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Invariant and coinvariant spaces for the algebra of symmetric polynomials in non-commuting variables

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Abstract. We analyze the structure of the algebra $\mathbb{K}\langle \mathbf{x}\rangle^{\mathfrak{S}_n}$ of symmetric polynomials in non-commuting variables in so far as it relates to $\mathbb{K}[\mathbf{x}]^{\mathfrak{S}_n}$, its commutative counterpart. Using the "place-action" of the symmetric group, we are able to realize the latter as the invariant polynomials inside the former. We discover a tensor product decomposition of $\mathbb{K}\langle \mathbf{x}\rangle^{\mathfrak{S}_n}$ analogous to the classical theorems of Chevalley, Shephard-Todd on finite reflection groups. In the case $|\mathbf{x}| = \infty$, our techniques simplify to a form readily generalized to many other familiar pairs of combinatorial Hopf algebras.

Résumé. Nous analysons la structure de l'algèbre $\mathbb{K}\langle \mathbf{x}\rangle^{\mathfrak{S}_n}$ des polynômes symétriques en des variables non-commutatives pour obtenir des analogues des résultats classiques concernant la structure de l'anneau $\mathbb{K}[\mathbf{x}]^{\mathfrak{S}_n}$ des polynômes symétriques en des variables commutatives. Plus précisément, au moyen de "l'action par positions", on réalise $\mathbb{K}[\mathbf{x}]^{\mathfrak{S}_n}$ comme sous-module de $\mathbb{K}\langle \mathbf{x}\rangle^{\mathfrak{S}_n}$. On découvre alors une nouvelle décomposition de $\mathbb{K}\langle \mathbf{x}\rangle^{\mathfrak{S}_n}$ comme produit tensorial, obtenant ainsi un analogues des théorèmes classiques de Chevalley et Shephard-Todd. Dans le cas $|\mathbf{x}| = \infty$, nos techniques se simplifient en une forme aisément généralisables à beaucoup d'autres paires d'algèbres de Hopf familières.

Keywords: Chevalley theorem, symmetric group, noncommutative symmetric polynomials, set partitions

1 Introduction

One of the more striking results of the invariant theory of reflection groups is certainly the following: if W is a finite group of $n \times n$ matrices, then there is a graded W-module decomposition of the polynomial ring $S = \mathbb{K}[\mathbf{x}]$, in variables $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$, as a tensor product[(i)]

$$S \simeq S_W \otimes S^W , \tag{1}$$

if and only if W is a group generated by (pseudo) reflections. As usual, S affords the natural W-module structure obtained by considering it as the symmetric space on the defining vector space X^* for W, e.g.,

[(i)] We assume throughout that $\mathbb{K}$ is a field containing $\mathbb{Q}$.

$w \cdot f(\mathbf{x}) = f(w \cdot \mathbf{x})$. It is customary to denote by S^W the ring of W-invariant polynomials for this action. To finish parsing (1), recall that S_W stands for the **coinvariant space**, i.e., the W-module defined as

$$S_W := S/\langle S_+^W \rangle, \tag{2}$$

the quotient of S by the ideal generated by constant-term free W-invariant polynomials. We give S, S^W, and S_W a grading by polynomial degree in $\mathbf{x}$ (the latter being well-defined because $\langle S_+^W \rangle$ is a homogeneous ideal). The motivation behind the quotient in (2) is to eliminate redundant copies of irreducible W-modules inside S. Indeed, if $\mathcal{V}$ is such a module and $f(\mathbf{x})$ is any W-invariant polynomial with no constant term, then $\mathcal{V} f(\mathbf{x})$ is an isomorphic copy of $\mathcal{V}$ living within $\langle S_+^W \rangle$. As a result, the coinvariant space S_W is the interesting part of the story.

The context for the present paper is the algebra $T = \mathbb{K}\langle \mathbf{x} \rangle$ of noncommutative polynomials, with W-module structure on T obtained by considering it as the tensor space on the defining space X^* for W. In the special case when W is the symmetric group $\mathfrak{S}_n$, we elucidate a relationship between the space S^W and the subalgebra T^W of W-invariants in T. The subalgebra T^W was first studied in [14, 5] with the aim of obtaining noncommutative analogs of classical results concerning symmetric function theory. Recent work in [12, 3] has extended a large part of the story surrounding (1) to this noncommutative context. In particular, there is an explicit $\mathfrak{S}_n$-module decomposition of the form $T \simeq T_{\mathfrak{S}_n} \otimes T^{\mathfrak{S}_n}$, cf. [3, Theorem 8.7].

By contrast, our work proceeds in a somewhat complementary direction. We consider $\mathcal{N} = T^{\mathfrak{S}_n}$ as a tower of $\mathfrak{S}_d$-modules under the "place-action" and realize $S^{\mathfrak{S}_n}$ inside $\mathcal{N}$ as a subspace Λ of invariants for this action. This leads to a decomposition of $\mathcal{N}$ analogous to (1). More explicitly, our main result is as follows.

Theorem 1 *There is an explicitly constructed subspace $\mathcal{C}$ of $\mathcal{N}$ so that $\mathcal{C}$ and the place-action invariants Λ exhibit a graded vector space isomorphism*

$$\mathcal{N} \simeq \mathcal{C} \otimes \Lambda. \tag{3}$$

As an immediate corollary we derive the Hilbert series formula

$$\mathrm{Hilb}_t(\mathcal{C}) = \mathrm{Hilb}_t(\mathcal{N}) \prod_{i=1}^{n} (1 - t^i). \tag{4}$$

Here, as usual, the **Hilbert series** of a graded space $\mathcal{V} = \bigoplus_{d \geq 0} \mathcal{V}_d$ is the formal power series defined as

$$\mathrm{Hilb}_t(\mathcal{V}) = \sum_{d \geq 0} \dim \mathcal{V}_d \, t^d,$$

where $\mathcal{V}_d$ is the **homogeneous degree d component** of $\mathcal{V}$. The fact that (4) expands as a series in $\mathbb{N}[\![t]\!]$ is not at all obvious, as one may check that the Hilbert series of $\mathcal{N}$ is

$$\mathrm{Hilb}_t(\mathcal{N}) = 1 + \sum_{k=1}^{n} \frac{t^k}{(1-t)(1-2\,t)\cdots(1-k\,t)} \tag{5}$$

(taking $n = |\mathbf{x}|$). We underline that the harder part of our work lies in working out the case $n < \infty$. This is accomplished in Section 6. If we restrict ourselves to the case $n = \infty$, both $\mathcal{N}$ and Λ become Hopf algebras and things are much simpler. Our results are then consequences of a general theorem of Blattner, Cohen and Montgomery. As we will see in Section 5, stronger results hold in this simpler context. For example, (4) may be refined to a statement about "shape" enumeration.

2 The algebra $S^{\mathfrak{S}}$ of symmetric polynomials

2.1 Vector space structure of $S^{\mathfrak{S}}$

We specialize our introductory discussion to the group $W = \mathfrak{S}_n$ of permutation matrices. The action on $S = \mathbb{K}[\mathbf{x}]$ is simply the **permutation action** $\sigma \cdot x_i = x_{\sigma(i)}$ and $S^{\mathfrak{S}_n}$ comprises the usual symmetric polynomials. We suppress n in the notation and denote the subring of symmetric polynomials by $S^{\mathfrak{S}}$. (Note that upon sending n to ∞, the elements of $S^{\mathfrak{S}}$ become formal series in $\mathbb{K}[\![\mathbf{x}]\!]$ of bounded degree; we still call them polynomials to affect a uniform discussion.) A monomial in S of degree d may be written as follows: given an r-subset $\mathbf{y} = \{y_1, y_2, \dots, y_r\}$ of $\mathbf{x}$ and a **composition** of d into r parts, $\boldsymbol{a} = (a_1, a_2, \dots, a_r)$ ($a_i > 0$), we write $\mathbf{y}^{\boldsymbol{a}}$ for $y_1^{a_1} y_2^{a_2} \cdots y_r^{a_r}$. We assume that the variables y_i are naturally ordered, so that whenever $y_i = x_j$ and $y_{i+1} = x_k$ we have $j < k$. Reordering the entries of a composition $\boldsymbol{a}$ in decreasing order results in a partition $\lambda(\boldsymbol{a})$ called the **shape** of $\boldsymbol{a}$. Summing over monomials $\mathbf{y}^{\boldsymbol{a}}$ with the same shape leads to the monomial symmetric polynomial

$$m_\mu = m_\mu(\mathbf{x}) := \sum_{\lambda(\boldsymbol{a})=\mu,\ \mathbf{y}\subseteq\mathbf{x}} \mathbf{y}^{\boldsymbol{a}}.$$

Letting $\mu = (\mu_1, \dots, \mu_r)$ run over all partitions of $d = |\mu| = \mu_1 + \cdots + \mu_r$ gives a basis for $S_d^{\mathfrak{S}}$. As usual, we set $m_0 := 1$ and agree that $m_\mu = 0$ if μ has too many parts (i.e., $n < r$).

2.2 Dimension enumeration

A fundamental result in the invariant theory of $\mathfrak{S}_n$ is that $S^{\mathfrak{S}}$ is generated by a family $\{f_k\}_{1\le k\le n}$ of algebraically independent symmetric polynomials, having respective degrees $\deg f_k = k$. (One may choose $\{m_k\}_{1\le k\le n}$ for such a family.) It follows immediately that the Hilbert series of $S^{\mathfrak{S}}$ is

$$\mathrm{Hilb}_t(S^{\mathfrak{S}}) = \prod_{i=1}^{n} \frac{1}{1-t^i}. \tag{6}$$

Recalling that the Hilbert series of S is $(1-t)^{-n}$, we see from (1) and (6) that the Hilbert series for the coinvariant space $S_{\mathfrak{S}}$ is the well-known t-analog of $n!$:

$$\prod_{i=1}^{n} \frac{1-t^i}{1-t} = \prod_{i=1}^{n} (1 + t + \cdots + t^{i-1}). \tag{7}$$

In particular, contrary to the situation in (4), the series $\mathrm{Hilb}_t(S)/\mathrm{Hilb}_t(S^{\mathfrak{S}})$ in $\mathbb{Z}[\![t]\!]$ is *obviously* positive.

2.3 Algebra and coalgebra structures of $S^{\mathfrak{S}}$

Given partitions μ and ν, there is an explicit formula for computing the product $m_\mu \cdot m_\nu$. In lieu of giving the formula, we refer the reader to [3, §4.1] and simply give an example:

$$m_{21} \cdot m_{11} = 3\, m_{2111} + 2\, m_{221} + 2\, m_{311} + m_{32}. \tag{8}$$

The extremal terms above are relevant to our coming discussion. Note that if $n < 4$, then the first term disappears. However, if n is sufficiently large then analogs of these terms always appear with positive integer coefficients for a given pair (μ, ν). If $\mu = (\mu_1, \ldots, \mu_r)$ and $\nu = (\nu_1, \ldots, \nu_s)$ with $r \leq s$, then the partition indexing the left-most term is denoted by $\mu \cup \nu$ and is given by sorting the list $(\mu_1, \ldots, \mu_r, \nu_1, \ldots, \nu_s)$ in increasing order; the right-most term is indexed by $\mu + \nu := (\mu_1 + \nu_1, \ldots, \mu_r + \nu_r, \nu_{r+1}, \ldots, \nu_s)$. Taking $\mu = 31$ and $\nu = 221$, we would have $\mu \cup \nu = 32211$ and $\mu + \nu = 531$.

The ring $S^{\mathfrak{S}}$ is also afforded a coalgebra structure with coproduct $\Delta : S^{\mathfrak{S}}_d \to \bigoplus_{k=0}^{d} S^{\mathfrak{S}}_k \otimes S^{\mathfrak{S}}_{d-k}$ and counit $\varepsilon : S^{\mathfrak{S}} \to \mathbb{K}$ given, respectively, by

$$\Delta(m_\mu) = \sum_{\theta \cup \nu = \mu} m_\theta \otimes m_\nu \quad \text{and} \quad \varepsilon(m_\mu) = \delta_{\mu,0}.$$

In the case $n = \infty$, Δ and ε are algebra maps, making $S^{\mathfrak{S}}$ a connected graded (by degree) Hopf algebra.

3 The algebra $\mathcal{N}$ of noncommutative symmetric polynomials

3.1 Vector space structure of $\mathcal{N}$

Suppose now that $\mathbf{x}$ denotes a set of non-commuting variables. The algebra $T = \mathbb{K}\langle \mathbf{x} \rangle$ of noncommutative polynomials is graded by degree. A degree d **noncommutative monomial** $\mathbf{z} \in T_d$ is simply a length-d "word":

$$\mathbf{z} = z_1 z_2 \cdots z_d, \quad \text{with each} \quad z_i \in \mathbf{x}.$$

In other terms, $\mathbf{z}$ is a function $\mathbf{z} : [d] \to \mathbf{x}$, with $[d]$ denoting the set $\{1, \ldots, d\}$. The permutation-action on $\mathbf{x}$ clearly extends to T, giving rise to the subspace $\mathcal{N} = T^{\mathfrak{S}}$ of noncommutative $\mathfrak{S}$-invariants. With the aim of describing a linear basis for the homogeneous component $\mathcal{N}_d$, we next introduce set partitions of $[d]$ and the type of a monomial $\mathbf{z} : [d] \to \mathbf{x}$. We write $\mathbf{A} \vdash [d]$ when $\mathbf{A} = \{A_1, \ldots, A_r\}$ is a **set partition** of $[d]$, i.e., $A_1 \cup \ldots \cup A_r = [d]$, with $A_i \neq \emptyset$ and $A_i \cap A_j = \emptyset$ whenever $i \neq j$. The **type** $\tau(\mathbf{z})$ of a degree d monomial $\mathbf{z} : [d] \to \mathbf{x}$ is the set partition

$$\tau(\mathbf{z}) := \{\mathbf{z}^{-1}(x) \mid x \in \mathbf{x}\} \setminus \{\emptyset\} \quad \text{of} \quad [d],$$

whose parts are the non-empty fibers of the function $\mathbf{z}$. For instance,

$$\tau(x_1 x_8 x_1 x_5 x_8) = \{\{1,3\},\{2,5\},\{4\}\}.$$

In the sequel, we lighten the heavy notation for set partitions, writing, e.g., $\{\{1,3\},\{2,5\},\{4\}\}$ as $13.25.4$. Clearly the type of a monomial is a finite set partition with at most n parts. Note that we may always order the parts in increasing order of their minimum elements. The **shape** $\lambda(\mathbf{A})$ of a set partition $\mathbf{A} = \{A_1, \ldots, A_r\}$ is the (integer) partition $\lambda(|A_1|, \ldots, |A_r|)$ obtained by sorting the part sizes of $\mathbf{A}$ in increasing order. Observing that the permutation-action is *type preserving*, we are led to consider the **monomial** linear basis for the space $\mathcal{N}_d$:

$$m_{\mathbf{A}} = m_{\mathbf{A}}(\mathbf{x}) := \sum_{\tau(\mathbf{z}) = \mathbf{A}} \mathbf{z}$$

For example, with $n = 2$, we have $m_{\emptyset} = 1$, $m_1 = x_1 + x_2$, $m_{12} = x_1^2 + x_2^2$, $m_{1.2} = x_1x_2 + x_2x_1$, $m_{123} = {x_1}^3 + {x_2}^3$, $m_{12.3} = {x_1}^2 x_2 + {x_2}^2 x_1$, $m_{13.2} = x_1x_2x_1 + x_2x_1x_2$, $m_{1.23} = x_1{x_2}^2 + x_2{x_1}^2$, $m_{1.2.3} = 0, \ldots$ (Note that we set $m_{\emptyset} := 1$, taking $\emptyset$ as the unique set partition of the empty set, and we agree that $m_{\mathbf{A}} = 0$ if $\mathbf{A}$ is a set partition with more than n parts.)

3.2 Dimension enumeration and shape grading

Above, we determined that $\dim \mathcal{N}_d$ is the number of set partitions of d into at most n parts. These are counted by the (length restricted) **Bell numbers** $B_d^{(n)}$. Then (5) follows from the fact that its right-hand side is the ordinary generating function for length restricted Bell numbers. See [9, §2]. We next highlight a finer enumeration, where we grade $\mathcal{N}$ by shape rather than degree.

For each partition μ, we may consider the submodule $\mathcal{N}_\mu$ spanned by those $m_{\mathbf{A}}$ for which $\lambda(\mathbf{A}) = \mu$. This results in a direct sum decomposition $\mathcal{N}_d = \bigoplus_{\mu \vdash d} \mathcal{N}_\mu$. A simple dimension description for $\mathcal{N}_d$ takes the form of a **shape Hilbert series** in the following manner. View commuting variables q_i as marking parts of size i and set $\boldsymbol{q}_\mu := q_{\mu_1} q_{\mu_2} \cdots q_{\mu_r}$. Then

$$\mathrm{Hilb}_{\boldsymbol{q}}(\mathcal{N}_d) = \sum_{\mu \vdash d} \dim \mathcal{N}_\mu \, \boldsymbol{q}_\mu, = \sum_{\mathbf{A} \vdash [d]} q_{\lambda(\mathbf{A})}. \tag{9}$$

Here, $\boldsymbol{q}_\mu$ is a marker for set partitions of shape $\lambda(\mathbf{A}) = \mu$ and the sum is over all partitions into at most n parts. Such a shape grading also makes sense for $S_d^{\mathfrak{S}}$. Summing over all $d \geq 0$ and all μ, we get

$$\mathrm{Hilb}_{\boldsymbol{q}}(S^{\mathfrak{S}}) = \sum_{\mu} \boldsymbol{q}_\mu = \prod_{i \geq 1}^{n} \frac{1}{1 - q_i}. \tag{10}$$

Using classical combinatorial arguments (cf. Chapter 2.3 of [2], Example 13), we see that the enumerator polynomials $\mathrm{Hilb}_{\boldsymbol{q}}(\mathcal{N}_d)$ are naturally collected in the **exponential generating function**

$$\sum_{d=0}^{\infty} \mathrm{Hilb}_{\boldsymbol{q}}(\mathcal{N}_d) \, \frac{t^d}{d!} = \sum_{m=0}^{n} \frac{1}{m!} \left(\sum_{k=1}^{\infty} q_k \frac{t^k}{k!} \right)^m. \tag{11}$$

For example, with $n = 3$, we have

$$\mathrm{Hilb}_{\boldsymbol{q}}(\mathcal{N}_6) = q_6 + 6\, q_5 q_1 + 15\, q_4 q_2 + 15\, q_4 q_1^2 + 10\, q_3^2 + 60\, q_3 q_2 q_1 + 15\, q_2{}^3,$$

thus $\dim \mathcal{N}_{222} = 15$ when $n \geq 3$. Evidently, the $\boldsymbol{q}$-polynomials $\mathrm{Hilb}_{\boldsymbol{q}}(\mathcal{N}_d)$ specialize to the length restricted Bell numbers $B_d^{(n)}$ when we set all q_k equal to 1.

In view of (10), (11), and Theorem 1, we are led to claim the following refinement of (4).

Corollary 2 *For $n = \infty$, the shape Hilbert series of the space $\mathcal{C}$ is given by the expression*

$$\mathrm{Hilb}_{\boldsymbol{q}}(\mathcal{C}) = \sum_{d \geq 0} d! \, \exp\left(\sum_{k=1}^{\infty} q_k \, \frac{t^k}{k!} \right) \bigg|_{t^d} \prod_{i \geq 1} (1 - q_i), \tag{12}$$

with $(-)|_{t^d}$ standing for the operation of taking the coefficient of t^d.

Thus we have the expansion

$$\begin{aligned} \mathrm{Hilb}_{\boldsymbol{q}}(\mathcal{C}) \quad = \quad & 1 + 2\, q_2 q_1 + \big(3\, q_3 q_1 + 2\, q_2{}^2 + 3\, q_2 q_1{}^2\big) \\ & + \big(4\, q_4 q_1 + 9\, q_3 q_2 + 6\, q_3 q_1{}^2 + 10\, q_2{}^2 q_1 + 4\, q_2 q_1{}^3\big) + \ldots \end{aligned}$$

Corollary 2 will follow immediately from the explicit description of $\mathcal{C}$ and the isomorphism $\mathcal{C} \otimes \Lambda \to \mathcal{N}$ in Section 5, which is not only degree preserving, but shape preserving as well.

3.3 Algebra and coalgebra structures of $\mathcal{N}$

Since the action of $\mathfrak{S}$ on T is multiplicative, it is straightforward to see that $\mathcal{N}$ is an subalgebra of T. The *multiplication rule* in $\mathcal{N}$, expressing a product $m_{\mathbf{A}} \cdot m_{\mathbf{B}}$ as a sum of basis vectors $\sum_{\mathbf{C}} m_{\mathbf{C}}$, is easy to describe. Since we make heavy use of the rule later, we develop it carefully here. We begin with an example (the digits corresponding to $\mathbf{B} = \mathbf{1.2}$ appear in bold):

$$\begin{aligned} m_{13.2} \cdot m_{\mathbf{1.2}} \quad = \quad & m_{13.2.\mathbf{4}.\mathbf{5}} + m_{13\mathbf{4}.2.\mathbf{5}} + m_{13\mathbf{5}.2.\mathbf{4}} \\ & + m_{13.2\mathbf{4}.\mathbf{5}} + m_{13.2\mathbf{5}.\mathbf{4}} + m_{13\mathbf{5}.2\mathbf{4}} + m_{13\mathbf{4}.2\mathbf{5}} \end{aligned} \tag{13}$$

Compare this to (8). Notice that the shapes indexing the first and last terms in (13) are the partitions $\lambda(13.2) \cup \lambda(1.2)$ and $\lambda(13.2) + \lambda(1.2)$. As was the case in $S^{\mathfrak{S}}$, one of these shapes, namely $\lambda(\mathbf{A}) + \lambda(\mathbf{B})$, will always appear in the product, while appearance of the shape $\lambda(\mathbf{A}) \cup \lambda(\mathbf{B})$ depends on the cardinality of $\mathbf{x}$.

Let us now describe the multiplication rule. Given any $D \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, we write D^{+k} for the set

$$D^{+k} := \{a + k \mid a \in D\}.$$

By extension, for any set partition $\mathbf{A} = \{A_1, \ldots, A_r\}$ we set $\mathbf{A}^{+k} := \{A_1^{+k}, A_2^{+k}, \ldots, A_r^{+k}\}$. These definitions allow for the introduction of a bilinear (non-commutative) operation denoted by "$\Cup$" on formal linear combinations of set partitions. Given partitions $\mathbf{A} = \{A_1, A_2, \ldots, A_r\}$ of $[c]$ and a partition $\mathbf{B} = \{B_1, B_2, \ldots, B_s\}$ of $[d]$, the summands of $\mathbf{A} \Cup \mathbf{B}$ are set partitions of $[c + d]$. The operation $\Cup$ is recursively defined by the rules:

(a) $\mathbf{A} \Cup \emptyset = \emptyset \Cup \mathbf{A} = \mathbf{A}$, with $\emptyset$ denoting the unique set partition of the empty set;

(b) $\mathbf{A} \Cup \mathbf{B} = \{A_1\} \cup (\mathbf{A}' \Cup \mathbf{B}^{+c}) + \sum_{i=1}^{s} \{A_1 \cup B_i^{+c}\} \cup (\mathbf{A}' \Cup (\mathbf{B} \setminus \{B_i\})^{+c})$,

with union $\cup$ extended bilinearly and $\mathbf{A}'$ denoting $\{A_2, \ldots, A_r\}$.

As shown in [3, Prop. 3.2], the multiplication rule for $m_{\mathbf{A}}$ and $m_{\mathbf{B}}$ in $\mathcal{N}$, is

$$m_{\mathbf{A}} \cdot m_{\mathbf{B}} = \sum_{\mathbf{C} \in \mathbf{A} \Cup \mathbf{B}} m_{\mathbf{C}}\,. \tag{14}$$

The subalgebra $\mathcal{N}$, like its commutative analog, is freely generated by certain monomial symmetric polynomials $\{m_{\mathbf{A}}\}_{\mathbf{A} \in \mathcal{A}}$, where $\mathcal{A}$ is some carefully chosen collection of set partitions. This is the main theorem of Wolf [14]. See also [3, §7]. We use two such collections later, our choice depending on whether or not $n < \infty$.

The operation $(–)^{+k}$ has a left inverse called the **standardization** operator and denoted by "$(–)^{\downarrow}$". It maps set partitions $\mathbf{A}$ of any cardinality-d subset $D \subseteq \mathbb{N}$ to set partitions of $[d]$, with $\mathbf{A}^{\downarrow}$ defined as the pullback of $\mathbf{A}$ along the unique increasing bijection from $[d]$ to D. For example, $(18.4)^{\downarrow} = 13.2$ and $(18.4.67)^{\downarrow} = 15.2.34$. The coproduct Δ and counit ε on $\mathcal{N}$ are given, respectively, by

$$\Delta(m_{\mathbf{A}}) = \sum_{\mathbf{B} \Cup \mathbf{C} = \mathbf{A}} m_{\mathbf{B}\downarrow} \otimes m_{\mathbf{C}\downarrow} \qquad \text{and} \qquad \varepsilon(m_{\mathbf{A}}) = \delta_{\mathbf{A},\emptyset},$$

where $\mathbf{B} \Cup \mathbf{C} = \mathbf{A}$ means that $\mathbf{B}$ and $\mathbf{C}$ form complementary subsets of $\mathbf{A}$. In the case $n = \infty$, the maps Δ and ε are algebra maps, making $\mathcal{N}$ a graded connected Hopf algebra.

4 The place-action of $\mathfrak{S}$ on $\mathcal{N}$

4.1 Swapping places in T_d and $\mathcal{N}_d$

On top of the permutation-action of the symmetric group $\mathfrak{S}_{\mathbf{x}}$ on T, we also consider the "place-action" of $\mathfrak{S}_d$ on the degree d homogeneous component T_d. Observe that the permutation-action of $\sigma \in \mathfrak{S}_{\mathbf{x}}$ on a monomial $\mathbf{z}$ corresponds to the functional composition

$$\sigma \circ \mathbf{z} : [d] \xrightarrow{\mathbf{z}} \mathbf{x} \xrightarrow{\sigma} \mathbf{x}.$$

By contrast, the **place-action** of $\rho \in \mathfrak{S}_d$ on $\mathbf{z}$ gives the monomial

$$\mathbf{z} \circ \rho : [d] \xrightarrow{\rho} [d] \xrightarrow{\mathbf{z}} \mathbf{x}$$

composing ρ with $\mathbf{z}$ on the right. In the linear extension of this action to all of T_d, it is easily seen that $\mathcal{N}_d$ (even each $\mathcal{N}_\mu$) is an invariant subspace of T_d. Indeed, for any set partition $\mathbf{A} = \{A_1, \ldots, A_r\} \vdash [d]$ and $\rho \in \mathfrak{S}_d$, one has (see [12, §2])

$$m_{\mathbf{A}} \cdot \rho = m_{\rho^{-1} \cdot \mathbf{A}} \,, \tag{15}$$

where as usual $\rho^{-1} \cdot \mathbf{A} := \{\rho^{-1}(A_1), \rho^{-1}(A_2), \ldots, \rho^{-1}(A_r)\}$.

4.2 The place-action structure of $\mathcal{N}$

Notice that the action in (15) is transitive on set partitions and is shape-preserving. It follows that a basis for the place-action invariants in $\mathcal{N}_d$ is indexed by partitions. For such a basis we choose the polynomials

$$\mathbf{m}_\mu := \frac{1}{(\dim \mathcal{N}_\mu)\, \mu^!} \sum_{\lambda(\mathbf{A}) = \mu} m_{\mathbf{A}}, \tag{16}$$

with $\mu^! = a_1! a_2! \cdots$ whenever $\mu = 1^{a_1} 2^{a_2} \cdots$. The normalizing coefficient will be explained in (19).

To simplify our discussion of the structure of $\mathcal{N}$ in this context, we will say that $\mathfrak{S}$ acts on $\mathcal{N}$ rather than being fastidious about underlying in each situation that individual $\mathcal{N}_d$'s are being acted upon on the right by the corresponding group $\mathfrak{S}_d$. We also denote the set $\mathcal{N}^{\mathfrak{S}}$ of **place-invariants** by Λ. To summarize,

$$\Lambda = \mathrm{span}\{\mathbf{m}_\mu : \mu \text{ a partition of } d,\ d \in \mathbb{N}\} \,. \tag{17}$$

The pair $(\mathcal{N}, \Lambda)$ begins to look like the pair $(S, S^{\mathfrak{S}})$ from the introduction. This was the observation that originally motivated our search for Theorem 1.

We next decompose $\mathcal{N}$ into irreducible place-action representations. Although this can be worked out for any value of n, the results are more elegant when we send n to infinity. Recall that the **Frobenius characteristic** of a $\mathfrak{S}_d$-module $\mathcal{V}$ is the symmetric function

$$\mathrm{Frob}(\mathcal{V}) = \sum_{\mu \vdash d} v_\mu \, s_\mu,$$

where s_μ is a Schur function—the character of "the" irreducible $\mathfrak{S}_d$ representation $\mathcal{V}_\mu$ indexed by μ—and v_μ is the multiplicity of $\mathcal{V}_\mu$ in $\mathcal{V}$. To reveal the $\mathfrak{S}_d$-module structure of $\mathcal{N}_\mu$ we may use (15) and standard techniques from the theory of combinatorial species, cf. [2]. The Frobenius characteristic of $\mathcal{N}_\mu$ is given by the following lemma.

Lemma 3 *For a partition $\mu = 1^{a_1} 2^{a_2} \cdots k^{a_k}$, having a_i parts of size i, we have*

$$\mathrm{Frob}(\mathcal{N}_\mu) = h_{d_1}[h_1]\, h_{d_2}[h_2] \cdots h_{d_k}[h_k], \tag{18}$$

with $f[g]$ denoting plethysm of f and g, and h_i denoting the i^{th} homogeneous symmetric function.

Recall that the **plethysm** $f[g]$ of two symmetric functions is obtained by linear and multiplicative extension of the rule $p_k[p_\ell] := p_{k\,\ell}$, where the p_k's denote the usual power sum symmetric functions (see [10, I.8] for notations and more details). For instance, one finds that $h_3[h_2] = s_6 + s_{42} + s_{222}$. That is, $\mathcal{N}_{222}$ decomposes into 3 irreducible components, with the trivial representation s_6 coming from $\mathbf{m}_{222}$ inside Λ.

4.3 Λ meets $S^{\mathfrak{S}}$

We begin by explaining the choice of coefficient in (16). From [12, Thm. 2.1], one learns that the restriction to $\mathcal{N}$ of the **abelianization** map **ab** : $T \to S$ (the map making the variables commute) satisfies:

(a) **ab**$(\mathcal{N}) = S^{\mathfrak{S}}$, and

(b) **ab**$(m_{\mathbf{A}})$ is a multiple of $m_{\lambda(\mathbf{A})}$ depending only on $\mu = \lambda(\mathbf{A})$, more precisely

$$\mathbf{ab}(\mathbf{m}_\mu) = m_\mu. \tag{19}$$

Formula (19) suggests that a natural right-inverse to **ab**(–) is given by

$$\iota : S^{\mathfrak{S}} \hookrightarrow \mathcal{N}, \quad \text{with} \quad \iota(m_\mu) := \mathbf{m}_\mu. \tag{20}$$

The fact that the image of $S^{\mathfrak{S}}$ in $\mathcal{N}$ is exactly the subspace Λ affords us a quick proof of Theorem 1 in the case $n = \infty$. The isomorphism we construct for $n < \infty$ still uses the map ι, but in a less essential way.

5 The coinvariant space of $\mathcal{N}$ (Case: $n = \infty$)

5.1 Proof of main result

Suppose $n = \infty$. Combining results of [3] and a theorem of Blattner, Cohen, and Montgomery [6], we may immediately deduce the existence of a subspace $\mathcal{C}$ of $\mathcal{N}$ together with a vector space isomorphism $\mathcal{N} \simeq \mathcal{C} \otimes \Lambda$. Indeed, from Propositions 4.3 and 4.5 of [3], we get that the map ι is a **coalgebra splitting** of **ab** : $\mathcal{N} \to S^{\mathfrak{S}} \to 0$, i.e.,

$$\mathbf{ab} \circ \iota = \mathrm{id} \quad \text{and} \quad \Delta_{\mathcal{N}} \circ \iota = (\iota \otimes \iota) \circ \Delta_{S^{\mathfrak{S}}}.$$

Moreover **ab** is a morphism of Hopf algebras. In this context, Theorem 4.14 of [6] suggests that we let $\mathcal{C}$ be the **left Hopf kernel** of the Hopf map **ab**,

$$\mathcal{C} = \{h \in \mathcal{N} : (\mathrm{id} \otimes \mathbf{ab}) \circ \Delta(h) = h \otimes 1\}.$$

This theorem gives an algebra isomorphism between $\mathcal{N}$ and the *crossed product* $\mathcal{C} \#_\sigma S^{\mathfrak{S}}$. In fact, since $\Delta_{\mathcal{N}}$ is cocommutative, it is an isomorphism of Hopf algebras. We refer the interested reader to [6, §4] for the technical details. We mention only that: (i) the space $\mathcal{C}$ is actually a Hopf subalgebra of $\mathcal{N}$ by construction; (ii) the crossed product $\mathcal{C} \#_\sigma S^{\mathfrak{S}}$ is a certain algebra structure built on the tensor product $\mathcal{C} \otimes S^{\mathfrak{S}}$ using a cocycle $\sigma : S^{\mathfrak{S}} \times S^{\mathfrak{S}} \to \mathcal{C}$; and (iii) the isomorphism amounts to a cocyle twisting of simple multiplication: $\mathcal{C} \otimes S^{\mathfrak{S}} \mapsto \mathcal{C} \cdot \Lambda$. This completes the proof of Theorem 1. Moreover, since all spaces and morphisms are graded by degree, the Hilbert series for $\mathcal{C}$ is the quotient of that for $\mathcal{N}$ by that for Λ. This demonstrates (4).

5.2 Atomic set partitions.

Recall the result of Wolf that $\mathcal{N}$ is a polynomial algebra, i.e., $\mathcal{N}$ is freely generated by some collection of polynomials. We announce our first choice for this collection now, following the terminology of [4]. Let Π denote the set of all set partitions (of $[d]$, $\forall\, d \geq 0$). We introduce the **atomic set partitions** $\dot{\Pi}$. A set partition $\mathbf{A} = \{A_1, \ldots, A_r\}$ of $[d]$ is atomic if there does not exist a pair (s, c) $(1 \leq s < r, 1 \leq c < d)$ such that $\{A_1, \ldots, A_s\}$ is a set partition of $[c]$. Conversely, $\mathbf{A}$ is not atomic if there are set partitions $\mathbf{B}$ of $[d']$ and $\mathbf{C}$ of $[d'']$ splitting $\mathbf{A}$ in two: $\mathbf{A} = \mathbf{B} \cup \mathbf{C}^{+d'}$. We write $\mathbf{A} = \mathbf{B}|\mathbf{C}$ in this situation. A **maximal splitting** $\mathbf{A} = \mathbf{A}'|\mathbf{A}''|\cdots|\mathbf{A}^{(r)}$ of $\mathbf{A}$ is one where each $\mathbf{A}^{(i)}$ is atomic. For example, the partition $17.235.4.68$ is atomic, while $12.346.57.8$ is not. The maximal splitting of the latter would be $12|124.35|1$, but we abuse notation and write $12|346.57|8$ to improve legibility.

It is proven in [4] that $\mathcal{N}$ is freely generated by the atomic polynomials. To get a better sense of the structure, let us order Π by giving $\dot{\Pi}$ a total order "$\prec$" and then extending lexicographically. Given two atomic set partitions $\mathbf{A}$ and $\mathbf{B}$, we demand that $\mathbf{A} \prec \mathbf{B}$ if $\mathbf{A} \vdash [c]$ and $\mathbf{B} \vdash [d]$ with $c < d$. In case $\mathbf{A}, \mathbf{B}$ are partitions of the same set $[d]$, then any ordering will do for the current purpose… one interesting choice is to order $\mathbf{A}$ and $\mathbf{B}$ by ordering lexicographically their associated **rhyme scheme words.**[(ii)] Our convention for writing set partitions provides a bijection between set partitions and this special class of words, sending $\mathbf{A} = \{A_1, A_2, \ldots, A_r\} \in \Pi_d$ to $w(\mathbf{A}) = w_1 w_2 \cdots w_d$ defined by $w_i := k$ if and only if $i \in A_k$. For example, $w(13.2) = 121$ and $w(17.235.4.68) = 12232414$. Using this ordering on $\dot{\Pi}$, we have the following chain within the set partitions of shape 3221:

$$1|23|45|678 \prec 13.2|456|78 \prec 13.24|578.6 \prec 14.23|578.6 \prec 17.235.4.68 \prec 17.236.4.58.$$

In fact, $1|23|45|678$ is the unique minimal element of $\Pi_{(3221)}$.

Define the **leading term** of a sum $\sum_{\mathbf{C}} \alpha_{\mathbf{C}}\, m_{\mathbf{C}}$ to be the monomial $m_{\mathbf{C}_0}$ such that $\mathbf{C}_0$ is lexicographically least among all $\mathbf{C}$ with $\alpha_{\mathbf{C}} \neq 0$. Combined with (14), our choice for $\prec$ makes it clear that the leading term of $m_{\mathbf{A}} \cdot m_{\mathbf{B}}$ is $m_{\mathbf{A}|\mathbf{B}}$. That is, multiplication in $\mathcal{N}$ is *shape-filtered*. Since the left Hopf kernel $\mathcal{C}$ is a subalgebra, it is shape-filtered as well. Finally, the isomorphism $\mathcal{C} \otimes \Lambda \to \mathcal{N}$ respects the shape structures on either side. This completes the proof of Corollary 2.

It is proven in [8] that $\mathcal{N}$ is not only freely generated by the *atomic polynomials* $\left\{m_{\mathbf{A}} | \mathbf{A} \in \dot{\Pi}\right\}$, but co-freely generated by them as well. By a classic theorem of Milnor and Moore [11], this means that $\mathcal{N}$ is isomorphic to the universal enveloping algebra $\mathfrak{U}(\mathfrak{L}(\dot{\Pi}))$ of the free Lie algebra $\mathfrak{L}(\dot{\Pi})$ on the set $\dot{\Pi}$. This description will be useful in the next subsection. Let us finish this section with a few final remarks on atomic set partitions. First, note that set partitions with one part are trivially atomic. The set of these is denoted by $\dot{\Pi}_\flat$. They are analogs of the generators m_k for the algebra $S^{\mathfrak{S}}$. The remaining atomic set partitions

$$\dot{\Pi}_\sharp := \Big\{ \{A_1, \ldots, A_r\} \in \dot{\Pi} : r > 1 \Big\}$$

are more interesting. They index a large portion of the generators for $\mathcal{C}$. They are also the subject of an open question formulated at the end of Section 5.3.

(ii) Quoting Bill Blewett from [13, A000110], "a rhyme scheme is a string of letters (eg, $abba$) such that the leftmost letter is always a and no letter may be greater than one more than the greatest letter to its left. Thus aac is not valid since c is more than one greater than a. For example, $[\#\Pi_3 = 5]$ because there are 5 rhyme schemes on 3 letters: aaa, aab, aba, abb, abc."

5.3 Explicit description of the Hopf algebra structure of $\mathcal{C}$

It is not too hard to find elements in the left Hopf kernel of the abelianization map **ab**. Consider the following simple calculation. The sum of monomials $\tilde{m}_{13.2} := m_{13.2} - m_{12.3}$ is primitive. Indeed,

$$\begin{aligned}\Delta(\tilde{m}_{13.2}) &= 1 \otimes m_{13.2} + m_{12} \otimes m_1 + m_1 \otimes m_{12} + m_{13.2} \otimes 1 \\ &\quad - 1 \otimes m_{12.3} - m_{12} \otimes m_1 - m_1 \otimes m_{12} - m_{12.3} \otimes 1 \\ &= 1 \otimes \tilde{m}_{13.2} + \tilde{m}_{13.2} \otimes 1.\end{aligned}$$

We conclude that $(\mathrm{id} \otimes \mathbf{ab}) \circ \Delta(\tilde{m}_{13.2}) = \tilde{m}_{13.2} \otimes 1$. In other terms, $\tilde{m}_{13.2} \in \mathcal{C}$. The linear map Δ may be split as $\Delta = \Delta^{\mathrm{P}} + \Delta^{\mathrm{I}}$, the sum of its **primitive** and **imprimitive** parts respectively. What we have just done in the example is to find a modification $\tilde{m}_{13.2}$ of $m_{13.2}$ satisfying $\Delta^{\mathrm{I}}(\tilde{m}_{13.2}) = 0$. This suggests the following proposition.

Proposition 4 *There is a primitive element*

$$\tilde{m}_{\mathbf{A}} = m_{\mathbf{A}} + \sum_{\mathbf{B}\,:\,\lambda(\mathbf{B})=\lambda(\mathbf{A})} \alpha_{\mathbf{B}}\, m_{\mathbf{B}}$$

associated to each $\mathbf{A} \in \dot{\Pi}_\sharp$ *such that* $\sum_{\mathbf{B}} \alpha_{\mathbf{B}} = -1$ *and* $\mathbf{B} \in \dot{\Pi} \Rightarrow \alpha_{\mathbf{B}} = 0$.

The existence of primitives comes from the Milnor-Moore isomorphism of $\mathcal{N}$ with $\mathfrak{U}(\mathfrak{L}(\dot{\Pi}))$. Showing that they can be chosen with the above properties is a simple calculation, inducting on the number of parts r of an atomic set partition $\mathbf{A} = \{A_1, \ldots, A_r\}$ and applying $(\Delta^{\mathrm{I}})^r$.

The ideas behind the proposition and the preceding example yield several immediate corollaries: (i) each $\tilde{m}_{\mathbf{A}}$ from Proposition 4 belongs to $\mathcal{C}$; (ii) $\mathcal{C}$ is shape-graded, i.e., if $h \in \mathcal{C}$ is written as $\sum_\mu h_\mu$, then each h_μ belongs to $\mathcal{C}$ as well; (iii) for any $g \in \mathcal{N}$ and $h \in \mathcal{C}$, we have that $[g, h] = gh - hg$ also belongs to $\mathcal{C}$; (iv) if $\mathbf{A}$ and $\mathbf{B}$ belong to $\dot{\Pi}_\flat$, then $[m_{\mathbf{A}}, m_{\mathbf{B}}]$ belongs to $\mathcal{C}$. These points essentially account for all of $\mathcal{C}$, as the next result suggests. First, recall that $S^{\mathfrak{S}}$ is also a universal enveloping algebra of a Lie algebra. Namely, the abelian Lie algebra $\mathfrak{A}(\{m_1, m_2, \ldots\})$, where all Lie brackets $[m_j, m_k]$ are zero. Since the integers $k = 1, 2, \ldots$ are in 1-1 correspondence with $\dot{\Pi}_\flat$, we have a natural map from $\mathfrak{L}(\dot{\Pi})$ to $\mathfrak{A}(\{m_1, m_2 \ldots\})$. Our final characterization of $\mathcal{C}$ is as follows.

Corollary 5 *Let* $\mathfrak{C}$ *be the kernel of the map* π *from the free Lie algebra on* $\dot{\Pi}$ *to the free abelian Lie algebra on* $\dot{\Pi}_\flat$*. Then the coinvariant space* $\mathcal{C}$ *is the universal enveloping algebra of the Lie algebra* $\mathfrak{C}$*.*

Before turning to the case $n < \infty$, we remark that we have left unanswered the question of finding a systematic procedure (e.g., a closed formula in the spirit of Möbius inversion) that constructs a primitive element $\tilde{m}_{\mathbf{A}}$ for each $\mathbf{A} \in \dot{\Pi}_\sharp$.

6 The coinvariant space of $\mathcal{N}$ (Case: $n < \infty$)

We repeat our example of Section 3.3 in the case $n = 3$. The leading term with respect to our previous order would be $m_{13.2.4.5}$, except that this term does not appear because 13.2.4.5 has more than $n = 3$ parts. Fortunately, the rhyme scheme bijection *w* reveals a more useful leading term:

$$m_{121} \cdot m_{\mathbf{12}} = 0 + m_{12113} + m_{12131} + m_{12123} + m_{12132} + m_{12121} + m_{12112}\,.$$

The concatenation $121|12$ is the lexicographically smallest word appearing above. This is generally true: if $\mathsf{w}(\mathbf{A}) = u$ and $\mathsf{w}(\mathbf{B}) = v$, then uv is the smallest element of $\mathsf{w}(\mathbf{A} \sqcup\!\sqcup\, \mathbf{B})$. Let us call a rhyme scheme word a **verse** if it cannot be written as the concatenation of two shorter rhyme schemes. The **splitting** of a rhyme scheme w is the maximal deconcatenation $w = w'|w''|\cdots|w^{(r)}$ of w into verses $w^{(i)}$. For example, 12314 is a verse while 11232411 is a string of four versus $1|12324|1|1$. It is easy to see that if a, b, c, and d are verses, then $a|c = b|d$ if and only if $a = b$ and $c = d$. The preceding observations make it clear that $\mathcal{N}$ is *verse-filtered* and that $\mathcal{N}$ is freely generated by the monomials $\{m_{\mathsf{w}(\mathbf{A})} \mid \mathsf{w}(\mathbf{A}) \text{ is a verse}\}$. This is the collection of monomials originally chosen by Wolf, cf. [3, §7] for details.

Toward locating $\mathcal{C}$ within $\mathcal{N}$, we first locate $S^{\mathfrak{S}}$. Consider the partition $\mu = 32211$. Note that the lexicographically least rhyme scheme word of shape μ is $\mathsf{w}(123.45.67.8.9) = 111223345$. We are led to introduce the words

$$\mathsf{w}(\mu) := 1^{\mu_1} 2^{\mu_2} \cdots k^{\mu_k}$$

asociated to partitions $\mu = (\mu_1, \mu_2, \cdots, \mu_k)$; we call these **descending rhymes** since $\mu_1 \geq \cdots \geq \mu_k$. Finally, we want to view $\mathcal{C}$ as the rhymes that don't involve a descending rhyme. Then, by the fact that $\mathcal{N}$ is verse-filtered, we will get an easy vector space isomorphism $\mathcal{C} \otimes \Lambda \to \mathcal{N}$ given by multiplication. Toward that end, we introduce the notion of vexillary rhymes.

A **vexillary rhyme** is a word that begins with a maximal (but possibly empty) descending rhyme, followed by one extra verse. The **vexillary decomposition** of a rhyme scheme w is the expression of w as a product $w = w'|w''|\cdots|w^{(r)}|w^{(r+1)}$, where $w', \ldots, w^{(r)}$ are vexillary rhymes and $w^{(r+1)}$ is a possibly empty descending rhyme (which we call a **tail**). For a given word w, this decomposition is accomplished by first splitting w into verses, then recombining, from left to right, consecutive verses to form vexillary rhymes. For instance, the splitting of 112212 is $1|1222|12$. The first two factors combine to make one vexillary rhyme; the last factor is a descending tail: $1122212 \mapsto \overbrace{1\;1222}\;\overbrace{12}$. Similarly,

$$12312314111122311 \mapsto 123|12314|1|1|1223|1|1 \mapsto \overbrace{123\;12314}\;\overbrace{1\;1\;1223\;1}\;\overbrace{1}.$$

Suppose now that u and v are rhyme schemes and that the vexillary decomposition of u is tail-free. Then by construction, the vexillary decomposition of uv is the concatenation of the respective vexillary decompositions of u and v. We are ready to identify $\mathcal{C}$ as a subalgebra of $\mathcal{N}$.

Theorem 6 *Let $\mathcal{C}$ be the subalgebra of $\mathcal{N}$ generated by vexillary rhymes. Then $\mathcal{C}$ has a basis indexed by rhyme scheme words w whose vexillary decompositions are tail-free. Moreover, the map $\mathcal{C} \otimes \Lambda \to \mathcal{N}$ given by $m_{w'} m_{w''} \cdots m_{w^{(r)}} \otimes m_{(\mu_1 \cdots \mu_k)} \mapsto m_{w'|w''|\cdots|w^{(r)}|\mathsf{w}(\mu)}$ is a vector space isomorphism.*

7 Other directions

We conclude with another advertisement for the Blattner-Cohen-Montgomery theorem. The authors' present investigation into coinvariant spaces began by moving vertically within the commuting diagram (cube) of Hopf algebras depicted in Figure 1 (whereas in previous work, it was customary to move from left to right, cf. [1]). One may just as well move in other directions within the cube. To illustrate, we apply the Blattner-Cohen-Montgomery theorem to two other edges of interest (leaving aside any comments on group actions). The first of these concerns the downward arrow on the front-right side of the cube. Recall that, from a purely combinatorial perspective, bases in $\mathbb{K}\langle\mathbf{x}\rangle^{\sim\mathfrak{S}}$ are indexed by "set compositions" (ordered set partitions), and those in $\mathbb{K}[\mathbf{x}]^{\sim\mathfrak{S}}$ by integer compositions (here "$\sim$" indicates the quasi-action

$$\begin{tikzcd}
& \mathcal{C} \arrow[dd] & & \\
\mathbb{K}\langle \mathbf{x},\mathbf{y}\rangle^{\mathfrak{S}} \arrow[rr, hook] \arrow[dr, two heads] \arrow[dd, two heads] & & \mathbb{K}\langle \mathbf{x},\mathbf{y}\rangle^{\sim\mathfrak{S}} \arrow[dr, two heads] \arrow[dd, two heads] & \\
& \mathbb{K}\langle \mathbf{x}\rangle^{\mathfrak{S}} \arrow[rr, hook] \arrow[dd, two heads] & & \mathbb{K}\langle \mathbf{x}\rangle^{\sim\mathfrak{S}} \arrow[dd, two heads] \\
\mathbb{K}[\mathbf{x},\mathbf{y}]^{\mathfrak{S}} \arrow[rr, hook] \arrow[dr, two heads] & & \mathbb{K}[\mathbf{x},\mathbf{y}]^{\sim\mathfrak{S}} \arrow[dr, two heads] & \\
& \mathbb{K}[\mathbf{x}]^{\mathfrak{S}} \arrow[rr, hook] & & \mathbb{K}[\mathbf{x}]^{\sim\mathfrak{S}}
\end{tikzcd}$$

FIG. 1: The Hopf algebras of symmetric and quasisymmetric functions in one and two sets of commuting and noncommuting variables.

of Hivert, cf. [7, §3]). One may find a coalgebra splitting from $\mathbb{K}[\mathbf{x}]^{\sim\mathfrak{S}}$ to $\mathbb{K}\langle\mathbf{x}\rangle^{\sim\mathfrak{S}}$ and an associated coinvariant subalgebra in the spirit of our $(\mathcal{N}, S^{\mathfrak{S}})$ investigation.

Another direction is to consider the Hopf algebra morphism $\mathbf{sp} : \mathbb{K}[\mathbf{x},\mathbf{y}]^{\sim\mathfrak{S}} \twoheadrightarrow \mathbb{K}[\mathbf{x}]^{\sim\mathfrak{S}}$ (the bottom-right arrow going from NW to SE in Figure 1). These are the **diagonally quasi-symmetric functions** and **quasi-symmetric functions** respectively. For details omitted below, we refer the reader to [1]. The space $\mathbb{K}[\mathbf{x},\mathbf{y}]^{\mathfrak{S}}$ is defined as the $\mathfrak{S}$-invariants, inside $\mathbb{K}[\mathbf{x},\mathbf{y}]$, under the diagonal embedding of $\mathfrak{S}$ in $\mathfrak{S}\times\mathfrak{S}$. (The quasi-action of Hivert passes easily through this diagonal embedding.) A basis for $\mathbb{K}[\mathbf{x},\mathbf{y}]^{\sim\mathfrak{S}}$ is given by the "monomial functions" $m_{\boldsymbol{a},\boldsymbol{b}}$, indexed by "bicompositions", i.e., elements $(\boldsymbol{a},\boldsymbol{b})$ in $\mathbb{N}^{2\times r}$ such that $a_i + b_i > 0$. These $m_{\boldsymbol{a},\boldsymbol{b}}$ conveniently map to the quasi-symmetric function $m_{\boldsymbol{a}+\boldsymbol{b}}$ under the specialization map $\mathbf{sp}$ sending y_i to x_i. It is straightforward to show that the map sending $m_{\boldsymbol{a}}$ to $m_{\boldsymbol{a},\mathbf{0}}$, is a coalgebra splitting. We may thus analyze this situation in a manner analogous to our main result. Perhaps more surprising than the fact that the quotient

$$\mathrm{Hilb}_t\big(\mathbb{K}[\mathbf{x},\mathbf{y}]^{\sim\mathfrak{S}}\big)\Big/\mathrm{Hilb}_t\big(\mathbb{K}[\mathbf{x}]^{\sim\mathfrak{S}}\big)$$

belongs to $\mathbb{N}[\![t]\!]$ is the fact that the objects it counts have already been named. We discover a connection between compositions, set compositions, and "L-convex polyominoes." See [13, A003480].

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