The first ascent of size d or more in compositions

Charlotte Brennan, Arnold Knopfmacher

The first ascent of size d or more in compositions

Charlotte Brennan[1] and Arnold Knopfmacher[1 †]

[1] *The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg, South Africa.*

A composition of a positive integer n is a finite sequence of positive integers $a_1, a_2, \ldots, a_k$ such that $a_1 + a_2 + \cdots + a_k = n$. Let d be a fixed nonnegative integer. We say that we have an ascent of size d or more at position i, if $a_{i+1} \geq a_i + d$. We study the average position, initial height and end height of the first ascent of size d or more in compositions of n as $n \to \infty$.

Keywords: compositions, ascents, generating functions

---

1 Introduction

A composition of a positive integer n is a finite sequence of positive integers $a_1, a_2, \ldots, a_k$ such that $a_1 + a_2 + \cdots + a_k = n$. Compositions are basic combinatorial objects which arise in a number of statistical applications. A comprehensive treatment of compositional data in statistics can be found in Aitchison (1).

Let d be a fixed nonnegative integer. We say that we have an ascent of size d or more at position i if $a_{i+1} \geq a_i + d$. In (3) the present authors found the distribution of the number of ascents of size d or more in compositions of n. In this paper we study various statistics relating to the *first ascent* of size d or more in compositions of n. In particular we find the average initial height, end height and position of the first ascent of size greater than or equal to d.

For example, consider the following composition $2+3+1+2+1+4+1$ of $n = 15$. Then the first ascent of size greater than or equal to 2 has initial height 1, end height 4 and (initial) position 5. We assign the value zero to these parameters if the composition has no such first ascent. We also consider the size of the composition preceding the first ascent of size d or more. In our example this parameter equals $2+3+1+2+1 = 9$.

We show that as $n \to \infty$ the asymptotic mean value of the initial height μ_I, end height μ_E, position μ_P and size of composition preceding the first ascent μ_C all tend to certain constant depending on d. The explicit formulas for these constants which involve complicated sums of q-binomial coefficients are given in Theorems 1 to 4 in subsequent sections of the paper. In principle the same technique could also be used to compute higher moments.

For the introduction we will rather present a table of numerical values for the asymptotic statistics μ_E, μ_I, μ_P and μ_C, for $d = 0$ up to 10.
In addition the behaviour of these statistics as $d \to \infty$ is fairly simple. We have $\mu_I \sim 4/3$, $\mu_E \sim d + 7/3$, $\mu_P \sim 3 \times 2^{d-1}$ and $\mu_C \sim 6 \times 2^{d-1}$.

The case $d = 1$ is of special interest as it corresponds to the first occurrence of $a_{i+1} > a_i$ in the composition. An ocurrence of $a_{i+1} > a_i$ is also called a *rise* in the literature. The number of rises in compositions has been studied by Carlitz (4) and more recently by Chinn, Heubach and Grimaldi in (5). In particular, they found generating functions and formulas for the total number of rises in all compositions of n. However, statistics relating to the first rise have not previously been considered.

---

†This material is based upon work supported by the National Research Foundation under grant number 2053740

d	μ_E	μ_I	μ_P	μ_C
0	2.2661	1.2661	1.3842	2.5024
1	3.168	1.168	2.4627	3.7574
2	4.2553	1.2553	5.2203	8.1853
3	5.3051	1.3051	11.108	18.911
4	6.3241	1.3241	23.054	41.783
5	7.3305	1.3305	47.027	88.723
6	8.3325	1.3325	95.013	183.69
7	9.3331	1.3331	191.01	374.68
8	10.333	1.3333	383.	757.67
9	11.333	1.3333	767.	1524.7
10	12.333	1.3333	1535.	3059.7

Fig. 1: Statistics for the first ascent of size d or more.

2 Compositions of n having no ascents of size d or more

Our approach to the statistics mentioned in Section 1 requires that we first derive the generating function of compositions that have *no* ascents of size d or more, which we shall denote by $F_d(z)$.

We use the "adding-the-slice" technique which was originally used by Flajolet and Prodinger in (7) and more recently, for example, by Knopfmacher and Prodinger in (8).

Let j be the value of the last component of the composition $a_1+a_2+\cdots+a_k=n$, i.e. $a_k=j$. We proceed from a composition with k parts to a composition with $k+1$ parts. We denote by $f_k(z,u)$ the generating function where z counts the size of the composition and u the value of j . In moving from a composition with k parts to a composition of $k+1$ parts, where j is coded by u^j, we have no ascent of size d or more, provided that the new last letter has any value less than $j+d$. This gives the following rule for adding a new part ("slice") to the end of a composition:

$$u^j \longrightarrow zu+(zu)^2+(zu)^3+\cdots+(zu)^{j+d-1}=zu\frac{1-(zu)^{j+d-1}}{1-zu}\,.$$

This implies that

$$f_{k+1}(z,u)=\frac{zu}{1-zu}f_k(z,1)-\frac{(zu)^d}{1-zu}f_k(z,zu)\,. \tag{2.1}$$

Now we set

$$F_d(z,u):=\sum_{k\geq 1}f_k(z,u).$$

Summing (2.1) over $k\geq 1$:

$$F_d(z,u)-f_1(z,u)=\frac{zu}{1-zu}F_d(z,1)-\frac{(zu)^d}{1-zu}F_d(z,zu)\,,$$

so that

$$F_d(z,u)=\frac{zu}{1-zu}F_d(z,1)+\frac{zu}{1-zu}-\frac{(zu)^d}{1-zu}F_d(z,zu)\,, \tag{2.2}$$

since

$$f_1(z,u)=zu+(zu)^2+(zu)^3+\cdots=\frac{zu}{1-zu}\,.$$

Now we iterate the functional equation (2.2).

$$\begin{aligned}
F_d(z,u)&=\frac{zu}{1-zu}F_d(z,1)+\frac{zu}{1-zu}-\frac{(zu)^d}{1-zu}\times\\
&\qquad\times\left\{\frac{z^2u}{1-z^2u}F_d(z,1)+\frac{z^2u}{1-z^2u}-\frac{(z^2u)^d}{1-z^2u}F_d(z,z^2u)\right\}\\
&=\left[\frac{zu}{1-zu}-\frac{(zu)^dz^2u}{(1-zu)(1-z^2u)}+\frac{(zu)^d(z^2u)^dz^3u}{(1-zu)(1-z^2u)(1-z^3u)}\right][F_d(z,1)+1]\\
&\quad-\frac{(zu)^d(z^2u)^d(z^3u)^d}{(1-zu)(1-z^2u)(1-z^3u)}F_d(z,z^3u)\,.
\end{aligned}$$

We continue the iterations and then substitute $u = 1$ to obtain

$$F_d(z,1) = \sum_{i\geq 1} \frac{(-1)^{i-1}\, z^{d\binom{i}{2}}\, z^i}{(1-z)(1-z^2)\cdots(1-z^i)}[F_d(z,1)+1]\,.$$

Hence, if we add the term 1 for the empty composition we get the following formula for the generating function of compositions with no ascents of size d or more,

$$F_d(z) := 1 + F_d(z,1) = \frac{1}{1-\tau_d(z)}\,, \tag{2.3}$$

where

$$\tau_d(z) := \sum_{i\geq 1} \frac{(-1)^{i-1} z^i\, z^{d\binom{i}{2}}}{(1-z)(1-z^2)\cdots(1-z^i)}. \tag{2.4}$$

Remarks If $d = 1$, then by definition, $F_1(z)$ is the generating function for partitions of n, given by $F_1(z) = \prod_{i\geq 1} \frac{1}{1-z^i}$ and if $d = 0$, then $F_0(z)$ is the generating function for partitions of n with distinct parts, given by $F_0(z) = \prod_{i\geq 1}(1+z^i)$. These product expressions for $d = 0$ and $d = 1$ can also be derived from (2.3) by means of Euler's partition identities (see e.g. (2)).

As we shall see, all our statistics of interest in the following sections can be expressed in terms of the special value $F_d(1/2)$ as well as the values of the partial derivatives $\frac{\partial}{\partial u} F_d(1/2, 1/2)$ and $\frac{\partial}{\partial z} F_d(1/2, 1/2)$. In general, it does not seem possible to express these particular constants in a simpler form.

3 End height of the first ascent of size d or more in compositions

In this section we are interested in finding the average end height of the first ascent of size d or more. To do this we use the following decomposition for the set of all compositions, where we use h as the value of the starting letter of the first ascent.

$$\begin{aligned}\{all\ compositions\} = \bigcup_{h\geq 1} &\{composition\ with\ no\ ascent\ of\ size\ d\ or\ more\ ending\ with\ h\}\\ &\{integer \geq h+d\}\{any\ composition\}\\ &\cup \{composition\ with\ no\ ascent\ of\ size\ d\ or\ more\}.\end{aligned} \tag{3.1}$$

In order to translate this decomposition into the appropriate generating function for the end height, we need to find the generating function $F_d(z,u)$ for a word with no ascent of size d or more, but this time ending with the letter h. We return to the step just before the iteration in the "adding the slice" technique in Section 2.

$$\begin{aligned}F_d(z,u) = &\left[\frac{zu}{1-zu} - \frac{z^2u(zu)^d}{(1-zu)(1-z^2u)} + \frac{z^3u(zu)^d(z^2u)^d}{(1-zu)(1-z^2u)(1-z^3u)}\right] F_d(z)\\ &- \frac{(zu)^d(z^2u)^d(z^3u)^d}{(1-zu)(1-z^2u)(1-z^3u)}F_d(z,z^3u).\end{aligned}$$

We keep iterating but, this time, we do not set $u = 1$ as we are interested in the value of the last part. This gives

$$F_d(z,u) = \sum_{i\geq 1} \frac{(-1)^{i-1} z^{i+d\binom{i}{2}} u^{(i-1)d+1}}{(1-zu)(1-z^2u)\cdots(1-z^iu)}\ F_d(z)\,. \tag{3.2}$$

Here we are interested in the coefficient of u^h in $F_d(z,u)$. For this it is convenient to use the q-series notation:

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})\,.$$

Now

$$[u^h]F_d(z,u) = [u^h]F_d(z)\sum_{i\geq 1}\frac{(-1)^{i-1}z^{i+d\binom{i}{2}}u^{(i-1)d+1}}{(zu;z)_i}$$
$$= F_d(z)\sum_{i\geq 1}(-1)^{i-1}z^{i+d\binom{i}{2}}\,[u^{h-(i-1)d-1}]\frac{1}{(zu;z)_i}.$$

To extract the coefficient of $\frac{1}{(zu;z)_i}$ we use the q-binomial theorem (see e.g. (2)),

$$\sum_{k=0}^{\infty}\begin{bmatrix}N+k-1\\k\end{bmatrix}_q x^k = \frac{1}{(x;q)_N} = \frac{1}{(1-x)\cdots(1-xq^{N-1})}, \qquad |x|<1$$

where the q-binomial coefficient is given by

$$\begin{bmatrix}n\\k\end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

Hence, replacing q by z, N by i, and x by zu:

$$[u^{h-(i-1)d-1}]\frac{1}{(zu;z)_i} = [u^{h-(i-1)d-1}]\sum_{k=0}^{\infty}\begin{bmatrix}i+k-1\\k\end{bmatrix}_z (zu)^k$$
$$= \begin{bmatrix}i+h-(i-1)d-2\\h-(i-1)d-1\end{bmatrix}_z z^{h-(i-1)d-1}.$$

So

$$[u^h]F_d(z,u) = F_d(z)\sum_{i\geq 1}(-1)^{i-1}z^{i+d\binom{i}{2}}\begin{bmatrix}i+h-(i-1)d-2\\h-(i-1)d-1\end{bmatrix}_z z^{h-(i-1)d-1},$$

or

$$F_d(z,u) = F_d(z)\sum_{h\geq 1}\sum_{i\geq 1}(-1)^{i-1}z^{i+d\binom{i}{2}}\begin{bmatrix}i+h-(i-1)d-2\\h-(i-1)d-1\end{bmatrix}_z z^{h-(i-1)d-1}\,u^h. \tag{3.3}$$

Putting all this together into our decomposition (3.1), we can now derive the probability generating function $F_{end}(z,u)$, where z labels the value of n, and u marks the end height of the first ascent of size d or more.

$$F_{end}(z,u) = \sum_{h\geq 1}\left\{F_d(z)\sum_{i\geq 1}(-1)^{i-1}z^{i+d\binom{i}{2}}\begin{bmatrix}i+h-(i-1)d-2\\h-(i-1)d-1\end{bmatrix}_z z^{h-(i-1)d-1}\times\right.$$
$$\left.\times\sum_{j\geq h+d}u^j\,z^j.\,\frac{1-z}{1-2z}\right\} + F_d(z)$$
$$= \frac{F_d(z)(1-z)}{1-2z}\sum_{h\geq 1}\left\{\sum_{i\geq 1}(-1)^{i-1}z^{i+d\binom{i}{2}}\begin{bmatrix}i+h-(i-1)d-2\\h-(i-1)d-1\end{bmatrix}_z\times\right.$$
$$\left.\times\quad z^{h-(i-1)d-1}\frac{(uz)^{h+d}}{1-uz}\right\} + F_d(z)\,.$$

For the mean end height of the first ascent we need to compute $\frac{\partial}{\partial u}F_{end}(z,1)$.

$$\frac{\partial}{\partial u}F_{end}(z,1)$$
$$= \frac{F_d(z)}{1-2z}\sum_{h\geq 1}\left\{\sum_{i\geq 1}(-1)^{i-1}z^{i+d\binom{i}{2}}\begin{bmatrix}i+h-(i-1)d-2\\h-(i-1)d-1\end{bmatrix}_z z^{h-(i-1)d-1}\times\right.$$
$$\left.\times\quad z^{h+d}\,\frac{(1-z)(h+d)+z}{1-z}\right\}$$
$$= \frac{(1-z)d+z}{(1-z)(1-2z)}\,z^d\,F_d(z,z) + \frac{z}{1-2z}\,z^d\frac{\partial}{\partial u}\,F_d(z,u)\bigg|_{u=z}.$$

Since this function has a dominant simple pole at $z = 1/2$ we may apply singularity analysis (see (6)) to deduce that

$$[z^n]\frac{\partial}{\partial u}F_{end}(z,1) \sim (d+1)\,2^{n-d}\,F_d\left(\frac{1}{2},\frac{1}{2}\right) + 2^{n-d-1}\frac{\partial}{\partial u}\,F_d\left(\frac{1}{2},\frac{1}{2}\right). \tag{3.4}$$

To evaluate $F_d\left(\frac{1}{2},\frac{1}{2}\right)$ we use (2.2) with $u = 1$ to get

$$F_d(z,1) = \frac{z}{1-z}F_d(z,1) + \frac{z}{1-z} - \frac{z^d}{1-z}F_d(z,z)\,.$$

This, combined with (2.3), gives

$$F_d(z,z) = \frac{(2z-1)F_d(z)+1-z}{z^d}\,. \tag{3.5}$$

Therefore

$$F_d\left(\frac{1}{2},\frac{1}{2}\right) = 2^{d-1}\,.$$

Thus after dividing (3.3) by 2^{n-1}, the total number of compositions of n, we obtain

Theorem 1 *The mean end height of the first ascent of size* d *or more in compositions of* n *tends to*

$$\mu_E := d+1+2^{-d}\frac{\partial}{\partial u}\,F_d\left(\frac{1}{2},\frac{1}{2}\right) \quad \text{as} \quad n\to\infty.$$

4 Initial height of the first ascent of size d or more in compositions

In this section we are interested in finding the initial height of the first ascent of size d or more. We make use of the decomposition from the previous section that we used for the end height. The generating function is very similar; we just shift u which will now mark the initial height, to the first term. This time we want the initial height to have a value of h. Hence

$$\begin{aligned} F_{initial}(z,u) &= \sum_{h\geq 1}\left\{F_d(z)\sum_{i\geq 1}(-1)^{i-1}u^h\, z^{i+d\binom{i}{2}}\begin{bmatrix} i+h-(i-1)d-2 \\ h-(i-1)d-1\end{bmatrix}_z z^{h-(i-1)d-1}\,\times\right.\\ &\quad\left.\times\sum_{j\geq h+d} z^j\cdot\frac{1-z}{1-2z}\right\} + F_d(z)\\ &= \frac{F_d(z)}{1-2z}\sum_{h\geq 1}\left\{\sum_{i\geq 1}(-1)^{i-1}u^h z^{i+d\binom{i}{2}}\begin{bmatrix} i+h-(i-1)d-2 \\ h-(i-1)d-1\end{bmatrix}_z z^{h-(i-1)d-1}\, z^{h+d}\right\} + F_d(z). \end{aligned}$$

For the mean initial height we must compute $\frac{\partial}{\partial u}F_{initial}(z,1)$. Now

$$\begin{aligned} &\frac{\partial}{\partial u}F_{initial}(z,1)\\ &= \frac{F_d(z)}{1-2z}\sum_{h\geq 1}\left\{\sum_{i\geq 1}(-1)^{i-1}\,h\, z^{i+d\binom{i}{2}}\begin{bmatrix} i+h-(i-1)d-2 \\ h-(i-1)d-1\end{bmatrix}_z z^{h-(i-1)d-1}\, z^{h+d}\right\}\\ &= \frac{z}{1-2z}\,z^d\frac{\partial}{\partial u}F_d(z,z)\,. \end{aligned}$$

We apply singularity analysis to the dominant pole at $z = 1/2$ to deduce that

$$[z^n]\frac{\partial}{\partial u}F_{initial}(z,1) \sim 2^{n-d-1}\frac{\partial}{\partial u}F_d\left(\frac{1}{2},\frac{1}{2}\right)\,.$$

Again after dividing by 2^{n-1} we obtain

Theorem 2 *The mean initial height of the first ascent of size d or more in compositions of n tends to*

$$\mu_I := 2^{-d} \frac{\partial}{\partial u} F_d\left(\frac{1}{2}, \frac{1}{2}\right), \text{ as } n \to \infty.$$

The *length* of the first ascent of size d or more is defined to be the corresponding end height minus the initial height. From the above results we have

Corollary 1 *The mean length of the first ascent of size d or more in compositions of n tends to $d+1$ as $n \to \infty$.*

5 Position of the first ascent of size d or more in compositions

We return to the enumeration of compositions with no ascents of size d or more in Section 2, but now introduce a new parameter namely v, which will count the number of parts. The new "adding-a-slice" rule is

$$u^j \longrightarrow zuv + (zu)^2 v + (zu)^3 v + \cdots + (zu)^{j+d-1} v = zuv \frac{1-(zu)^{j+d-1}}{1-zu}.$$

This implies that

$$f_{k+1}(z,u,v) = \frac{zuv}{1-zu} f_k(z,1,v) - \frac{(zu)^d v}{1-zu} f_k(z,zu,v). \tag{5.1}$$

As before we set

$$F_d(z,u,v) := \sum_{k \ge 1} f_k(z,u,v).$$

Summing (5.1) over $k \ge 1$:

$$F_d(z,u,v) = \frac{zuv}{1-zu} F_d(z,1,v) + \frac{zuv}{1-zu} - \frac{(zu)^d v}{1-zu} F_d(z,zu,v), \tag{5.2}$$

since

$$f_1(z,u,v) = zuv + (zu)^2 v + (zu)^3 v + \cdots = \frac{zuv}{1-zu}.$$

We iterate as in Section 2 and then set $u=1$ to find that the bivariate generating function for compositions with no ascents of size d or more, where v marks the number of parts is

$$G_d(z,v) := 1 + F_d(z,1,v) = \frac{1}{1-\tau_d(z,v)}, \tag{5.3}$$

where

$$\tau_d(z,v) := \sum_{i \ge 1} \frac{(-1)^{i-1} z^i \, z^{d\binom{i}{2}} \, v^i}{(1-z)(1-z^2)\cdots(1-z^i)}. \tag{5.4}$$

We also need the corresponding generating function where the last part of the composition is h. Hence if we do not put $u=1$ after iterating, we obtain

$$F_d(z,u,v) = \sum_{i \ge 1} \frac{(-1)^{i-1} z^{i+d\binom{i}{2}} u^{(i-1)d+1} \, v^i}{(1-zu)(1-z^2u)\cdots(1-z^iu)} \, G_d(z,v). \tag{5.5}$$

We are interested in the coefficient of u^h in $F_d(z,u,v)$. Using results from Section 3, we have

$$F_d(z,u,v) = G_d(z,v) \sum_{h \ge 1} \sum_{i \ge 1} (-1)^{i-1} z^{i+d\binom{i}{2}} \begin{bmatrix} i+h-(i-1)d-2 \\ h-(i-1)d-1 \end{bmatrix}_z z^{h-(i-1)d-1} \, u^h \, v^i. \tag{5.6}$$

Putting all these results together into our decomposition (3.1), we find the generating function for the (initial) position of the first ascent is

$$F_{position}(z,v) = \sum_{h\geq 1}\left\{G_d(z,v)\sum_{i\geq 1}(-1)^{i-1}\, z^{i+d\binom{i}{2}} \begin{bmatrix} i+h-(i-1)d-2 \\ h-(i-1)d-1 \end{bmatrix}_z (z)^{h-(i-1)d-1}\, v^i \times \right.$$

$$\left. \times \sum_{j\geq h+d} z^j \cdot \frac{1-z}{1-2z}\right\} + F_d(z)$$

$$= \frac{G_d(z,v)}{1-2z}\sum_{h\geq 1}\left\{\sum_{i\geq 1}(-1)^{i-1}\, z^{i+d\binom{i}{2}} \begin{bmatrix} i+h-(i-1)d-2 \\ h-(i-1)d-1 \end{bmatrix}_z z^{h-(i-1)d-1}\, z^{h+d}\, v^i\right\} + F_d(z)\,.$$

For the mean position we need to compute $\frac{\partial}{\partial v}F_{position}(z,v)\Big|_{v=1}$.

$$\frac{\partial}{\partial v}F_{position}(z,v)\Big|_{v=1}$$
$$= \frac{G_d(z,v)}{1-2z}\sum_{h\geq 1}\left\{\sum_{i\geq 1}(-1)^{i-1}\, z^{i+d\binom{i}{2}} \begin{bmatrix} i+h-(i-1)d-2 \\ h-(i-1)d-1 \end{bmatrix}_z z^{h-(i-1)d-1}\, z^{h+d}\, i\right\}$$
$$= \frac{z^d}{1-2z}\frac{\partial}{\partial v}F_d(z,z,v)\Big|_{v=1}$$

which by singularity analysis at the dominant pole $z = 1/2$ gives

$$[z^n]\frac{\partial}{\partial v}F_{position}(z,1) \sim 2^{n-d}\frac{\partial}{\partial v}F_d(1/2,1/2,v)\Big|_{v=1}\,.$$

Hence the asymptotic mean position satisfies

$$\mu_P := 2^{1-d}\frac{\partial}{\partial v}F_d(1/2,1/2,v)\Big|_{v=1}\,. \tag{5.7}$$

To simplify this we use (5.2) with $z = 1/2$ and $u = 1$

$$F_d(1/2,1,v) = vF_d(1/2,1,v) + v - 2^{-d+1}vF_d(1/2,1/2,v)\,.$$

This leads to

$$F_d(1/2,1/2,v) = 2^{d-1}\frac{(v-1)F_d(1/2,v)+1}{v}\,.$$

Differentiating this with respect to v gives

$$\frac{\partial}{\partial v}F_d(1/2,1/2,v)\Big|_{v=1} = 2^{d-1}(F_d(1/2,1)-1)\,.$$

Hence we have shown

Theorem 3 *The mean position of the first ascent of size d or more in compositions of n tends to*

$$\mu_P := F_d(1/2) - 1 \quad \textit{as} \quad n\to\infty.$$

6 Size of the composition preceding the first ascent of size d or more

The last parameter we consider is the size of the initial composition preceding the first ascent of size d or more. For example, in the composition $3+1+1+2+5+1$, the first ascent of size two or more precedes the integer 5 and the size of the composition preceding this ascent is $3+1+1+2 = 7$.
We use the same decomposition as for the initial height where now v marks the initial position. We replace z by zv for each integer in the initial constrained composition that ends with h. This gives

$$
\begin{aligned}
F_{size}(z,v) &= \sum_{h\ge 1}\left\{F_d(zv)\sum_{i\ge 1}(-1)^{i-1}(zv)^{i+d\binom{i}{2}}\begin{bmatrix} i+h-(i-1)d-2 \\ h-(i-1)d-1\end{bmatrix}_{zv}(zv)^{h-(i-1)d-1}\times\right.\\
&\quad\left.\times\sum_{j\ge h+d} z^j\cdot\frac{1-z}{1-2z}\right\}+F_d(z)\\
&=\frac{F_d(zv)}{1-2z}\sum_{h\ge 1}\left\{\sum_{i\ge 1}(-1)^{i-1}(zv)^{i+d\binom{i}{2}}\begin{bmatrix} i+h-(i-1)d-2 \\ h-(i-1)d-1\end{bmatrix}_{zv}(zv)^{h-(i-1)d-1}z^{h+d}\right\}+F_d(z)\,.
\end{aligned}
$$

To obtain the mean size we need $\frac{\partial}{\partial v}F_{size}(z,1)$.

$$
\frac{\partial}{\partial v}F_{size}(z,1)=\frac{z^d}{1-2z}\frac{\partial}{\partial v}F_d(zv,z)\Big|_{v=1}=\frac{z^{d+1}}{1-2z}\frac{\partial}{\partial z}F_d(z,u)\Big|_{u=z}.
$$

Now for the coefficient of z^n, we once again apply singularity analysis to the dominant pole at $z=1/2$:

$$
[z^n]\frac{z^{d+1}}{1-2z}\frac{\partial}{\partial z}F_d(z,z)\sim 2^{n-d-1}\frac{\partial}{\partial z}F_d(1/2,1/2)\quad\text{as}\quad n\to\infty.
$$

After dividing by 2^{n-1} we find that the asymptotic mean composition size is

$$
\mu_C:=2^{-d}\frac{\partial}{\partial z}F_d(1/2,1/2)\quad\text{as}\quad n\to\infty.
$$

We can derive an alternative expression for μ_C in terms of $\frac{\partial}{\partial u}F_d(1/2,1/2)$ if we differentiate (2.2) with respect to z and put $u=1$,

$$
\begin{aligned}
\frac{\partial}{\partial z}F_d(z,1) &= -\frac{dF_d(z,z)z^{d-1}}{1-z}-\frac{F_d(z,z)z^d}{(1-z)^2}-\frac{\left(\frac{\partial}{\partial u}F_d(z,z)+\frac{\partial}{\partial z}F_d(z,z)\right)z^d}{1-z}+\frac{F_d(z,1)z}{(1-z)^2}\\
&\quad+\frac{\frac{\partial}{\partial z}F_d(z,1)z}{1-z}+\frac{z}{(1-z)^2}+\frac{F_d(z,1)}{1-z}+\frac{1}{1-z}.
\end{aligned}
$$

If we set $z=1/2$ and use $F_d(1/2,1/2)=2^{d-1}$,

$$
\begin{aligned}
&\frac{\partial}{\partial z}F_d\left(\frac{1}{2},1\right)=8\left(\frac{1}{2}\left(-\frac{d}{2}-\frac{1}{2}\right)+\frac{1}{2}F_d\left(\frac{1}{2},1\right)\right)\\
&+4\left(-2^{-d-1}\frac{\partial}{\partial u}F_d\left(\frac{1}{2},\frac{1}{2}\right)+2^{-d-1}\frac{\partial}{\partial z}F_d\left(\frac{1}{2},\frac{1}{2}\right)-2^{-d}\frac{\partial}{\partial z}F_d\left(\frac{1}{2},\frac{1}{2}\right)+\frac{1}{4}\frac{\partial}{\partial z}F_d\left(\frac{1}{2},1\right)+1\right).
\end{aligned}
$$

It follows that

$$
\frac{\partial}{\partial z}F_d\left(\frac{1}{2},\frac{1}{2}\right)=-2^d d+2^d+2^{d+1}F_d\left(\frac{1}{2},1\right)-\frac{\partial}{\partial u}F_d\left(\frac{1}{2},\frac{1}{2}\right).
$$

Hence we have shown

Theorem 4 *The mean size of the composition preceding the first ascent of size d or more in compositions of n tends to*

$$
\mu_C:=2^{-d}\frac{\partial}{\partial z}F_d(1/2,1/2)=2F_d(1/2)-1-d-2^{-d}\frac{\partial}{\partial u}F_d(1/2,1/2)\quad \textit{as}\quad n\to\infty.
$$

Remark We see from our results that

$$
\mu_E=\mu_I+d+1\quad\text{and}\quad\mu_C=2\mu_P-\mu_E+2. \tag{6.1}
$$

From Figure 1 it appears that as $d\to\infty$, the asymptotic mean value of the initial height of the first ascent of size d or more, μ_I, tends to $\frac{4}{3}$. In view of (6.1) we can determine the growth of all four statistics as $d\to\infty$ by considering just μ_P and μ_I.

As $d \to \infty$,

$$\tau_d(z) \sim \frac{z}{1-z} - \frac{z^{d+2}}{(1-z)(1-z^2)}$$

so that

$$F_d(z) \sim \frac{(1-z)(1-z^2)}{(1-2z)(1-z^2)+z^{d+2}}.$$

It follows that $\mu_P = F_d(1/2) - 1 \sim 3 \times 2^{d-1}$. Also, by (3.2) as $d \to \infty$,

$$F_d(z,u) \sim F_d(z)\frac{zu}{1-zu}$$

so that

$$\frac{\partial}{\partial u}F_d(z,u) \sim F_d(z)\frac{z}{(1-zu)^2}.$$

Thus

$$\mu_I = 2^{-d}\frac{\partial}{\partial u}F_d\left(\frac{1}{2},\frac{1}{2}\right) \sim 2^{-d} \times 3 \times 2^{d-1} \times \frac{1}{2} \times \frac{4^2}{3^2} = \frac{4}{3},$$

as was suggested by Figure 1.

References

[1] J. Aitchison. *The statistical analysis of of compositional data*. Chapman and Hall, 1986.

[2] G. E. Andrews, R. Askey, and R. Roy. *Special functions*. Cambridge University Press, 1999.

[3] C. Brennan and A. Knopfmacher. The distribution of ascents of size d or more in compositions. *Preprint*, 2005.

[4] L. Carlitz. Enumeration of compositions by rises, falls and levels. *Mathematische Nachrichten*, 77(3):254–264, 1977.

[5] P. Chinn, S. Heubach, and R. Grimaldi. Rises, levels, drops and"+" signs in compositions: extensions of a paper by Alladi and Hoggart. *The Fibonacci Quarterly*, 41(3):229–239, 2003.

[6] P. Flajolet and A. M. Odlyzko. Singularity analysis of generating functions. *SIAM Journal on Discrete Mathematics*, 3(2):216–240, 1990.

[7] P. Flajolet and H. Prodinger. Level number sequences for trees. *Discrete Mathematics*, 65:149–156, 1997.

[8] A. Knopfmacher and H. Prodinger. On Carlitz compositions. *European Journal of Combinatorics*, 19:579–589, 1998.