If $m = m_n$ tends to infinity, this sum tends to 0 and the first term dominates.

$$\mathbb{P}(\overline{C_{\mathbf{j}}}) = s(1 - q^{-d})^{m_n} (1 + o(1)) .$$

Recall that $\alpha_d = -d/\log(1 - q^{-d})$. For $m_n = \alpha_d \log n + c + o(1)$,

$$\mathbb{P}(\overline{C_{\mathbf{j}}}) = s(1 - q^{-d})^c n^{-d} (1 + o(1)) . \tag{5}$$

Let us now introduce the integer-valued random variable X which counts the number of failed events among the $C_{\mathbf{j}}$'s.

$$X = \sum_{|\mathbf{j}|=d} \mathbb{I}_{\overline{C_{\mathbf{j}}}} ,$$

where \mathbb{I}_B denotes the indicator function of an event B . Obviously, a random matrix is S -constrained, iff $X = 0$. The expectation of X is

$$\mathbb{E}(X) = \binom{n}{d} \mathbb{P}(\overline{C}_j). \tag{6}$$

As n tends to infinity,

$$\binom{n}{d} = \frac{n^d}{d!} (1 + o(1)).$$

Therefore, for $m_n = \alpha_d \log n + c + o(1)$,

$$\lim_{n \rightarrow \infty} \mathbb{E}(X) = \frac{s}{d!} (1 - q^{-d})^c. \tag{7}$$

Comparing (7) with (3), there remains to prove that

$$\lim_{n \rightarrow \infty} \mathbb{P}(X = 0) = \lim_{n \rightarrow \infty} \exp(-\mathbb{E}(X)),$$

We will prove that a Poisson approximation holds for X . The family of indicators $(\mathbb{I}_{\overline{C}_j})$ is *dissociated*, in the sense of Janson (14) p. 10: the two sets of random variables $\{\mathbb{I}_{\overline{C}_j}, j \in J_1\}$ and $\{\mathbb{I}_{\overline{C}_{j'}}, j' \in J_2\}$ are independent whenever every $j \in J_1$ is disjoint from every $j' \in J_2$. Denote by Γ the set of all $j \subset [n]$ with $|j| = d$. For $j \in \Gamma$, denote by Γ_j the set of all j' such that $j \cap j' \neq \emptyset$. By Theorem 4 p. 10 of (14), the total variation distance between the distribution of X and the Poisson distribution with parameter $\mathbb{E}(X)$ is bounded above by

$$(1 \wedge \mathbb{E}(X))^{-1} \left(\sum_{j \in \Gamma} \sum_{j' \in \Gamma_j} \mathbb{P}(\overline{C}_j) \mathbb{P}(\overline{C}_{j'}) + \sum_{j \in \Gamma} \sum_{j' \in \Gamma_j \setminus \{j\}} \mathbb{P}(\overline{C}_j \cap \overline{C}_{j'}) \right) \tag{8}$$

The result will follow by proving that each of the two sums in (8) converges to zero. The first sum has $O(n^{2d-1})$ terms, each of order $O(n^{-2d})$, by (5). We decompose the second sum according to the number of elements in $j \cap j'$ as follows:

$$\sum_{j \in \Gamma} \sum_{j' \in \Gamma_j \setminus \{j\}} \mathbb{P}(\overline{C}_j \cap \overline{C}_{j'}) = \sum_{h=1}^{d-1} \Delta_h,$$

where

$$\Delta_h = \sum_{|j \cap j'|=h} \mathbb{P}(\overline{C}_j \cap \overline{C}_{j'}).$$

Clearly, there are $O(n^{2d-h})$ terms in Δ_h . We have

$$\overline{C}_j \cap \overline{C}_{j'} = \left(\bigcup_{\eta \in S} \overline{C}_{j,\eta} \right) \cap \left(\bigcup_{\zeta \in S} \overline{C}_{j',\zeta} \right) = \bigcup_{\eta,\zeta} (\overline{C}_{j,\eta} \cap \overline{C}_{j',\zeta}).$$

The probability of any of these intersections is:

$$\mathbf{P}(\overline{C}_{j,\eta} \cap \overline{C}_{j',\zeta}) = \begin{cases} (1 - 2q^{-d} + q^{-2d+h})^m & \text{if } \eta \equiv \zeta \text{ on } j \cap j' \\ (1 - 2q^{-d})^m & \text{otherwise.} \end{cases}$$

Hence

$$\Delta_h \leq a n^{2d-h} (1 - 2q^{-d} + q^{-2d+h})^m$$

for some positive a , not depending on n and m . For $m = m_n = \alpha_d \log n + c + o(1)$, there exists a positive constant b such that

$$\Delta_h \leq b n^{2d-h+\alpha_d \log(1-2q^{-d}+q^{-2d+h})}.$$

That Δ_h tends to zero follows from having a negative exponent, *i.e.*,

$$2d - h + \alpha_d \log(1 - 2q^{-d} + q^{-2d+h}) < 0 \tag{9}$$

for $d \geq 2$ and $h = 1, \dots, d - 1$. Indeed, the left hand side of (9) vanishes both for $h = 0$ and $h = d$. As a function of h , its second derivative is positive on $[0, d]$ hence it is strictly convex. Therefore it is strictly negative for all $h = 1, \dots, d - 1$. Hence Δ_h tends to zero with increasing n , which concludes the proof. \square

3 Sidon families

Let $\mathcal{R} = \{R_1, \dots, R_n\}$ be a family of subsets of $[m]$. To each of them, one can associate a m -dimensional binary column vector, whose i -th entry is 1 if $i \in R_j$ and 0 else. This defines the *incidence matrix* of the family, denoted by $M = (M_{i,j})$.

$$M_{i,j} = \mathbb{I}_{R_j}(i) .$$

Superimposed codes translate into *cover-free families*: \mathcal{R} is a (w, r) -cover-free family, if no intersection of w members of the family is covered by a union of r others (see (18; 9; 7; 16)). As already observed, the incidence matrix of a (w, r) -cover-free family is S -superimposed, where S is the set of vectors having w ones and r zeros. The notion we are studying in this section does not translate as straightforwardly.

Definition 2 A set family \mathcal{R} is a k -Sidon family if all the k -term unions are distinct :

$$\bigcup_{h=1}^k R_{j_h} \neq \bigcup_{h=1}^k R_{j'_h} ,$$

whenever $\{j_1, \dots, j_k\} \neq \{j'_1, \dots, j'_k\}$.

The notion was defined in (15) where the authors used the terminology ‘ UD_k code’. One can also find in the literature the term ‘ k -superimposed family’ in (8). We adopt here the terminology of (13).

Contrarily to cover-free families, there is no set of vectors S such that the family is k -Sidon if and only if its incidence matrix is S -constrained. Proposition 1 gives a necessary and a sufficient condition.

Proposition 1 Let k be a positive integer. Let S_0 be the singleton containing only the null vector with k entries. Let S_1 be the set of all vectors of length $(k + 1)$ having a single one and k zeros.

1. If a family of sets is k -Sidon, then in its incidence matrix, at most one set of k columns does not contain S_0 .
2. If the incidence matrix is S_1 -constrained, then the family of sets is k -Sidon.

Proof: Consider first the necessary condition. A set of k columns does not contain the null vector, if and only if the union of the k sets is $[m]$. If the family is k -Sidon, this can happen at most once.

Let us now turn to the sufficient condition. Consider two sets of indices j_1, \dots, j_k and j'_1, \dots, j'_k . With no loss of generality, assume that $j_1 \neq j'_1$. Since the incidence matrix is S_1 -constrained, there exists an index $i \in [m]$ such that $M_{i,j_1} = 1$ and $M_{i,j'_1} = \dots = M_{i,j'_k} = 0$: the element i belongs to R_{j_1} , but not to $R_{j'_1} \cup \dots \cup R_{j'_k}$. Hence the family is k -Sidon. \square

Consider a random binary matrix in the sense of Theorem 1, with m_n rows and n columns. It is the incidence matrix of a family of n sets, provided its columns are distinct. This holds *w.h.p.* if m_n is large compared to $(2/\log 2) \log n$. As a consequence of point 2 of Proposition 1 and Theorem 1, if $m_n - \alpha_{k+1} \log n$ tends to $+\infty$, then the family is k -Sidon *w.h.p.* From point 1 of Proposition 1, for a k -Sidon family, the variable X counting the number of failures of the property of having a copy of S_0 is either 0 or 1. It follows from the Poisson approximation of Theorem 1, that the threshold for ‘ $X \leq 1$ ’ is the same as for ‘ $X = 0$ ’, *i.e.* $\alpha_k \log n$: if $m_n - \alpha_k \log n$ tends to $-\infty$, then the family will not be k -Sidon *w.h.p.*

4 Explicit construction

Given a set $S = \{\eta_1, \dots, \eta_s\}$ of s 2-dimensional vectors with entries in a q -ary alphabet, we will construct a $m \times n$ S -constrained matrix M with $m \leq \frac{q^2}{\log 2} \log n$. Notice that the size of this matrix is an improvement of the probabilistic bound obtained in Theorem 1. Indeed, the ratio

$$\frac{q^2/\log 2}{\alpha_2} = \frac{q^2/\log 2}{-2/\log(1 - q^{-2})}$$

is smaller than 1 for all q and tends to $1/(2 \log 2) \simeq 0.72$ as q increases.

Let k be a positive integer. We start with a pattern matrix P having $k + 1$ rows and 2^k columns. Its first row is null. Its rows with indices $2, \dots, k + 1$ are formed by all 2^k binary column vectors, ranked in alphabetical order. Here is the matrix P for $k = 3$.

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

We denote by \overline{P} the matrix $1 - P$.

Let S_1 be the set of all vectors (a, b) in S such that (b, a) also belongs to S , and $S_2 = S \setminus S_1$. Let s_1 and s_2 denote the respective cardinalities of S_1 and S_2 . We will use as an example

$$S = \{(0, 2), (2, 0), (2, 1)\}, \quad S_1 = \{(0, 2), (2, 0)\}, \quad S_2 = \{(2, 1)\}.$$

Given a set X of two dimensional vectors, we denote by $A(X)$ (resp.: $B(X)$) the column vector of their first (resp.: second) coordinates. In our example:

$$A(S_1) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad B(S_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad A(S_2) = (2), \quad B(S_2) = (1).$$

Let U be the $(s_1 + s_2)(k + 1) \times 2^{k+1}$ matrix obtained from the pattern matrix P by replacing all 0's (resp. all 1's) by the $(s_1 + s_2) \times 2$ block

$$\begin{pmatrix} A(S_1) & B(S_1) \\ A(S_2) & B(S_2) \end{pmatrix} \quad (\text{resp.: } \begin{pmatrix} B(S_1) & A(S_1) \\ B(S_2) & A(S_2) \end{pmatrix}).$$

In our example:

$$U = \begin{pmatrix} 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ \hline 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 2 & 2 & 0 \\ 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ \hline 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 2 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ 2 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ \hline 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 \\ 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 0 \\ 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 1 \end{pmatrix}$$

Similarly, if S_2 is nonempty, we define V as the $s_2(k + 1) \times 2^{k+1}$ matrix obtained by replacing in \overline{P} all 0's (respectively 1's) by the $s_2 \times 2$ block:

$$(A(S_2), B(S_2)) \quad (\text{resp.: } (B(S_2), A(S_2))).$$

In our example:

$$V = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 \end{pmatrix}$$

If S_2 is empty, let $M_S = U$. If S_2 is not empty, let M_S be the matrix $\begin{pmatrix} U \\ V \end{pmatrix}$. The matrix M_S has $(s_1 + 2s_2)(k + 1)$ rows and 2^{k+1} columns. In any case, $(s_1 + 2s_2) \leq q^2$. Hence if $n = 2^{k+1}$ is the number of columns, the number of rows is no larger than $(q^2 / \log 2) \log n$. Observe moreover that contrarily to α_2 , the bound now depends on S : if S is a singleton, the matrix M_S only has $(2 / \log 2) \log n$ rows.

Theorem 2 *The matrix M_S is S -constrained.*

Proof: We must prove that any two columns contain a copy of all row vectors of S . Let j_1 and j_2 be two column indices. We will prove that there exist $s_1 + s_2$ row indices $i_1, \dots, i_{s_1}, i'_1, \dots, i'_{s_2}$ such that the submatrix of M_S indexed by $\{i_1, \dots, i_{s_1}, i'_1, \dots, i'_{s_2}\} \times \{j_1, j_2\}$ is either

$$\begin{pmatrix} A(S_1) & B(S_1) \\ A(S_2) & B(S_2) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} B(S_1) & A(S_1) \\ A(S_2) & B(S_2) \end{pmatrix}.$$

Since, up to permutation of rows, $(A(S_1), B(S_1))$ is the same as $(B(S_1), A(S_1))$, this will be enough to prove that M_S is S -constrained.

Assume first that j_1 and j_2 have distinct parities. Then the submatrix indexed by $\{1, \dots, s_1\} \times \{j_1, j_2\}$ is either $(A(S_1), B(S_1))$ or $(A(S_1), B(S_1))$. If j_1 is odd then the submatrix indexed by $\{s_1 + 1, \dots, s_1 + s_2\} \times \{j_1, j_2\}$ is $(A(S_2), B(S_2))$. If j_1 is even, then the submatrix indexed by $\{s(k + 1) + 1, \dots, s(k + 1) + s_2\} \times \{j_1, j_2\}$ is $(A(S_2), B(S_2))$.

Assume now that j_1 and j_2 have the same parity. Let $j'_1 = \lceil j_1/2 \rceil$ and $j'_2 = \lceil j_2/2 \rceil$. By definition of P , there exists an index i , $1 < i \leq k + 1$, such that its coefficients of order (i, j'_1) and (i, j'_2) are distinct. So, the submatrix of M_S indexed by $\{(i - 1)s + 1, \dots, (i - 1)s + s_1\} \times \{j_1, j_2\}$ is either $(A(S_1), B(S_1))$ or $(B(S_1), A(S_1))$. Moreover, if the coefficient of order (i, j'_1) of P is 0, then the submatrix of M_S indexed by $\{(i - 1)s + s_1 + 1, \dots, (i - 1)s + s_1 + s_2\} \times \{j_1, j_2\}$ is $(A(S_2), B(S_2))$. Otherwise, the coefficient of order $(i + k + 1, j'_1)$ of P is 0, and the submatrix of M_S indexed by $\{(k + 1)s + (i - 1)s_2 + 1, \dots, (k + 1)s + is_2\} \times \{j_1, j_2\}$ is $(A(S_2), B(S_2))$. \square

5 Concluding remarks

The bounds deduced from Theorem 1 are probabilistic. Possibly, one could obtain a deterministic algorithm to construct such S -constrained matrices using a derandomization procedure which would lead to an algorithm similar to that of Hwang and Sós (13). Nevertheless, it would be even more interesting to get an ‘explicit’ construction for all d as we did in Section 4 for $d = 2$. Indeed, such a construction gives more information on the structure of extremal matrices.

Hwang and Sós (13) used the concept of *part-intersecting family*. Given a set S of cardinality N , a t -part-intersecting family is a family \mathcal{F} of subsets of S such that for all $A, B \in \mathcal{F}$

$$|A \cap B| < \frac{1}{t} \min_{A, B \in \mathcal{F}} \{|A|, |B|\}.$$

To understand how this problem relates with ours, consider the simpler constraint that every intersection $A \cap B$ has cardinality 2. For the incidence matrix, this means that any two columns must contain two copies of the row vector $(1, 1)$. The technique that was used in the proof of Theorem 1 relied on the fact that the elements of S were distinct, and it cannot be directly adapted to counting copies of a given row vector. So a natural generalisation of our setting would be to consider matrices such that any set of d columns contains, up to permutation, a copy of some fixed matrix, which could have distinct rows or not.

Another interesting extension would be to consider matrices such that any set of d columns contains a copy of at least one set of vectors among different sets S_1, \dots, S_k . This is related to separating codes, in a similar way as S -constrained matrices are related to superimposed codes.

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