Quivers and the Euclidean algebra (Extended abstract)

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Abstract. We show that the category of representations of the Euclidean group $E(2)$ is equivalent to the category of representations of the preprojective algebra of the quiver of type A_∞. Furthermore, we consider the moduli space of $E(2)$-modules along with a set of generators. We show that these moduli spaces are quiver varieties of the type considered by Nakajima. These identifications allow us to draw on known results about preprojective algebras and quiver varieties to prove various statements about representations of $E(2)$. In particular, we show that $E(2)$ has wild representation type but that if we impose certain combinatorial restrictions on the weight decompositions of a representation, we obtain only a finite number of indecomposable representations.

Résumé. Nous montrons que la catégorie des représentations du groupe d'Euclide $E(2)$ est équivalente à la catégorie des représentations de l'algèbre préprojective de type A_∞. De plus, nous considérons l'espace classifiant de modules de $E(2)$ avec un ensemble de générateurs. Nous montrons que ces espaces sont de variétés de carquois de Nakajima. Cette identification nous permet d'utiliser des résultats des algèbres préprojectives et des variétés de carquois pour prouver des affirmations sur des représentations de $E(2)$. En particulier, nous montrons que le type de répresentations de $E(2)$ est sauvage mais si nous imposons des restrictions aux poids d'une représentation, il y a seulement un nombre fini de représentations qui ne sont pas décomposables.

Keywords: Quiver, quiver variety, directed graph, Euclidean algebra, Euclidean group, preprojective algebra

1 Introduction

The Euclidean group $E(n) = \mathbb{R}^n \rtimes SO(n)$ is the group of isometries of n-dimensional Euclidean space. The study of these objects, at least in cases $n = 2, 3$, predates even the concept of a group. In this paper we will focus on the Euclidean group $E(2)$. Even in this case, much is still unknown about the representation theory.

All finite-dimensional irreducible unitary representations of $E(2)$ are one-dimensional. The infinite-dimensional unitary irreducible representations have received considerable attention (see [1, 2, 3]). There also exist finite-dimensional nonunitary indecomposable representations and much less is known about these. However, they play an important role in mathematical physics and the representation theory of the Poincaré group. The group $E(2)$ also appears in the Chern-Simons formulation of Einstein gravity in

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$2+1$ dimensions. In the case when the space-time has Euclidean signature and the cosmological constant vanishes, the phase space of gravity is the moduli space of flat $E(2)$-connections.

In the current paper, we relate the representation theory of the Euclidean group $E(2)$ to the representation theory of preprojective algebras of quivers of type A_∞. In fact, we show that the categories of representations of the two are equivalent. Furthermore, we consider the moduli space of representations of $E(2)$ along with a set of generators. We show that these moduli spaces are quiver varieties of the type considered by Nakajima in [12, 13]. These identifications allow us to draw on known results about preprojective algebras and quiver varieties to prove various statements about representations of $E(2)$. In particular, we show that the Lie algebra of $E(2)$ is of wild representation type but that if we impose certain restrictions on the weight decomposition of a representation, we obtain only a finite number of indecomposable representations.

2   The Euclidean algebra

Let $E(2) = \mathbb{R}^2 \rtimes SO(2)$ be the Euclidean group of motions in the plane and let $\mathfrak{e}(2)$ be the complexification of its Lie algebra. We call $\mathfrak{e}(2)$ the (three-dimensional) Euclidean algebra. It has basis $\{p_+, p_-, l\}$ and commutation relations

$$[p_+, p_-] = 0, \quad [l, p_\pm] = \pm p_\pm.$$

Since $SO(2)$ is compact, the category of finite-dimensional $E(2)$-modules is equivalent to the category of finite dimensional $\mathfrak{e}(2)$-modules in which l acts semisimply with integer eigenvalues. Will will use the term $\mathfrak{e}(2)$-module to refer only to such modules. For $k \in \mathbb{Z}$, we shall write V_k to indicate the eigenspace of l with eigenvalue k (the *k-weight space*). Thus, for an $\mathfrak{e}(2)$-module V, we have the weight space decomposition

$$V = \bigoplus_k V_k, \quad V_k = \{v \in V \mid l \cdot v = kv\}, \quad k \in \mathbb{Z}$$

and

$$p_+ V_k \subseteq V_{k+1}, \quad p_- \subseteq V_{k-1}.$$

We may form the tensor product of any representation V with the character χ_n for $n \in \mathbb{Z}$. Here χ_n is the one-dimensional module $\mathbb{C}$ on which $p_\pm$ act by zero and l acts by multiplication by n. Then a weight space V_k of weight k becomes a weight space $V_k \otimes \chi_n$ of weight $k + n$. In this way, we may "shift weights" as we please.

For $k \in \mathbb{Z}$, let $\mathbf{e}^k$ be the element of $(\mathbb{Z}_{\geq 0})^{\mathbb{Z}}$ with kth component equal to one and all others equal to zero. For an $\mathfrak{e}(2)$-module V we define

$$\mathbf{dim}\, V = \sum_{k \in \mathbb{Z}} (\dim V_k) \mathbf{e}^k.$$

3   Preprojective algebras

For $a, b \in \mathbb{Z}$ with $a \leq b$, let $Q_{a,b}$ be the quiver with vertex set $I = \{k \in \mathbb{Z} \mid a \leq k \leq b\}$ and arrows $H = \{h_i \mid a \leq i \leq b-1\}$ with $\mathrm{out}(h_i) = i$ and $\mathrm{in}(h_i) = i+1$. The quiver Q_∞ has vertex set $I = \mathbb{Z}$ and arrows $H = \{h_i \mid i \in \mathbb{Z}\}$ with $\mathrm{out}(h_i) = i$ and $\mathrm{in}(h_i) = i+1$.

Let $Q = (I, H, \text{out}, \text{in})$ be a quiver without loops and let $Q^* = (I, H^*, \text{out}^*, \text{in}^*)$ be the *double quiver* of Q. By definition,

$$H^* = \{h \mid h \in H\} \cup \{\bar{h} \mid h \in H\},$$
$$\text{out}^*(h) = \text{out}(h), \quad \text{in}^*(h) = \text{in}(h), \quad \text{out}^*(\bar{h}) = \text{in}(h), \quad \text{in}^*(\bar{h}) = \text{out}(h).$$

From now on, we will write in and out for in^* and out^* respectively. Since $\text{in}^*|_H = \text{in}$ and $\text{out}^*|_H = \text{out}$, this should cause no confusion.

For $i \in I$ let

$$r_i = \sum_{h \in H,\, out(h)=i} \bar{h}h - \sum_{h \in H,\, \text{in}(h)=i} h\bar{h}$$

be the *Gelfand-Ponomarev relation* associated to i. The *preprojective algebra* $P(Q)$ corresponding to Q is defined to be

$$P(Q) = \mathbb{C}Q^*/J$$

where J is the two-sided ideal generated by the relations r_i for $i \in I$.

Let $\mathcal{V}(I)$ denote the category of finite-dimensional I-graded vector spaces with morphisms being linear maps respecting the grading. A *representation* of the quiver Q^* is an element $\mathbf{V} \in \mathcal{V}(I)$ along with a linear map $x_h : V_{\text{out}(h)} \to V_{\text{in}(h)}$ for each $h \in H^*$. We let

$$\text{rep}(Q^*, V) = \bigoplus_{h \in H^*} \text{Hom}_{\mathbb{C}}(V_{\text{out}(h)}, V_{\text{in}(h)})$$

be the affine variety consisting of representations of Q^* with underlying vector space $\mathbf{V}$. A representation of a quiver can be naturally interpreted as a $\mathbb{C}Q^*$-module structure on $\mathbf{V}$. For a path $p = h_n h_{n-1} \ldots h_1$ in Q^*, we let

$$x_p = x_{h_n} x_{h_{n-1}} \cdots x_{h_1}.$$

We say a representation $x \in \text{rep}(Q^*, V)$ *satisfies the relation* $\sum_{j=1}^{k} a_j p_j$, if

$$\sum_{j=1}^{k} a_j x_{p_j} = 0.$$

If R is a set of relations, we denote by $\text{rep}(Q^*, R, \mathbf{V})$ the set of all representations in $\text{rep}(Q^*, \mathbf{V})$ satisfying all relations in R. This is a closed subvariety of $\text{rep}(Q^*, \mathbf{V})$. Every element of $\text{rep}(Q^*, J, \mathbf{V})$ can be naturally interpreted as a $P(Q)$-module structure on $\mathbf{V}$ and so we also write

$$\text{mod}(P(Q), \mathbf{V}) = \text{rep}(Q, J, \mathbf{V})$$

for the affine variety of $P(Q)$-modules with underlying vector space $\mathbf{V}$.

The algebraic group $G_{\mathbf{V}} = \prod_{i \in I} GL(V_i)$ acts on $\text{mod}(P(Q), \mathbf{V})$ by

$$g \cdot x = (g_i) \cdot (x_h) = (g_{\text{in}(h)} x_h g_{\text{out}(h)}^{-1})_h.$$

Two $P(Q)$-modules are isomorphic if and only if they lie in the same orbit. For a dimension vector $\mathbf{v} \in (Z_{\geq 0})^I$, let

$$\mathbf{V^v} = \bigoplus_{i \in I} \mathbb{C}^{\mathbf{v}_i}, \quad \mathrm{mod}(P(Q), \mathbf{v}) = \mathrm{mod}(P(Q), \mathbf{V^v}), \quad G_{\mathbf{v}} = G_{\mathbf{V^v}}.$$

Then we have that $\mathrm{mod}(P(Q), \mathbf{V}) \cong \mathrm{mod}(P(Q), \dim \mathbf{V})$ for all $\mathbf{V} \in \mathcal{V}(I)$. Therefore, we will blur the distinction between $\mathrm{mod}(P(Q), \mathbf{V})$ and $\mathrm{mod}(P(Q), \dim \mathbf{V})$.

We say an element $x \in \mathrm{mod}(P(Q), \mathbf{V})$ is *nilpotent* if there exists an $N \in \mathbb{Z}_{>0}$ such that for any path p of length greater than N, we have $x_p = 0$. Denote the closed subset of nilpotent elements of $\mathrm{mod}(P(Q), \mathbf{V})$ by $\Lambda_{\mathbf{V},Q}$ and let $\Lambda_{\mathbf{v},Q} = \Lambda_{\mathbf{V^v},Q}$. The varieties $\Lambda_{\mathbf{V},Q}$ are called *nilpotent varieties* or *Lusztig quiver varieties*. Lusztig [11] has shown that the $\Lambda_{\mathbf{V},Q}$ have pure dimesion $\dim(\mathrm{rep}(Q, \mathbf{V}))$ and that their irreducible components are in one-to-one correspondence with a basis of a certain weight space of the lower half of the enveloping algebra of $\mathfrak{g}$, the Kac-Moody algebra whose Dynkin graph is the underlying graph of Q.

Proposition 3.1 *For a quiver Q, the following are equivalent:*

1. *$P(Q)$ is finite-dimensional,*

2. *$\Lambda_{\mathbf{V},Q} = \mathrm{mod}(P(Q), \mathbf{V})$ for all $\mathbf{V} = \mathcal{V}(I)$,*

3. *Q is a Dynkin quiver (i.e. its underlying graph is of ADE type).*

Proof: The equivalence of (1) and (3) is well-known (see for example [14]). That (2) implies (3) was proven by Crawley-Boevey [4] and the converse was proven by Lusztig [11, 14.2]. □

Thus, for a Dynkin quiver Q, nilpotency holds automatically and $\Lambda_{\mathbf{V},Q}$ is just the variety of representations of the preprojective algebra $P(Q)$ with underlying vector space $\mathbf{V}$.

The representation type of the preprojective algebras is known.

Proposition 3.2 ([5, 9]) *Let Q be a finite quiver. Then the following hold:*

1. *$P(Q)$ is of finite representation type if and only if Q is of Dynkin type A_n, $n \leq 4$,*

2. *$P(Q)$ is of tame representation type if and only if Q is of Dynkin type A_5 or D_4.*

Thus $P(Q)$ is of wild representation type if Q is not of Dynkin type A_n, $n \leq 5$, or D_4.

In the sequel, we will refer to the preprojective algebra $P(Q_\infty)$. While Q_∞ is not a finite quiver, any finite-dimensional representation is supported on finitely many vertices and thus is a representation of a quiver of type A_n for sufficiently large n. Thus we deduce the following.

Corollary 3.3 *All finite-dimensional representations of Q_∞ are nilpotent and $P(Q_\infty)$ is of wild representation type.*

4 Representations of the Euclidean algebra and preprojective algebras

In this section we examine the close relationship between representations of the Euclidean algebra $\mathfrak{e}(2)$ and the preprojective algebras of type A_n. Let $\mathbf{Mod}\,\mathfrak{e}(2)$ be the category of $\mathfrak{e}(2)$-modules. For $a \leq b$, let $\mathbf{Mod}_{a,b}\,\mathfrak{e}(2)$ be the full subcategory consisting of representations V such that $V_k = 0$ for $k < a$ or $k > b$. Then $\mathbf{Mod}\,\mathfrak{e}(2)$ is the limit of the categories $\mathbf{Mod}_{a,b}\,\mathfrak{e}(2)$. For $\mathbf{v} \in (\mathbb{Z}_{\geq 0})^{\mathbb{Z}}$, we also define $\mathbf{Mod}_{a,b}^{\mathbf{v}}\,\mathfrak{e}(2)$ to be the full subcategory of $\mathbf{Mod}_{a,b}\,\mathfrak{e}(2)$ consisting of representations V such that $\dim V = \mathbf{v}$.

Let $\mathbf{Mod}\,P(Q)$ be the category of finite-dimensional $P(Q)$-modules and for $\mathbf{v} \in (\mathbb{Z}_{\geq 0})^I$, let $\mathbf{Mod}^{\mathbf{v}}\,P(Q)$ be the full subcategory consisting of modules of graded dimension $\mathbf{v}$.

Proposition 4.1 *We have the following equivalences of categories.*

1. $\mathbf{Mod}_{a,b}^{\mathbf{v}}\,\mathfrak{e}(2) \cong \mathbf{Mod}^{\mathbf{v}}\,P(Q_{a,b})$, $\mathbf{Mod}_{a,b}\,\mathfrak{e}(2) \cong \mathbf{Mod}\,P(Q_{a,b})$,

2. $\mathbf{Mod}^{\mathbf{v}}\,\mathfrak{e}(2) \cong \mathbf{Mod}^{\mathbf{v}}\,P(Q_\infty)$ $\mathbf{Mod}\,\mathfrak{e}(2) \cong \mathbf{Mod}\,P(Q_\infty)$.

Theorem 4.2 *The following statements hold.*

1. *The Euclidean algebra $\mathfrak{e}(2)$ has wild representation type, and*

2. *for $a, b \in \mathbb{Z}$ with $0 \leq b - a \leq 3$, there are a finite number isomorphism classes of indecomposable $\mathfrak{e}(2)$-modules V whose weights lie between a and b; that is, such that $V_k = 0$ for $k < a$ or $k > b$.*

Corollary 4.3 *Let A be a finite subset of $\mathbb{Z}$ with the property that A does not contain any five consecutive integers. Then there are a finite number of isomorphism classes of indecomposable $\mathfrak{e}(2)$-modules V with the property that $V_k = 0$ if $k \notin A$.*

Douglas [7] has shown that there are finitely many indecomposable $\mathfrak{e}(2)$-modules (up to isomorphism) of dimensions five and six. Corollary 4.3 can be used to simply such proofs by allowing one to restrict attention to modules with certain weight space dimensions. We also point out that the graphs appearing in [7] roughly correspond, under the equivalence of categories in Proposition 4.1, to the diagrams appearing in the enumeration of irreducible components of quiver varieties given in [8].

5 Nakajima quiver varieties

Let Q be the quiver Q_∞ or $Q_{a,b}$ for some $a \leq b$. For $\mathbf{V}, \mathbf{W} \in \mathcal{V}(I)$ define

$$L_Q(\mathbf{V}, \mathbf{W}) = \Lambda_{\mathbf{V},Q} \oplus \bigoplus_{i \in I} \mathrm{Hom}_{\mathbb{C}}(W_i, V_i).$$

We denote points of $L_Q(\mathbf{V}, \mathbf{W})$ by (x, s) where $x = (x_h)_{h \in H^*} \in \Lambda_{\mathbf{V},Q}$ and $s = (s_i)_{i \in I} \in \mathrm{Hom}_{\mathbb{C}}(W_i, V_i)$. We say an I-graded subspace $\mathbf{U}$ of $\mathbf{V}$ is *x-invariant* if $x_h(U_{\mathrm{out}(h)}) \subseteq U_{\mathrm{in}(h)}$ for all $h \in H^*$. We say a point $(x, s) \in L_Q(\mathbf{V}, \mathbf{W})$ is *stable* if the following property holds: If $\mathbf{U}$ is an I-graded x-invariant subspace of $\mathbf{V}$ containing $\mathrm{im}\,s$, then $\mathbf{U} = \mathbf{V}$. We denote by $L_Q(\mathbf{V}, \mathbf{W})^{\mathrm{st}}$ the set of stable points.

The group $G_{\mathbf{V}}$ acts on $L_Q(\mathbf{V}, \mathbf{W})$ by

$$g \cdot (x, s) = (g_i)_{i \in I} \cdot ((x_h)_{h \in H^*}, (s_i)_{i \in I}) = ((g_{\mathrm{in}(h)} x_h g_{\mathrm{out}(h)}^{-1})_{h \in H^*}, (g_i s_i)_{i \in I}).$$

The action of $G_{\mathbf{V}}$ preserves the stability condition and the stabilizer in $G_{\mathbf{V}}$ of a stable point is trivial. We form the quotient

$$\mathcal{L}_Q(\mathbf{V}, \mathbf{W}) = L_Q(\mathbf{V}, \mathbf{W})^{\mathrm{st}}/G_{\mathbf{V}}.$$

The $\mathcal{L}_Q(\mathbf{V}, \mathbf{W})$ are called *Nakajima quiver varieties*. For $\mathbf{v}, \mathbf{w} \in (\mathbb{Z}_{\geq 0})^I$, we set

$$L_Q(\mathbf{v}, \mathbf{w}) = L_Q(\mathbf{V}^{\mathbf{v}}, \mathbf{V}^{\mathbf{w}}), \quad L_Q(\mathbf{v}, \mathbf{w})^{\mathrm{st}} = L_Q(\mathbf{V}^{\mathbf{v}}, \mathbf{V}^{\mathbf{w}})^{\mathrm{st}}, \quad \mathcal{L}_Q(\mathbf{v}, \mathbf{w}) = \mathcal{L}_Q(\mathbf{V}^{\mathbf{v}}, \mathbf{V}^{\mathbf{w}}).$$

We then have

$$L_Q(\mathbf{V}, \mathbf{W}) \cong L_Q(\dim \mathbf{V}, \dim \mathbf{W}), \quad L_Q(\mathbf{V}, \mathbf{W})^{\mathrm{st}} \cong L_Q(\dim \mathbf{V}, \dim \mathbf{W})^{\mathrm{st}},$$
$$\mathcal{L}_Q(\mathbf{V}, \mathbf{W}) \cong \mathcal{L}_Q(\dim \mathbf{V}, \dim \mathbf{W}),$$

and so we often blur the distinction between these pairs of isomorphic varieties.

6   Moduli spaces of representations of the Euclidean algebra

Given that $\mathfrak{e}(2)$ has wild representation type, it is prudent to restrict one's attention to certain subclasses of modules and to attempt a classification of the modules belonging to these classes. One possible approach is to impose a restriction on the number of generators of a representation (see [6, 7]). In this section we will examine the relationship between moduli spaces of representations of the Euclidean algebra along with a set of generating vectors and Nakajima quiver varieties.

Let V be a finite-dimensional $\mathfrak{e}(2)$-module. For $u_1, u_2, \ldots, u_n \in V$, we denote by $\langle u_1, \ldots, u_n \rangle$ the submodule of V generated by $\{u_1, \ldots, u_n\}$. It is defined to be the smallest submodule of V containing all the u_i. A element $u \in \mathbf{V}$ is called a *weight vector* if it lies in some weight space V_k of V. For a weight vector u, we let $\mathrm{wt}\, u = k$ where $u \in V_k$. We say that $\{u_1, \ldots, u_n\}$ is a set of *generators* of V if each u_i is a weight vector and $\langle u_1, \ldots, u_n \rangle = V$. For $\mathbf{v} \in (\mathbb{Z}_{\geq 0})^{\mathbb{Z}}$, we let $|\mathbf{v}| = \sum_{k \in \mathbb{Z}} v_k$.

Definition 6.1 *For $\mathbf{v}, \mathbf{w} \in (\mathbb{Z}_{\geq 0})^{\mathbb{Z}}$, let $E(\mathbf{v}, \mathbf{w})$ be the set of all*

$$(V, (u_k^j)_{k \in \mathbb{Z},\, 1 \leq j \leq \mathbf{w}_k})$$

where V is a finite-dimensional $\mathfrak{e}(2)$-module with $\dim V = \mathbf{v}$ and $(u_k^j)_{k \in \mathbb{Z},\, 1 \leq j \leq \mathbf{w}_k}$ is a set of generators of V such that $\mathrm{wt}\, u_k^j = \mathbf{e}^k$. We say that two elements $(V, (u_k^j))$ and $(\tilde{V}, (\tilde{u}_k^j))$ of $E(\mathbf{v}, \mathbf{w})$ are equivalent *if there exists a $\mathfrak{e}(2)$-module isomorphism $\phi : V \to \tilde{V}$ such that $\phi(u_k^j) = \tilde{u}_k^j$. We denote the set of equivalence classes by $\mathcal{E}(\mathbf{v}, \mathbf{w})$.*

Theorem 6.2 *There is a natural one-to-one correspondence between $\mathcal{E}(\mathbf{v}, \mathbf{w})$ and $\mathcal{L}_{Q^\infty}(\mathbf{v}, \mathbf{w})$.*

A *partition* is a sequence of non-increasing natural numbers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$. The corresponding *Young diagram* is a collection of rows of square boxes which are left justified, with λ_i boxes in the ith row, $1 \leq i \leq l$. We will identify a partition and its Young diagram and we denote by $\mathcal{Y}$ the set of all partitions (or Young diagrams). If b is a box in a Young diagram λ, we write $x \in \lambda$ and we denote the box in the ith column and jth row of λ by $x_{i,j}$ (if such a box exists). The *residue* of $x_{i,j} \in \lambda$ is defined to be $\mathrm{res}\, x_{i,j} = i - j$. For $\lambda \in \mathcal{Y}$ and $a \in \mathbb{Z}$, define $\mathbf{v}^{\lambda,a} \in (\mathbb{Z}_{\geq 0})^{\mathbb{Z}}$ by setting $\mathbf{v}_{i+a}^{\lambda,a}$ to be the number of boxes in λ of residue i.

Proposition 6.3 *For* $\lambda \in \mathcal{Y}$, *there exists a unique* $\mathfrak{e}(2)$-*module* V *(up to isomorphism) with a single generator of weight* $a \in \mathbb{Z}$ *and* $\mathbf{dim}\,V = \mathbf{v}^{\lambda,a}$. *It is given by*

$$V = \mathrm{Span}_{\mathbb{C}}\{x \mid x \in \lambda\}$$
$$l(x_{i,j}) = \mathrm{res}\,x_{i,j} = a + i - j$$
$$p_+(x_{i,j}) = x_{i+1,j}$$
$$p_-(x_{i,j}) = x_{i,j+1},$$

where we set $x_{i,j} = 0$ *if there is no box of* λ *in the* ith *column and* jth *row.*

For $\mathbf{v} \in (\mathbb{Z}_{\geq 0})^{\mathbb{Z}}$ *such that* $\mathbf{v} \neq \mathbf{v}^{\lambda,a}$ *for all* $\lambda \in \mathcal{Y}$ *and* $a \in \mathbb{Z}$, *there are no* $\mathfrak{e}(2)$-*modules* V *with a single generator and* $\mathbf{dim}\,V = \mathbf{v}^{\lambda,a}$

Thus $\mathfrak{e}(2)$-modules with a single generator of a fixed weight are determined completely by the dimensions of their weight spaces. This was proven directly by Gruber and Henneberger in [10]. However, our knowledge of the precise relationship between quivers and the Euclidean algebra allows us to use known results about quivers and quiver varieties to simplify such proofs.

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