# The Sorting Order on a Coxeter Group

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# *The Sorting Order on a Coxeter Group*

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**Abstract.** Let $(W, S)$ be an arbitrary Coxeter system. For each sequence $\omega = (\omega_1, \omega_2, \ldots) \in S^*$ in the generators we define a partial order—called the $\omega$-sorting order—on the set of group elements $W_\omega \subseteq W$ that occur as finite subwords of $\omega$. We show that the $\omega$-sorting order is a supersolvable join-distributive lattice and that it is strictly between the weak and strong Bruhat orders on the group. Moreover, the $\omega$-sorting order is a "maximal lattice" in the sense that the addition of any collection of edges from the Bruhat order results in a nonlattice.

Along the way we define a class of structures called supersolvable antimatroids and we show that these are equivalent to the class of supersolvable join-distributive lattices.

**Keywords:** Coxeter group, join-distributive lattice, supersolvable lattice, antimatroid, convex geometry

## Extended Abstract

Let $(W, S)$ be an arbitrary Coxeter system and let $\omega = (\omega_1, \omega_2, \ldots) \in S^*$ be an arbitrary sequence in the generators, called the sorting sequence. We will identify a finite subword $\alpha \subseteq \omega$ with the pair $(\alpha, I(\alpha))$, where $I(\alpha) \subseteq I(\omega) = \{1, 2, \ldots\}$ is the index set encoding the positions of the letters. Given a word $\alpha = (\alpha_1, \ldots, \alpha_k) \in S^*$, let

$$\langle \alpha \rangle = \alpha_1 \cdots \alpha_k \in W$$

denote the corresponding group element. The subsets of the ground set $I(\omega)$ are ordered lexicographically: if $A$ and $B$ are subsets of $I(\omega)$ we say that $A \leq_{\mathsf{lex}} B$ if the minimum element of $(A \cup B) \setminus (A \cap B)$ is contained in $A$.

**Definition 1** *We say that a finite subword $\alpha \subseteq \omega$ of the sorting sequence is $\omega$-sorted if*

1. *$\alpha$ is a reduced word,*
2. *$I(\alpha) = \min_{\leq_{\mathsf{lex}}} \{I(\beta) \subseteq I(\omega) : \langle \beta \rangle = \langle \alpha \rangle\}$.*

*That is, $\alpha$ is $\omega$-sorted if it is the lexicographically-least reduced word for $\langle \alpha \rangle$ among subwords of $\omega$.*

Let $W_\omega \subseteq W$ denote the set of group elements that occur as subwords of the sorting sequence. Then $\omega$ induces a canonical reduced word for each element of $W_\omega$—its $\omega$-sorted word. This, in turn, induces a partial order on the set $W_\omega$ by subword containment of sorted words.

**Definition 2** *Given group elements $u, w \in W_\omega$, we write $u \leq_\omega w$ if the index set of $\omega$-$\mathsf{sort}(u)$ is contained in the index set of $\omega$-$\mathsf{sort}(w)$. This is called the $\omega$-sorting order on $W_\omega$.*

The sorting orders are closely related to other important orders on the group.

**Theorem 1** *Let $\leq_{\mathsf{R}}$ denote the right weak order and let $\leq_{\mathsf{B}}$ denote the Bruhat order on $W$. For all $u, w \in W_\omega$ we have*

$$u \leq_{\mathsf{R}} w \quad \Rightarrow \quad u \leq_\omega w \quad \Rightarrow \quad u \leq_{\mathsf{B}} w.$$

For example, let $W = \mathfrak{S}_4$ be the symmetric group of permutations of $\{1, 2, 3, 4\}$ with the generating set of adjacent transpositions

$$S = \{s_1 = (12), s_2 = (23), s_3 = (34)\}.$$

Figure 1 displays the Hasse diagrams of the weak order, $(s_1, s_2, s_3, s_2, s_1, s_2)$-sorting order and strong order on the symmetric group $\mathfrak{S}_4$—the weak order is indicated by the shaded edges; solid edges indicate the sorting order; solid and broken edges together give the Bruhat order.

It turns out that the collection of $\omega$-sorted words has a remarkable structure. Given a ground set $E$ and a collection of finite subsets $\mathscr{F} \subseteq 2^E$, the pair $(E, \mathscr{F})$ is called a set system. A set system $(E, \mathscr{F})$ is called an antimatroid (see (11)) if it satisfies

- For all nonempty $A \in \mathscr{F}$ there exists $x \in A$ such that $A \setminus \{x\} \in \mathscr{F}$,
- For all $A, B \in \mathscr{F}$ with $B \not\subseteq A$ there exists $x \in B \setminus A$ such that $A \cup \{x\} \in \mathscr{F}$.

Equivalently, $\mathscr{F}$ is the collection of open sets for a closure operator $\tau : 2^E \to 2^E$ that satisfies the anti-exchange property:

- If $x, y \notin \tau(A)$ then $x \in \tau(A \cup \{y\})$ implies $x \notin \tau(A \cup \{x\})$.

Such an operator $\tau$ models the notion of "convex hull", and so it is called a convex closure. Furthermore, we say that a lattice $L$ is join-distributive if it satisfies:

- For each $x \in L$, the interval $[x, y]$, where $y$ is the join of elements that cover $x$, is a boolean algebra.

Edelman (5) proved that a finite lattice is join-distributive if and only if it arises as the lattice of open sets of a convex closure. We will generalize Edelman's characterization to the case of supersolvable join-distributive lattices.

**Definition 3** *Consider a set system $(E, \mathscr{F})$ on a totally ordered ground set $(E, \leq_E)$. We say that $(E, \mathscr{F})$ is a* supersolvable antimatroid *if it satisfies:*

- $\emptyset \in \mathscr{F}$.
- *For all $A, B \in \mathscr{F}$ with $B \not\subseteq A$ and $x = \min_{\leq_E} B \setminus A$ we have $A \cup \{x\} \in \mathscr{F}$.*

**Theorem 2** *A (possibly infinite) lattice $P$ is join-distributive and every interval in $P$ is supersolvable if and only if $P$ arises as the lattice of feasible sets of a supersolvable antimatroid.*

Our main result is the following.

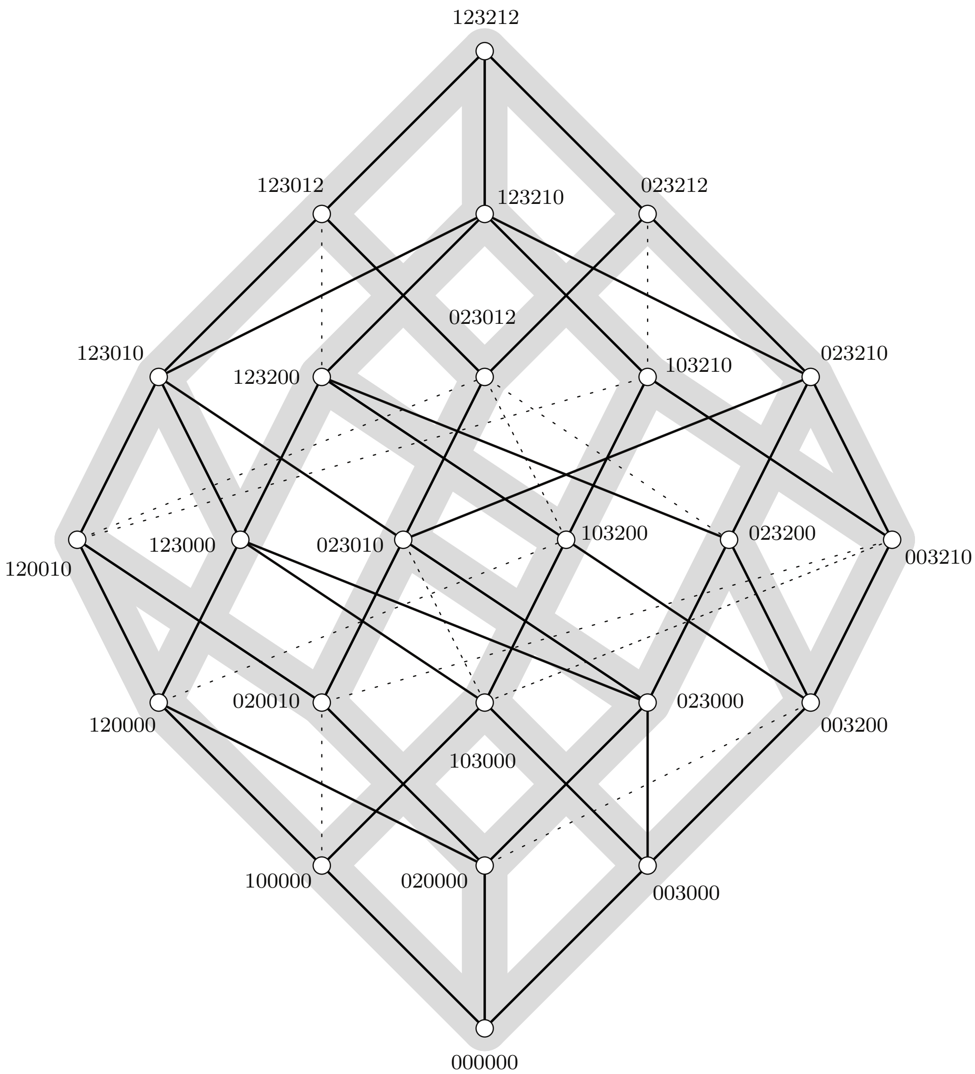


**Fig. 1:** The weak order, 123212-sorting order and strong order on $\mathfrak{S}_4$

**Theorem 3** *Let $(W,S)$ be an arbitrary Coxeter system and consider an arbitrary sequence $\omega \in S^*$. The collection of index sets of $\omega$-sorted subwords*

$$\mathscr{F} = \{ I(\alpha) \subseteq I(\omega) : \alpha \text{ is } \omega\text{-sorted } \}$$

*is a supersolvable antimatroid with respect to the natural order on the ground set $E = I(\omega)$.*

**Corollary 1** *The $\omega$-sorting order is a join-distributive lattice in which every interval is supersolvable and it is graded by the usual Coxeter length function $\ell : W \to \mathbb{Z}$.*

Note that this holds even when infinitely many group elements occur as subwords of the sorting sequence. This is remarkable because the weak order on an infinite group is *not* a lattice. Indeed, we do not know of any other natural source of lattice structures on the elements of an infinite Coxeter group.

We also have

**Corollary 2** *There exists a* reduced *sequence $\omega' \subseteq \omega$ (that is, every prefix of $\omega'$ is a reduced word) such that the $\omega'$-sorting order coincides with the $\omega$-sorting order.*

That is, we may assume that the sorting sequence is reduced. Finally, we have

**Lemma 1** *If $\omega$ and $\zeta$ are sequences that differ by the exchange of adjacent commuting generators, then the $\omega$-sorting order coincides with the $\zeta$-sorting order.*

In summary, for each commutation class of reduced sequences we obtain a supersolvable join-distributive lattice that is strictly between the weak and Bruhat orders. This is particularly interesting in the case that $\omega$ represents a commutation class of reduced words for the longest element $w_\circ$ in a finite Coxeter group.

We end by noting an important special case. Let $(W,S)$ be a Coxeter system with generators $S = \{s_1, \ldots, s_n\}$. Any word of the form $(s_{\sigma(1)}, \ldots, s_{\sigma(n)})$—where $\sigma \in \mathfrak{S}_n$ is a permutation—is called Coxeter word, and the corresponding element $\langle c \rangle \in W$ is a Coxeter element. We say that a cyclic sequence is any sequence of the form

$$c^\infty := ccc \ldots .$$

The case of $c^\infty$-sorted words was first considered by Reading (see (13; 14)), and this is the main motivation behind our work. However, Reading did not consider the structure of the collection of sorted words nor did he consider the sorting order.

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