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Abstract. Let (W, S) be an arbitrary Coxeter system. For each sequence $\omega = (\omega_1, \omega_2, \ldots) \in S^*$ in the generators we define a partial order—called the ω-sorting order—on the set of group elements $W_\omega \subseteq W$ that occur as finite subwords of ω. We show that the ω-sorting order is a supersolvable join-distributive lattice and that it is strictly between the weak and strong Bruhat orders on the group. Moreover, the ω-sorting order is a "maximal lattice" in the sense that the addition of any collection of edges from the Bruhat order results in a nonlattice.

Along the way we define a class of structures called supersolvable antimatroids and we show that these are equivalent to the class of supersolvable join-distributive lattices.

Keywords: Coxeter group, join-distributive lattice, supersolvable lattice, antimatroid, convex geometry

Extended Abstract

Let (W, S) be an arbitrary Coxeter system and let $\omega = (\omega_1, \omega_2, \ldots) \in S^*$ be an arbitrary sequence in the generators, called the sorting sequence. We will identify a finite subword $\alpha \subseteq \omega$ with the pair $(\alpha, I(\alpha))$, where $I(\alpha) \subseteq I(\omega) = \{1, 2, \ldots\}$ is the index set encoding the positions of the letters. Given a word $\alpha = (\alpha_1, \ldots, \alpha_k) \in S^*$, let

$$\langle\alpha\rangle = \alpha_1 \cdots \alpha_k \in W$$

denote the corresponding group element. The subsets of the ground set $I(\omega)$ are ordered lexicographically: if A and B are subsets of $I(\omega)$ we say that $A \leq_{\mathsf{lex}} B$ if the minimum element of $(A \cup B) \setminus (A \cap B)$ is contained in A.

Definition 1 *We say that a finite subword $\alpha \subseteq \omega$ of the sorting sequence is ω-sorted if*

1. *α is a reduced word,*
2. *$I(\alpha) = \min_{\leq_{\mathsf{lex}}}\{I(\beta) \subseteq I(\omega) : \langle\beta\rangle = \langle\alpha\rangle\}$.*

That is, α is ω-sorted if it is the lexicographically-least reduced word for $\langle\alpha\rangle$ among subwords of ω.

Let $W_\omega \subseteq W$ denote the set of group elements that occur as subwords of the sorting sequence. Then ω induces a canonical reduced word for each element of W_ω—its ω-sorted word. This, in turn, induces a partial order on the set W_ω by subword containment of sorted words.

Definition 2 *Given group elements $u, w \in W_\omega$, we write $u \leq_\omega w$ if the index set of ω-$\mathsf{sort}(u)$ is contained in the index set of ω-$\mathsf{sort}(w)$. This is called the ω-sorting order on W_ω.*

The sorting orders are closely related to other important orders on the group.

Theorem 1 *Let $\leq_{\mathsf{R}}$ denote the right weak order and let $\leq_{\mathsf{B}}$ denote the Bruhat order on W. For all $u, w \in W_\omega$ we have*

$$u \leq_{\mathsf{R}} w \quad \Rightarrow \quad u \leq_\omega w \quad \Rightarrow \quad u \leq_{\mathsf{B}} w.$$

For example, let $W = \mathfrak{S}_4$ be the symmetric group of permutations of $\{1, 2, 3, 4\}$ with the generating set of adjacent transpositions

$$S = \{s_1 = (12), s_2 = (23), s_3 = (34)\}.$$

Figure 1 displays the Hasse diagrams of the weak order, $(s_1, s_2, s_3, s_2, s_1, s_2)$-sorting order and strong order on the symmetric group $\mathfrak{S}_4$—the weak order is indicated by the shaded edges; solid edges indicate the sorting order; solid and broken edges together give the Bruhat order.

It turns out that the collection of ω-sorted words has a remarkable structure. Given a ground set E and a collection of finite subsets $\mathscr{F} \subseteq 2^E$, the pair $(E, \mathscr{F})$ is called a set system. A set system $(E, \mathscr{F})$ is called an antimatroid (see (11)) if it satisfies

- For all nonempty $A \in \mathscr{F}$ there exists $x \in A$ such that $A \setminus \{x\} \in \mathscr{F}$,
- For all $A, B \in \mathscr{F}$ with $B \not\subseteq A$ there exists $x \in B \setminus A$ such that $A \cup \{x\} \in \mathscr{F}$.

Equivalently, $\mathscr{F}$ is the collection of open sets for a closure operator $\tau : 2^E \to 2^E$ that satisfies the anti-exchange property:

- If $x, y \notin \tau(A)$ then $x \in \tau(A \cup \{y\})$ implies $x \notin \tau(A \cup \{x\})$.

Such an operator τ models the notion of "convex hull", and so it is called a convex closure. Furthermore, we say that a lattice L is join-distributive if it satisfies:

- For each $x \in L$, the interval $[x, y]$, where y is the join of elements that cover x, is a boolean algebra.

Edelman (5) proved that a finite lattice is join-distributive if and only if it arises as the lattice of open sets of a convex closure. We will generalize Edelman's characterization to the case of supersolvable join-distributive lattices.

Definition 3 *Consider a set system $(E, \mathscr{F})$ on a totally ordered ground set $(E, \leq_E)$. We say that $(E, \mathscr{F})$ is a* supersolvable antimatroid *if it satisfies:*

- $\emptyset \in \mathscr{F}$.
- *For all $A, B \in \mathscr{F}$ with $B \not\subseteq A$ and $x = \min_{\leq_E} B \setminus A$ we have $A \cup \{x\} \in \mathscr{F}$.*

Theorem 2 *A (possibly infinite) lattice P is join-distributive and every interval in P is supersolvable if and only if P arises as the lattice of feasible sets of a supersolvable antimatroid.*

Our main result is the following.

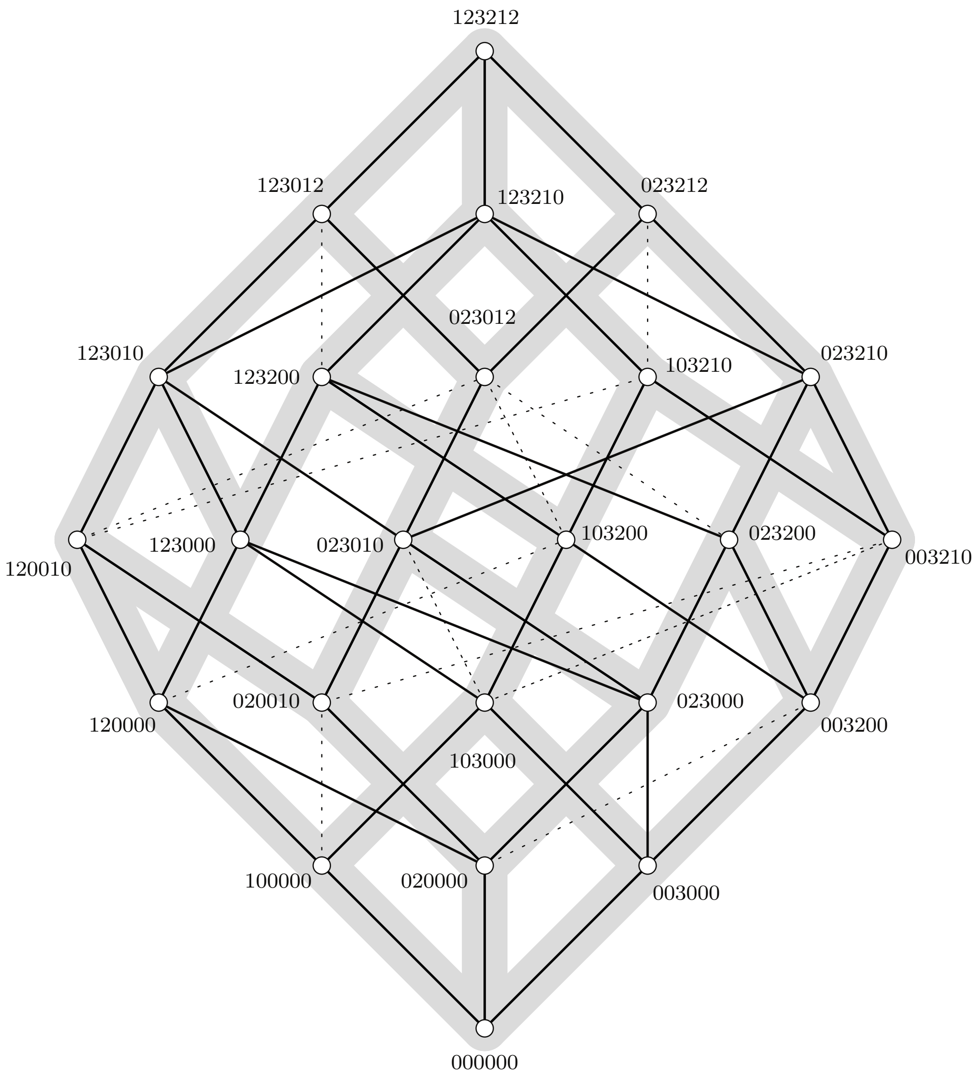

Fig. 1: The weak order, 123212-sorting order and strong order on $\mathfrak{S}_4$

Theorem 3 *Let (W,S) be an arbitrary Coxeter system and consider an arbitrary sequence $\omega \in S^*$. The collection of index sets of ω-sorted subwords*

$$\mathscr{F} = \{ I(\alpha) \subseteq I(\omega) : \alpha \text{ is } \omega\text{-sorted } \}$$

is a supersolvable antimatroid with respect to the natural order on the ground set $E = I(\omega)$.

Corollary 1 *The ω-sorting order is a join-distributive lattice in which every interval is supersolvable and it is graded by the usual Coxeter length function $\ell : W \to \mathbb{Z}$.*

Note that this holds even when infinitely many group elements occur as subwords of the sorting sequence. This is remarkable because the weak order on an infinite group is *not* a lattice. Indeed, we do not know of any other natural source of lattice structures on the elements of an infinite Coxeter group.

We also have

Corollary 2 *There exists a* reduced *sequence $\omega' \subseteq \omega$ (that is, every prefix of ω' is a reduced word) such that the ω'-sorting order coincides with the ω-sorting order.*

That is, we may assume that the sorting sequence is reduced. Finally, we have

Lemma 1 *If ω and ζ are sequences that differ by the exchange of adjacent commuting generators, then the ω-sorting order coincides with the ζ-sorting order.*

In summary, for each commutation class of reduced sequences we obtain a supersolvable join-distributive lattice that is strictly between the weak and Bruhat orders. This is particularly interesting in the case that ω represents a commutation class of reduced words for the longest element $w_\circ$ in a finite Coxeter group.

We end by noting an important special case. Let (W,S) be a Coxeter system with generators $S = \{s_1, \ldots, s_n\}$. Any word of the form $(s_{\sigma(1)}, \ldots, s_{\sigma(n)})$—where $\sigma \in \mathfrak{S}_n$ is a permutation—is called Coxeter word, and the corresponding element $\langle c \rangle \in W$ is a Coxeter element. We say that a cyclic sequence is any sequence of the form

$$c^\infty := ccc\ldots.$$

The case of c^∞-sorted words was first considered by Reading (see (13; 14)), and this is the main motivation behind our work. However, Reading did not consider the structure of the collection of sorted words nor did he consider the sorting order.

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