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Computing generating functions of ordered partitions with the transfer-matrix method

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An ordered partition of $[n] := \{1, 2, \dots, n\}$ is a sequence of disjoint and nonempty subsets, called blocks, whose union is $[n]$. The aim of this paper is to compute some generating functions of ordered partitions by the transfer-matrix method. In particular, we prove several conjectures of Steingrímsson, which assert that the generating function of some statistics of ordered partitions give rise to a natural q-analogue of $k!S(n,k)$, where $S(n,k)$ is the Stirling number of the second kind.

Keywords: Ordered partitions, Euler-Mahonian statistics, q-Stirling numbers of second kind, transfer-matrix method, determinants.

Contents

1 Introduction

1.1 Background

Definition 1.1 *A partition π_0 of $[n] = \{1, 2, \dots, n\}$ is a collection of disjoint and nonempty subsets, called* blocks, *whose union is $[n]$. By convention, we write $\pi_0 = B_1 - B_2 - \cdots - B_k$, where the blocks B_i are arranged in increasing order of their minimal elements and within each block the elements are arranged in increasing order.*

For instance, $\pi_0 = 1 - 2\,4\,6 - 3\,5 - 7\,8$ is a partition of [8] with 4 blocks.

Let $\mathcal{P}_n^k$ be the set of partitions of $[n]$ with k blocks. In the present paper we study statistics on *ordered partitions*, that is, partitions where the blocks are ordered arbitrarily.

Definition 1.2 *An* ordered partition *π of $[n]$ with k blocks is just a rearrangement of blocks of a partition in $\mathcal{P}_n^k$, that is there exist $\pi_0 = B_1 - B_2 - \cdots - B_k \in \mathcal{P}_n^k$ and σ a permutation of $[k]$ such that $\pi = B_{\sigma(1)} - B_{\sigma(2)} - \cdots - B_{\sigma(k)}$. We will say that σ is the* permutation induced *by π and set $\sigma = perm(\pi)$.*

For instance, $\pi = 3\,5 - 2\,4\,6 - 1 - 7\,8$ is an ordered partition of [8] with 4 blocks and we have $perm(\pi) = 3214$.Let $\mathcal{OP}_n^k$ be the set of ordered partitions of $[n]$ into k blocks. It is well known that the cardinality of $\mathcal{P}_n^k$ is the Stirling number of the second kind $S(n,k)$. It follows that $k!S(n,k)$ counts the ordered partitions of $[n]$ with k blocks.

Let $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ be a permutation of $[n]$, the integer $i \in [n-1]$ is called a *descent* of σ if $\sigma(i) > \sigma(i+1)$. The *Eulerian number* $A(n,k)$ counts permutations of $[n]$ with k descents. There is a basic identity relating the $S(n,k)$'s and the $A(n,k)$'s, namely

$$k!S(n,k) = \sum_{m=1}^{k} \binom{n-m}{n-k} \left\langle \begin{matrix} n \\ m-1 \end{matrix} \right\rangle, \tag{1}$$

which is easily proved combinatorially (for instance, see [10]).

In the present paper we will study some statistics originated from a q-analogue of (1) by means of the q-Eulerian numbers and q-Stirling numbers of the second kind, which were introduced by Carlitz [1, 2]. We need more definitions and notations.

Define the p,q-integer $[n]_{p,q} = \frac{p^n - q^n}{p-q}$, the p,q-factorial $[n]_{p,q}! = [1]_{p,q}[2]_{p,q}\cdots[n]_{p,q}$ and the p,q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!} \qquad n \geq k \geq 0.$$

If $p = 1$, we shall write $[n]_q$, $[n]_q!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q$ for $[n]_{1,q}$, $[n]_{1,q}!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_{1,q}$ respectively.

The q-Eulerian numbers $A_q(n,k)$ ($n \geq k \geq 0$) are defined by

$$A_q(n,k) = q^k [n-k]_q \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle_q + [k+1]_q \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle_q.$$

Let σ be a permutation of $[n]$, define the statistic $\operatorname{maj}\sigma = \sum_i i$ where the summation is over all descents of σ. Then

$$A_q(n,k) = \sum_{\sigma} q^{\operatorname{maj}\sigma},$$

where the summation is over all permutations of $[n]$ with k descents.

The q-Stirling numbers $S_q(n,k)$ of the second kind are defined by:

$$S_q(n,\, k) = q^{k-1} S_q(n-1,\, k-1) + [k]_q S_q(n-1,\, k) \qquad (n \geq k \geq 0), \tag{2}$$

where $S_q(n,k) = \delta_{n\,k}$ if $n = 0$ or $k = 0$.

A q-analogue of (1) has been found in [15] as follows:

$$[k]_q!\, S_q(n,k) = \sum_{m=1}^{k} q^{k(k-m)} \begin{bmatrix} n-m \\ n-k \end{bmatrix}_q \left\langle \begin{matrix} n \\ m-1 \end{matrix} \right\rangle_q. \tag{3}$$

There has been a considerable amount of recent interest in properties and combinatorial interpretations of the q-Stirling numbers and related numbers (see e.g. [1, 2, 4, 5, 6, 7, 8, 10, 11, 12, 13]). In the aim to give a combinatorial proof of (3), Steingrímsson [10] introduced the following definition.

Definition 1.3 *A statistic $Stat$ on $\mathcal{OP}_n^k$ is called* Euler-Mahonian *if its generating function is equal to $[k]_q!S_q(n,k)$, i.e.,*

$$\sum_{\pi \in \mathcal{OP}_n^k} q^{Stat\,\pi} = [k]_q!\, S_q(n,k).$$

Steingrímsson [10] has found a few of Euler-Mahonian statistics and has introduced new statistics on ordered partitions. Moreover, he has conjectured that all these new statistics are Euler-Mahonian. Wachs [11] has also obtained some *Euler-Mahonian* statistics on ordered partitions.

1.2 Definitions

Given a partition π in $\mathcal{OP}_n^k$, the elements of $[n]$ are divided into four classes:

- *singletons*: elements of the singleton blocks;
- *openers*: smallest elements of the non singleton blocks;
- *closers*: largest elements of the non singleton blocks;
- *transients*: all other elements, i.e., non extremal elements of non singleton blocks.

The sets of openers, closers, singletons and transients of π will be denoted by $\mathcal{O}(\pi)$, $\mathcal{C}(\pi)$, $\mathcal{S}(\pi)$ and $\mathcal{T}(\pi)$, respectively. The 4-tuple $(\mathcal{O}(\pi), \mathcal{C}(\pi), \mathcal{S}(\pi), \mathcal{T}(\pi))$ is called the *type* of π. For instance, if $\pi = 3\,5 - 2\,4\,6 - 1 - 7\,8$, then

$$\mathcal{O}(\pi) = \{2, 3, 7\}, \quad \mathcal{C}(\pi) = \{5, 6, 8\}, \quad \mathcal{S}(\pi) = \{1\} \quad \text{and} \quad \mathcal{T}(\pi) = \{4\}.$$

Definition 1.4 *Let σ be a permutation of $[n]$; the pair (i, j) is an* inversion *if $1 \le i < j \le n$ and $\sigma(i) > \sigma(j)$. Let $\operatorname{inv}\sigma$ be the number of inversions in σ. We also define the number of inversions of an ordered partition π, $\operatorname{inv}\,\pi$, by $\operatorname{inv}\,\pi = \operatorname{inv}(perm(\pi))$. For a partition π in $\mathcal{OP}^k$, we also set $\operatorname{cinv}\pi = \binom{\mathrm{k}}{2} - \operatorname{inv}\pi$.*

Let $\pi = B_1 - B_2 - \cdots - B_k$ be a partition in $\mathcal{OP}_n^k$. We define a *partial order* on blocks B_i's as follows : $B_i > B_j$ if all the letters of B_i are greater than those of B_j; in other words, if the opener of B_i is greater than the closer of B_j.

Definition 1.5 *A* block inversion *in π is a pair (i, j) such that $i < j$ and $B_i > B_j$. We denote by $\mathrm{bInv}\pi$ the number of block inversions in π. We also set $\mathrm{cbInv} = \binom{\mathrm{k}}{2} - \mathrm{bInv}$.*

Let w_i be the index of the block (counting from the left) containing i, namely the integer j such that $i \in B_j$. Following Steingrímsson [10], we define for $1 \le i \le n$ ten coordinate statistics on $\pi \in \mathcal{OP}_n^k$:

$$\begin{aligned}
\mathrm{ros_i}(\pi) &= \#\{j \in (\mathcal{O} \cup \mathcal{S})(\pi) \,|\, j < i,\ w_j > w_i\},\\
\mathrm{rob_i}(\pi) &= \#\{j \in (\mathcal{O} \cup \mathcal{S})(\pi) \,|\, j > i,\ w_j > w_i\},\\
\mathrm{rcs_i}(\pi) &= \#\{j \in (\mathcal{C} \cup \mathcal{S})(\pi) \,|\, j < i,\ w_j > w_i\},\\
\mathrm{rcb_i}(\pi) &= \#\{j \in (\mathcal{C} \cup \mathcal{S})(\pi) \,|\, j > i,\ w_j > w_i\},\\
\mathrm{los_i}(\pi) &= \#\{j \in (\mathcal{O} \cup \mathcal{S})(\pi) \,|\, j < i,\ w_j < w_i\},\\
\mathrm{lob_i}(\pi) &= \#\{j \in (\mathcal{O} \cup \mathcal{S})(\pi) \,|\, j > i,\ w_j < w_i\},\\
\mathrm{lcs_i}(\pi) &= \#\{j \in (\mathcal{C} \cup \mathcal{S})(\pi) \,|\, j < i,\ w_j < w_i\},\\
\mathrm{lcb_i}(\pi) &= \#\{j \in (\mathcal{C} \cup \mathcal{S})(\pi) \,|\, j > i,\ w_j < w_i\},
\end{aligned}$$

where $(\mathcal{O} \cup \mathcal{S})(\pi) = \mathcal{O}(\pi) \cup \mathcal{S}(\pi)$, and let $\mathrm{rsb_i}(\pi)$ (resp. $\mathrm{lsb_i}(\pi)$) be the number of blocks in π to the right (resp. left) of the block containing i such that the opener of B is smaller than i and the closer of B is greater than i. Then define $\mathrm{ros, rob, rcs, rcb, lob, los, lcs, lcb, lsb\ and\ rsb}$ as the sum of their coordinate statistics, e.g.

$$\mathrm{ros} = \sum_{\mathrm{i}} \mathrm{ros_i}.$$

Remark 1.6 *Note that* ros *is the abbreviation of "right, opener, smaller", while* lsb *is the abbreviation of "left, closer, bigger", etc.*

More generally, if stat is one of the above ten statistics, we define $\mathrm{stat}(\mathcal{O} \cup \mathcal{S})$ by

$$\mathrm{stat}(\mathcal{O} \cup \mathcal{S})(\pi) = \sum_{i \in (\mathcal{O} \cup \mathcal{S})(\pi)} \mathrm{stat}_i(\pi).$$

In the same way, we define $\mathrm{stat}(\mathcal{T} \cup \mathcal{C})$, $\mathrm{stat}(\mathcal{C} \cup \mathcal{S})$, etc. The following results illustrate the above notions.

Proposition 1.7 *The following functional identities hold on $\mathcal{OP}^k$:*

$$\mathrm{bInv} = \mathrm{rcs}(\mathcal{O} \cup \mathcal{S}), \quad \mathrm{inv} = \mathrm{ros}(\mathcal{O} \cup \mathcal{S}) \quad \textit{and} \quad \mathrm{cinv} = \mathrm{los}(\mathcal{O} \cup \mathcal{S}). \tag{4}$$

For instance, we give the values of the coordinate statistics computed on the partition $\pi = 6\ 8 - 5 - 1\ 4\ 7 - 3\ 9 - 2$:

$$\begin{array}{rccccccccc}
\pi = & 6\,8 & - & 5 & - & 1\,4\,7 & - & 3\,9 & - & 2 \\
\\
\mathrm{los_i}: & 0\,0 & - & 0 & - & 0\,0\,2 & - & 1\,3 & - & 1 \\
\mathrm{ros_i}: & 4\,4 & - & 3 & - & 0\,2\,2 & - & 1\,1 & - & 0 \\
\mathrm{lob_i}: & 0\,0 & - & 1 & - & 2\,2\,0 & - & 2\,0 & - & 3 \\
\mathrm{rob_i}: & 0\,0 & - & 0 & - & 2\,0\,0 & - & 0\,0 & - & 0 \\
\mathrm{lcs_i}: & 0\,0 & - & 0 & - & 0\,0\,1 & - & 0\,3 & - & 0 \\
\mathrm{rcs_i}: & 2\,3 & - & 1 & - & 0\,1\,1 & - & 1\,1 & - & 0 \\
\mathrm{lcb_i}: & 0\,0 & - & 1 & - & 2\,2\,1 & - & 3\,0 & - & 4 \\
\mathrm{rcb_i}: & 2\,1 & - & 2 & - & 2\,1\,1 & - & 0\,0 & - & 0 \\
\mathrm{lsb_i}: & 0\,0 & - & 0 & - & 0\,0\,1 & - & 1\,0 & - & 1 \\
\mathrm{rsb_i}: & 2\,1 & - & 2 & - & 0\,1\,1 & - & 0\,0 & - & 0
\end{array}$$

Since $\{6,8\} > \{5\}$, $\{6,8\} > \{2\}$, $\{5\} > \{2\}$ and $\{3,9\} > \{2\}$, we have $\mathrm{bInv}\,\pi = 4$ and $\mathrm{cbInv}\,\pi = \binom{5}{2} - 4 = 6$. Moreover, $perm(\pi) = 54132$, thus $\mathrm{inv}(\pi) = 8$ and $\mathrm{cinv}(\pi) = \binom{5}{2} - 8 = 2$.

Inspired by the statistic mak on the permutations due to Foata & Zeilberger [3] , Steingrímsson introduced its analogous on $\mathcal{OP}_n^k$ as follows:

$$\begin{aligned}
\mathrm{mak} &= \mathrm{ros} + \mathrm{lcs}, \\
\mathrm{lmak} &= n(k-1) - [\mathrm{los} + \mathrm{rcs}], \\
\mathrm{mak}' &= \mathrm{lob} + \mathrm{rcb}, \\
\mathrm{lmak}' &= n(k-1) - [\mathrm{lcb} + \mathrm{rob}].
\end{aligned}$$

The following proposition permits to reduce the conjecture 12 in [10] almost by half. This result was first proved in [4].

Proposition 1.8 *On $\mathcal{OP}_n^k$ the following functional identities hold:*

$$\mathrm{mak} = \mathrm{lmak}' \quad \textit{and} \quad \mathrm{mak}' = \mathrm{lmak}.$$

Define also the statistic cinvLSB on $\mathcal{OP}^k$ by

$$\mathrm{cinvLSB} := \mathrm{lsb} + \mathrm{cbInv} + \binom{\mathrm{k}}{2}. \tag{5}$$

1.3 Main results

Consider the following two generating functions of ordered partitions with $k \geq 0$ blocks:

$$\phi_k(a; x, y, t, u) := \sum_{\pi \in \mathcal{OP}^k} x^{(\mathrm{mak}+\mathrm{bInv})\pi}\, y^{\mathrm{cinvLSB}\,\pi}\, t^{\mathrm{inv}\,\pi}\, u^{\mathrm{cinv}\,\pi} a^{|\pi|}, \tag{6}$$

$$\varphi_k(a; z, t, u) := \sum_{\pi \in \mathcal{OP}^k} z^{(\mathrm{lmak}+\mathrm{bInv})\pi}\, t^{\mathrm{inv}\,\pi}\, u^{\mathrm{cinv}\,\pi}\, a^{|\pi|}, \tag{7}$$

where $|\pi| = n$ if π is a partition of $[n]$. The following Theorem is the main result of this paper.

Theorem 1.9 *We have*

$$\phi_k(a; x, y, t, u) = \frac{a^k\,(xy)^{\binom{k}{2}}[k]_{tx,uy}!}{\prod_{i=1}^k (1 - a[i]_{x,y})}, \tag{8}$$

$$\varphi_k(a; z, t, u) = \frac{a^k z^{\binom{k}{2}}\,[k]_{tz,u}!}{\prod_{i=1}^k (1 - a[i]_z)}. \tag{9}$$

The above theorem infers results on Euler-Mahonian statistics on ordered partitions. Indeed, it follows directly from (2) that

$$\sum_{n \geq k} [k]_q! S_q(n,k)\, a^n = \frac{a^k\, q^{\binom{k}{2}} [k]_q!}{\prod_{i=1}^k (1 - a[i]_q)}. \tag{10}$$

Then, by appropriate specializations in (6) and (7), we obtain the following result conjectured by Steingrímsson [10, conj.12 and 13].

Theorem 1.10 *The following statistics are Euler-Mahonian on $\mathcal{OP}_n^k$:*

$$\mathrm{mak} + \mathrm{bInv}, \quad \mathrm{lmak} + \mathrm{bInv}, \quad \mathrm{cinvLSB}.$$

In other words, the generating functions of the above statistics over $\mathcal{OP}_n^k$ are all equal to $[k]_q! S_q(n,k)$.

Others specializations give rise to new Euler-Mahonian statistics.

Theorem 1.11 *The following statistics are Euler-Mahonian on $\mathcal{OP}_n^k$:*

$$\mathrm{mak} + \mathrm{bInv} - (\mathrm{inv} - \mathrm{cinv}), \quad \mathrm{lmak} + \mathrm{bInv} + (\mathrm{inv} - \mathrm{cinv}), \quad \mathrm{cinvLSB} + (\mathrm{inv} - \mathrm{cinv}).$$

Theorem 1.9 provides also an alternative proof of the following combinatorial interpretations for q-Stirling numbers of the second kind, where the first two interpretations were proved by Ksavrelof and Zeng [4] and the third interpretation was first proved by Stanton (see [12]).

Corollary 1.12

$$S_q(n,k) = \sum_{\pi \in \mathcal{P}_n^k} q^{\mathrm{mak}\,\pi} = \sum_{\pi \in \mathcal{P}_n^k} q^{\mathrm{lmak}\,\pi} = \sum_{\pi \in \mathcal{P}_n^k} q^{\mathrm{lsb}\,\pi + \binom{k}{2}}.$$

Indeed, an (unordered) partition can be identified with an ordered partition without inversion, i.e.,

$$\mathcal{P}_n^k = \{\pi \in \mathcal{OP}_n^k \mid \mathrm{inv}\ \pi = 0\}.$$

Since the statistic bInv vanishes on (unordered) partitions and the identity $\mathrm{cinvLSB} = \mathrm{lsb} + 2\binom{k}{2}$ holds on $\mathcal{P}^k$, it is then trivial to obtain Corollary 1.12.

There are three main ingredients in the proof of Theorem 1.9: a bijection which maps ordered partitions to walks on some digraphs, the transfer-matrix method which permits to compute the generating function of walks using determinants, and the evaluation of determinants.

2 Ordered partitions and walks in digraphs

2.1 Encoding ordered partitions by walks

Let $\pi = B_1 - B_2 - \cdots - B_k$ in $\mathcal{OP}_n^k$ and i an integer in $[n]$. The restriction $B_j(\le i) := B_j \cap [i]$ of the block B_j is said to be *active* if $B_j \not\subseteq [i]$ and $B_j \cap [i] \neq \emptyset$, *complete* if $B_j \subseteq [i]$. We define the restriction of π on $[i]$, called the i-th *trace* of π, by

$$T_i(\pi) = B_1(\le i) - B_2(\le i) - \cdots - B_k(\le i),$$

with the empty restrictions being omitted. The sequence $(T_i(\pi))_{1\le i\le n}$ is called the *trace* of the partition π. We denote by $\mathrm{act}_i\ \pi$ and $\mathrm{com}_i\ \pi$ the numbers of active blocks and complete blocks, respectively, in $T_i(\pi)$ and set

$$\omega_i(\pi) = (\mathrm{com}_i \pi, \mathrm{act}_i \pi) \qquad \text{for} \quad 1 \le i \le n,$$

with $\omega_0(\pi) = (0,0)$. The sequence $(\omega_i(\pi))_{0\le i\le n}$ is called the *walk* of the partition π.

For instance, if $\pi = \{6\} - \{3,5,7\} - \{1,4,10\} - \{9\} - \{2,8\}$, then $T_6(\pi) = \{6\} - \{3,5,\cdots\} - \{1,4,\cdots\} - \{2,\cdots\}$, where each active block ends with an ellipsis, and we get $\omega_6(\pi) = (1,3)$.

Definition 2.1 *Let $D = (V,E)$ be the digraph on $V = \mathbb{N}^2$ with edges set E defined by*

$$E = \{(u,v) \in V^2 \mid u = v = (x,y) \text{ with } y > 0 \quad \text{or} \quad u - v = (0,1),\ (1,0),\ (1,-1)\}.$$

Clearly there are four types of edges. An edge (u,v) of D is called:

- North *if $v = u + (0,1)$;*
- East *if $v = u + (1,0)$;*
- South-East *if $v = u + (1,-1)$;*
- Null *if $v = u$.*

For any integer $k \ge 0$, let $V_k = \{(i,j) \in V \mid i + j \le k\}$ and D_k be the restriction of the digraph D on V_k . An illustration of D_k is given in Figure 1.

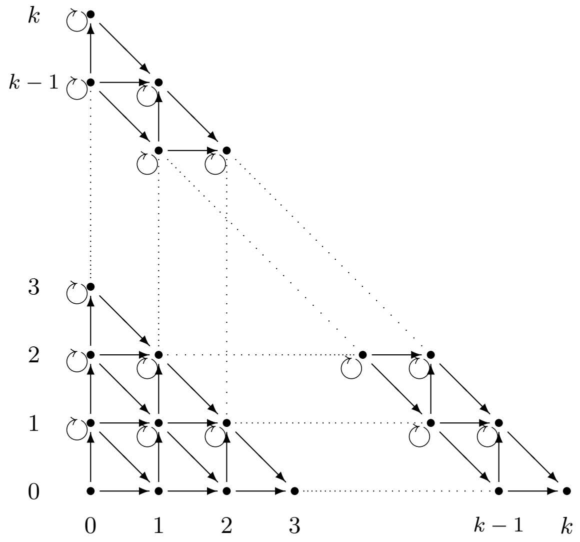

Fig. 1: The digraph D_k

Definition 2.2 *A walk of depth k and length n in D is a sequence $\omega = (v_0, v_1, \dots, v_n)$ of points in V such that $v_0 = (0,0)$, $v_n = (k, 0)$ and (v_i, v_{i+1}) is an edge of D for $i = 0, \dots, n-1$. Moreover, the* abscissa *and* height *of the step (v_i, v_{i+1}) are the* abscissa *and* ordinate *of v_i respectively. Let Ω_n^k be the set of walks of depth k and length n and Ω^k be the set of walks of depth k in D.*

The following result characterizes the walks of ordered partitions.

Proposition 2.3 *For $n \geq k \geq 0$, the walks of the partitions in $\mathcal{OP}_n^k$ are exactly the walks in Ω_n^k.*

We need a refinement of the notion of walk.

Definition 2.4 *A walk diagram of depth k and length n is a pair (ω, ξ), where $\omega = (v_0\ v_1 \cdots v_n)$ is a walk in Ω_n^k and $\xi = (\xi_i)_{1 \leq i \leq n}$ is a sequence of integers such that $1 \leq \xi_i \leq q$ if the i-th step is (Null or South-East) of height q, and $1 \leq \xi_i \leq p + q + 1$ if the i-th step is (North or East) of abscissa p and height q.*

Denote by Δ_n^k the set of path diagrams of depth k and length n. The main ingredient of our proof is a mapping from Δ_n^k to $\mathcal{OP}_n^k$ which keeps track of several statistics of Steingrímsson.

Theorem 2.5 *There exists a bijection $\psi : \Delta_n^k \to \mathcal{OP}_n^k$ such that if $h = (\omega, \xi)$ is in Δ_n^k and if the i-th step of w is of abscissa p and height q, and of type:*

(i) North or East : then, $i \in (\mathcal{O} \cup \mathcal{S})(\psi(h))$,
$(\mathrm{lcs} + \mathrm{rcs})_\mathrm{i}(\psi(\mathrm{h})) = \mathrm{q}$ *and* $(\mathrm{lsb} + \mathrm{rsb})_\mathrm{i}(\psi(\mathrm{h})) = \mathrm{p}$,
$\mathrm{los}_\mathrm{i}(\psi(\mathrm{h})) = \xi_\mathrm{i} - 1$ *and* $\mathrm{ros}_\mathrm{i}(\psi(\mathrm{h})) = \mathrm{p} + \mathrm{q} + 1 - \xi_\mathrm{i}$.

(ii) South-East or Null : then, $i \in (\mathcal{T} \cup \mathcal{C})(\psi(h))$,
$(\mathrm{lcs} + \mathrm{rcs})_\mathrm{i}(\psi(\mathrm{h})) = \mathrm{p}$ *and* $(\mathrm{lsb} + \mathrm{rsb})_\mathrm{i}(\psi(\mathrm{h})) = \mathrm{q} - 1$,
$\mathrm{lsb}_\mathrm{i}(\psi(\mathrm{h})) = \xi_\mathrm{i} - 1$ *and* $\mathrm{rsb}_\mathrm{i}(\psi(\mathrm{h})) = \mathrm{q} - \xi_\mathrm{i}$

2.2 Generating functions of walks

For $0 \leq k \leq n$, let $\mathbf{t} = (t_1, t_2, t_3, t_4, t_5, t_6, t_7)$ and

$$Q_{n,k}(\mathbf{t}) := \sum_{\pi \in \mathcal{OP}_n^k} t_1^{(\mathrm{lcs}+\mathrm{rcs})(\mathcal{O}\cup\mathcal{S})\pi}\, t_2^{(\mathrm{lcs}+\mathrm{rcs})(\mathcal{T}\cup\mathcal{C})\pi}\, t_3^{\mathrm{rsb}(\mathcal{T}\cup\mathcal{C})\pi} \times t_4^{\mathrm{lsb}(\mathcal{T}\cup\mathcal{C})\pi}\, t_5^{\mathrm{ros}(\mathcal{O}\cup\mathcal{S})\pi}\, t_6^{\mathrm{los}(\mathcal{O}\cup\mathcal{S})\pi}\, t_7^{(\mathrm{lsb}+\mathrm{rsb})(\mathcal{O}\cup\mathcal{S})\pi}. \tag{11}$$

Given a walk ω, define the weight $\mathrm{v}(\omega)$ of ω to be the product of the weights of all its steps, where the weight of a step of abscissa i and height j is:

$$\begin{cases} t_1^i\, t_7^j\, [i+j+1]_{t_5,t_6} & \text{if the step is North or East;} \\ t_2^i\, [j]_{t_3,t_4} & \text{if the step is Null or South-East.} \end{cases} \tag{12}$$

It follows easily from Theorem 2.5 that

$$\sum_{\omega\in\Omega_n^k} \mathrm{v}(\omega) = Q_{n,k}(\mathbf{t})\ .$$

Denote by $|\omega|$ the length of the walk ω. Then, using the above identity, we get

$$Q_k(a;\mathbf{t}) := \sum_{n\geq 0} Q_{n,k}(\mathbf{t})\, a^n = \sum_{w\in\Omega^k} \mathrm{v}(w)\, a^{|\omega|}\ . \tag{13}$$

It is obvious that the number of vertices of D_k is equal to

$$\widehat{k} := 1+2+\cdots+(k+1) = \frac{(k+1)(k+2)}{2}.$$

Let $v_1,\cdots,v_{\widehat{k}}$ be the vertices of D_k arranged according to the following order: $(i,j)\leq(i',j')$ if and only if $i+j<i'+j'$ or $(i+j=i'+j'$ and $j\geq j')$. For instance, we get $v_1=(0,0)$, $v_2=(0,1)$, $v_3=(1,0)$, $v_4=(0,2)$, $v_5=(1,1)$, $v_6=(2,0),\cdots,v_{\widehat{k}}=(k,0)$.

The *adjacency matrix* A_k of D_k relative to the valuation v is the $\widehat{k}\times\widehat{k}$ matrix defined by

$$A_k(i,j) = \begin{cases} \mathrm{v}(v_i,v_j) & \text{if } (v_i,v_j) \text{ is an edge of } D_k; \\ 0 & \text{otherwise.} \end{cases}$$

Applying the transfer-matrix method (see e.g. [9, Theorem 4.7.2]), we obtain

$$Q_k(a;\mathbf{t}) = \frac{(-1)^{1+\widehat{k}}\det(I-aA_k;\widehat{k},1)}{\det(I-aA_k)}, \tag{14}$$

where $(B;i,j)$ denotes the matrix obtained by removing the i-th row and j-th column of B and I is the $\widehat{k}\times\widehat{k}$ identity matrix.

In order to prove Theorem 1.9 it suffices to evaluate the following specializations of (14):

$$f_k(a;x,y,t,u) = Q_k(a;x,x,x,y,t,u,y), \tag{15}$$

$$g_k(a;z,t,u) = Q_k(a;1,z,1,z,t,u,1). \tag{16}$$

Let A'_k and A''_k be the adjacency matrix of D_k relative to the weight function v' and v'' obtained from the weight function v by making the substitution (15) and (16), respectively. Namely, the weights $\mathrm{v}'(e)$ and $\mathrm{v}''(e)$ of an edge $e=((i,j),(i',j'))$ of D_k with initial vertex (i,j) are :

$$\mathrm{v}'(e) = \begin{cases} x^i\, y^j\, [i+j+1]_{t,u} & \text{if } (i',j')=(i,j+1) \text{ or } (i+1,j)\ ; \\ x^i\, [j]_{x,y} & \text{if } (i',j')=(i,j) \text{ or } (i+1,j-1), \end{cases}$$

and

$$\mathrm{v}''(e) = \begin{cases} [i+j+1]_{t,u} & \text{if } (i',j')=(i,j+1) \text{ or } (i+1,j)\ ; \\ z^i[j]_z & \text{if } (i',j')=(i,j) \text{ or } (i+1,j-1). \end{cases}$$

Now, for each $k\geq 0$ let

$$M_k = I - aA'_k \quad\text{and}\quad N_k = I - aA''_k.$$

Then by (14), (15) and (16) we have

$$f_k(a;x,y,t,u) = \frac{(-1)^{1+\widehat{k}}\det(M_k;\widehat{k},1)}{\det M_k}, \tag{17}$$

$$g_k(a;z,t,u) = \frac{(-1)^{1+\widehat{k}}\det(N_k;\widehat{k},1)}{\det N_k} \tag{18}$$

for each $k \geq 1$.

Since M_n and N_n are upper triangular matrices it is easy to see that for each $n \geq 1$

$$\det M_n = \prod_{m=1}^{n}\prod_{i=0}^{m}(1 - ax^i[m-i]_{x,y}), \tag{19}$$

$$\det N_n = \prod_{m=1}^{n}\prod_{k=0}^{n-m}(1 - az^k[m]_q). \tag{20}$$

The evaluation of $\det(M_n; \widehat{n}, 1)$ and $\det(N_n; \widehat{n}, 1)$ is highly non trivial.

Theorem 2.6 *Let $n \geq 1$ be a positive integer. Then*

$$\det(M_n; \widehat{n}, 1) = (-1)^{\binom{n}{2}} a^n x^{\binom{n}{2}} [n]_{t,u}! \prod_{m=1}^{n-1}\prod_{i=1}^{m}(1 - ax^i[m-i+1]_{x,y}), \tag{21}$$

Theorem 2.7 *Let $n \geq 1$ be a positive integer. Then*

$$\det(N_n; \widehat{n}, 1) = (-1)^{\binom{n}{2}} a^n \, [n]_{t,u}! \prod_{m=1}^{n-1}\prod_{k=1}^{n-m}(1 - az^{k-1}[m]_z). \tag{22}$$

It is now trivial to obtain the following result.

Corollary 2.8 *For $k \geq 0$, we have*

$$f_k(a; x, y, t, u) = \frac{a^k x^{\binom{k}{2}} [k]_{t,u}!}{\prod_{i=1}^{k}(1 - a[i]_{x,y})}, \tag{23}$$

$$g_k(a; z, t, u) = \frac{a^k \, [k]_{t,u}!}{\prod_{i=1}^{k}(1 - az^{k-i}[i]_z)}. \tag{24}$$

Finally Theorem 1.9 follows immediately from Corollary 2.8 and the following lemma.

Lemma 2.9 *The following identities hold:*

$$\phi_k(a; x, y, t, u) = f_k(a; x, y, xyt, uy^2), \tag{25}$$

$$\varphi_k(a; z, t, u) = g_k(az^{k-1}; 1/z, t, u/z). \tag{26}$$

Therefore in order to prove Theorem 1.9 it remains to prove Theorems 2.6 and 2.7.

3 Determinantal computations

3.1 Proof of Theorem 2.6

The matrix M_n can be defined recursively by

$$M_0 = (1) \quad \text{and} \quad M_n = \left(\begin{array}{c|c} M_{n-1} & \overline{M}_{n-1} \\ \hline O_{n+1,\widehat{n-1}} & \widehat{M}_{n-1} \end{array}\right) \quad \text{for } n \geq 1, \tag{27}$$

where $\widehat{M}_{n-1}$ is the $(n+1) \times (n+1)$ matrix

$$\widehat{M}_{n-1} = \left(\delta_{ij} - ax^{i-1}[n+1-i]_{x,y}(\delta_{ij} + \delta_{i+1,j})\right)_{1 \leq i,j \leq n+1} \tag{28}$$

and $\overline{M}_{n-1}$ is the $\widehat{n-1} \times (n+1)$ matrix

$$\overline{M}_{n-1} = \left(\begin{array}{c} O_{\widehat{n-2},n+1} \\ \hline \check{M}_{n-1} \end{array}\right)$$

with the $n \times (n+1)$ matrix

$$\check{M}_{n-1} = \left(-ax^{i-1}y^{n-i}[n]_{t,u}(\delta_{ij} + \delta_{i+1,j})\right)_{1 \le i \le n,\, 1 \le j \le n+1}. \tag{29}$$

Here δ_{ij} stands for the Kronecker delta and $O_{m,n}$ denotes the $m \times n$ zero matrix. Let

$$\mathrm{K}_n = \widehat{n} - 1 = \frac{n(n+3)}{2},$$

and let $P_n = (M_n; \widehat{n}, 1)$, i.e the $\mathrm{K}_n \times \mathrm{K}_n$ matrix obtained from M_n by deleting the $\widehat{n}$th row and the first column. The problem is to compute $\det P_n$ for $n \ge 1$. The matrix P_n can also be defined recursively:

$$P_n = \left(\begin{array}{c|c} P_{n-1} & \overline{P}_{n-1} \\ \hline X_{n-1} & \widehat{P}_{n-1} \end{array}\right)$$

Here $\overline{P}_{n-1}$ is a $\mathrm{K}_{n-1} \times (n+1)$ matrix, X_{n-1} is a $(n+1) \times \mathrm{K}_{n-1}$ matrix, and $\widehat{P}_{n-1}$ is a $(n+1) \times (n+1)$ matrix. The idea here is to use the following well-known formula for any block matrix with an invertible square matrix A,

$$\det \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \det A \cdot \det\left(D - CA^{-1}B\right).$$

Since the entries of $CA^{-1}B$ are also written by minors, we guess these entries and prove it by induction.

3.2 Proof of Theorem 2.7

Let $F = \{F_n\}_{n=1}^{\infty}$ be a sequence of non-zero functions in finitely many variables $v_1, v_2, \ldots$. We use the convention that $F_n! = \prod_{k=1}^{n} F_k$ and

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \begin{cases} \frac{F_n!}{F_k! F_{n-k}!}, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

We prove Theorem 2.7 (22) by considering the following matrix $N_n(x,a)$, which generalize the matrix N_n (set $x = 1$ and $F_n = [n]_{t,u}$ to obtain N_n). Let $N_n(x,a)$ be the matrix defined inductively by:

$$N_0(x,a) = (x)$$

and

$$N_n(x,a) = \left(\begin{array}{c|c} N_{n-1}(x,a) & \overline{N}_{n-1}(x,a) \\ \hline O_{n+1,\widehat{n-1}} & \widehat{N}_{n-1}(x,a) \end{array}\right) \tag{30}$$

where $\widehat{N}_{n-1}(x,a)$ is the $(n+1) \times (n+1)$ matrix defined by

$$\widehat{N}_{n-1}(x,a) = \left(x\delta_{ij} - aq^{i-1}[n+1-i]_q(\delta_{ij} + \delta_{i+1,j})\right)_{1 \le i,j \le n+1} \tag{31}$$

and $\overline{N}_{n-1}(x,a)$ is the $\widehat{n-1} \times (n+1)$ matrix

$$\left(\begin{array}{c} O_{\widehat{n-2},n+1} \\ \hline \check{N}_{n-1} \end{array}\right)$$

with the $n \times (n+1)$ matrix

$$\check{N}_{n-1} = (-aF_n \cdot (\delta_{ij} + \delta_{i+1,j}))_{1 \le i \le n,\, 1 \le j \le n+1}. \tag{32}$$

Let $\dot{N}_n(x,a)$ denote the matrix obtained from $N_n(x,a)$ by deleting the $\widehat{n}$th row and the first column. Then the following theorem is sufficient to prove our result.

Theorem 3.1 *We have*

$$\det \dot{N}_n(x,a) = (-1)^{\frac{n(n-1)}{2}} a^n\, F_n!\, x^n \prod_{m=1}^{n-1} \prod_{k=1}^{n-m} \left(x - aq^{k-1}[m]_q\right). \tag{33}$$

Setting $x = 1$ and $F_n = [n]_{t,u}$ we obtain Theorem 2.7.

Here our strategy is as follows. We regard $\det N_n(x,a)$ as a polynomial in x and find all linear factors. Finally we check the leading coefficient in the both sides.

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