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Abstract. We investigate the homogeneous symmetric Macdonald polynomials $P_\lambda(\mathbb{X}; q, t)$ for the specialization $t = q^k$. We show an identity relying the polynomials $P_\lambda(\mathbb{X}; q, q^k)$ and $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right)$. As a consequence, we describe an operator whose eigenvalues characterize the polynomials $P_\lambda(\mathbb{X}; q, q^k)$.

Résumé. Nous nous intéressons aux propriétés des polynômes de Macdonald symétriques $P_\lambda(\mathbb{X}; q, t)$ pour la spécialisation $t = q^k$. En particulier nous montrons une égalité reliant les polynômes $P_\lambda(\mathbb{X}; q, q^k)$ et $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right)$. Nous en déduisons la description d'un opérateur dont les valeurs propres caractérisent les polynômes $P_\lambda(\mathbb{X}; q, q^k)$.

Keywords: Symmetric functions, Macdonald polynomials, q-discriminant

1 Introduction

The Macdonald polynomials are (q, t)-deformations of the Schur functions which play an important rôle in the representation theory of the double affine Hecke algebra [11, 13] since they are the eigenfunctions of the Cherednik elements. More precisely, the non-symmetric Macdonald polynomials are the eigenfunctions of the Cherednik elements, but the symmetric Macdonald polynomials are the eigenfunctions of the symmetric functions in the Cherednik elements. The polynomials considered here are the homogeneous symmetric Macdonald polynomials $P_\lambda(\mathbb{X}; q, t)$ and are the eigenfunctions of the Sekiguchi-Debiard-Macdonald operator $\mathfrak{M}_1$. For (q, t) generic, the dimension of each eigenspace equals 1 and each Macdonald polynomial is characterized (up to a multiplicative constant) by the associated eigenvalue of $\mathfrak{M}_1$. That is no longer true when t is specialized to a rational power of q (note that the case of the specialization $t^n q^m = 1$ - n and m being integer - has been investigated by Feigin et *al.* [4] in their study of ideals of symmetric functions defined by vanishing conditions). Hence, it is more convenient to characterize the Macdonald (homogeneous symmetric) polynomials by orthogonality (*w.r.t.* a (q, t)-deformation of the usual scalar product on symmetric functions) and by some conditions on their dominant monomials (see *e.g.* [12]). In this paper, we consider the specialization $t = q^k$ where k is a (strictly) positive integer. One of our motivations is to generalize an identity of [1], which shows that even powers of the discriminant are rectangular Jack polynomials. Here, we show that this property follows from deeper relations between the Macdonald polynomials $P_\lambda(\mathbb{X}; q, q^k)$ and $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right)$ (in the λ-ring notation). This result is interesting in the context of the fractional quantum Hall effect [8], since it implies properties of the expansion of the powers of the discriminant in the Schur basis [3, 6, 14]. It implies also that the

Macdonald polynomials (at $t = q^k$) are characterized by the eigenvalues of an operator $\mathfrak{M}$ (described in terms of isobaric divided differences) whose eigenspaces are of dimension 1.

The paper is organized as follows. After recalling notations and background (Section 2) related to Macdonald polynomials, we give, in Section 3, some properties of the operator which substitutes a complete function to each power of a letter. These properties allow us to show our main result in Section 4 which is an identity involving the polynomial $P_\lambda(\mathbb{X}; q, q^k)$ and $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right)$. As a consequence, we describe (Section 5) an operator $\mathfrak{M}$ whose eigenvalues characterize the Macdonald polynomials $P_\lambda(\mathbb{X}; q, q^k)$. Finally, in Section 6, we give an expression of $\mathfrak{M}$ in terms of the Cherednik elements.

2 Notations and background

We recall here the basic definitions and classical properties of the symmetric functions and the Macdonald polynomials.

2.1 Symmetric functions

Consider an alphabet $\mathbb{X}$ (potentially infinite). Following [10] we define the symmetric functions on $\mathbb{X}$ by the generating functions of the complete homogeneous functions $S^p(\mathbb{X})$,

$$\sigma_z(\mathbb{X}) := \sum_i S^i(\mathbb{X}) z^i = \prod_{x \in \mathbb{X}} \frac{1}{1 - xz}.$$

The algebra Sym of symmetric functions has a λ-ring structure [10] and many properties of that structure can be understood by manipulating σ_z. For example, the sum of two alphabets $\mathbb{X} + \mathbb{Y}$ is defined by the product

$$\sigma_z(\mathbb{X} + \mathbb{Y}) := \sigma_z(\mathbb{X})\sigma_z(\mathbb{Y}) = \sum_i S^i(\mathbb{X} + \mathbb{Y}) z^i.$$

In particular, if $\mathbb{X} = \mathbb{Y}$ one has $\sigma_z(2\mathbb{X}) = \sigma_z(\mathbb{X})^2$. This definition is extended to any complex number α by $\sigma_z(\alpha\mathbb{X}) = \sigma_z(\mathbb{X})^\alpha$. For example, the generating series of the elementary functions is

$$\begin{aligned} \lambda_z(\mathbb{X}) &:= \sum \Lambda_i(\mathbb{X}) z^i = \prod_{x \in \mathbb{X}} (1 + xz) \\ &= \sigma_{-z}(-\mathbb{X}) = \textstyle\sum_i (-1)^i S^i(-\mathbb{X}) z^i. \end{aligned}$$

The complete functions of the product of two alphabets $\mathbb{X}\mathbb{Y}$ are given by the Cauchy kernel

$$K(\mathbb{X}, \mathbb{Y}) := \sigma_1(\mathbb{X}\mathbb{Y}) = \sum_i S^i(\mathbb{X}\mathbb{Y}) = \prod_{x \in \mathbb{X}} \prod_{y \in \mathbb{Y}} \frac{1}{1 - xy} = \sum_\lambda S_\lambda(\mathbb{X}) S_\lambda(\mathbb{Y}),$$

where S_λ denotes, as in [10], a Schur function. More generally, one has

$$K(\mathbb{X}, \mathbb{Y}) = \sum_\lambda A_\lambda(\mathbb{X}) B_\lambda(\mathbb{Y})$$

for any pair of bases $(A_\lambda)_\lambda$ and $(B_\lambda)_\lambda$ in duality for the usual scalar product $\langle\, ,\, \rangle$, *i.e.* $K(\mathbb{X}, \mathbb{Y})$ is the reproducing kernel associated to $\langle\, ,\, \rangle$.

2.2 Macdonald polynomials

The usual scalar product on symmetric functions admits a (q,t)-deformation (see *e.g.* [12]) defined for a pair of power sum functions Ψ^λ and Ψ^μ (in the notation of [10]) by

$$\langle \Psi^\lambda, \Psi^\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}, \tag{1}$$

where $\delta_{\lambda,\mu} = 1$ if $\lambda = \mu$ and 0 otherwise. The family of (symmetric homogeneous) Macdonald polynomials $(P_\lambda(\mathbb{X}; q, t))_\lambda$ is the unique basis of the symmetric functions orthogonal *w.r.t.* $\langle\ ,\ \rangle_{q,t}$ verifying

$$P_\lambda(\mathbb{X}; q, t) = m_\lambda(\mathbb{X}) + \sum_{\mu \leq \lambda} u_{\lambda\mu} m_\mu(\mathbb{X}), \tag{2}$$

where m_λ denotes, as usual, a monomial function [10, 12]. The reproducing kernel associated to this scalar product is

$$K_{q,t}(\mathbb{X}, \mathbb{Y}) := \sum_\lambda \langle \Psi^\lambda, \Psi^\lambda \rangle_{q,t}^{-1} \Psi_\lambda(\mathbb{X}) \Psi_\lambda(\mathbb{Y}) = \sigma_1\left(\frac{1-t}{1-q}\mathbb{X}\mathbb{Y}\right)$$

see *e.g.* [12, VI.2]. In particular, one has

$$K_{q,t}(\mathbb{X}, \mathbb{Y}) = \sum_\lambda P_\lambda(\mathbb{X}; q, t) Q_\lambda(\mathbb{Y}; q, t), \tag{3}$$

where $Q_\lambda(\mathbb{X}; q, t)$ is the dual basis of $P_\lambda(\mathbb{Y}; q, t)$ with respect to $\langle\ ,\ \rangle_{q,t}$,

$$Q_\lambda(\mathbb{X}; q, t) = \langle P_\lambda, P_\lambda \rangle_{q,t}^{-1} P_\lambda(\mathbb{X}; q, t). \tag{4}$$

The coefficient $b_\lambda(q,t) = \langle P_\lambda, P_\lambda \rangle_{q,t}^{-1}$ is known to be

$$b_\lambda(q,t) = \prod_{(i,j)\in\lambda} \frac{1 - q^{\lambda_j - i + 1} t^{\lambda'_i - j}}{1 - q^{\lambda_j - i} t^{\lambda'_i - j + 1}} \tag{5}$$

see [12, VI.6]. Writing

$$K_{q,t}\left(\left(\frac{1-q}{1-t}\right)\mathbb{X}, \mathbb{Y}\right) = K(\mathbb{X}, \mathbb{Y}), \tag{6}$$

one finds that $\left(P_\lambda\left(\left(\frac{1-q}{1-t}\right)\mathbb{X}; q, t\right)\right)_\lambda$ is the dual basis of $(Q_\lambda(\mathbb{X}; q, t))_\lambda$ with respect to the usual scalar product $\langle\ ,\ \rangle$.

Note that there exists an other Kernel type formula which reads

$$\lambda_1(\mathbb{X}\mathbb{Y}) = \sum_\lambda P_{\lambda'}(\mathbb{X}; t, q) P_\lambda(\mathbb{Y}; q, t) = \sum_\lambda Q_{\lambda'}(\mathbb{X}; t, q) Q_\lambda(\mathbb{Y}; q, t). \tag{7}$$

where λ' denotes the conjugate partition of λ. This formula can be found in [12, VI.5 p329].

From equalities (6) and (3) , one has

$$\sigma_1(\mathbb{XY}) = K_{q,t}\left(\frac{1-q}{1-t}\mathbb{X},\mathbb{Y}\right) = \sum_\lambda Q_\lambda\left(\frac{1-q}{1-t}\mathbb{X};q,t\right)P_\lambda(\mathbb{Y};q,t). \tag{8}$$

Applying (7) to

$$\sigma_1(\mathbb{XY}) = \lambda_{-1}(-\mathbb{XY}),$$

one obtains

$$\sigma_1(\mathbb{XY}) = \sum_\lambda (-1)^{|\lambda|}Q_{\lambda'}(-\mathbb{X};t,q)Q_\lambda(\mathbb{Y};q,t). \tag{9}$$

Identifying the coefficient of $P_\lambda(\mathbb{Y};t,q)$ in (8) and (9), one finds the following property.

Lemma 2.1

$$Q_\lambda(-\mathbb{X};t,q) = (-1)^{|\lambda|}P_{\lambda'}\left(\frac{1-q}{1-t}\mathbb{X};q,t\right). \tag{10}$$

Unlike the usual ($q=t=1$) scalar product, there is no expression as a constant term for the product $\langle\, ,\,\rangle_{q,t}$ when $\mathbb{X}=\{x_1,\dots,x_n\}$ is finite. But the Macdonald polynomials are orthogonal with respect to an other scalar product defined by

$$\langle f,g\rangle'_{q,t;n} = \frac{1}{n!}\mathrm{C.T.}\{f(\mathbb{X})g(\mathbb{X}^\vee)\Delta_{q,t}(\mathbb{X})\} \tag{11}$$

where C.T. denotes the constant term *w.r.t.* the alphabet $\mathbb{X}$, $\Delta_{q,t}(\mathbb{X}) = \prod_{i\neq j}\frac{(x_ix_j^{-1};q)_\infty}{(tx_ix_j^{-1};q)_\infty}$, $(a;b)_\infty = \prod_{i\geq 0}(1-ab^i)$ and $\mathbb{X}^\vee = \{x_1^{-1},\dots,x_n^{-1}\}$. The expression of $\langle P_\lambda,Q_\lambda\rangle'_{q,t;n}$ is given by ([12, VI.9])

$$\langle P_\lambda,Q_\lambda\rangle'_{q,t;n} = \frac{1}{n!}\mathrm{C.T.}\{\Delta_{q,t}(\mathbb{X})\}\prod_{(i,j)\in\lambda}\frac{1-q^{i-1}t^{n-j+1}}{1-q^it^{n-j}}. \tag{12}$$

2.3 Skew symmetric functions

Let us define as in [12, VI.7], the skew Macdonald functions $Q_{\lambda/\mu}$ by

$$\langle Q_{\lambda/\mu},P_\nu\rangle_{q,t} := \langle Q_\lambda,P_\mu P_\nu\rangle_{q,t}. \tag{13}$$

Straightforwardly, one has

$$Q_{\lambda/\mu}(\mathbb{X};q,t) = \sum_\nu \langle Q_\lambda,P_\nu P_\mu\rangle_{q,t}Q_\nu(\mathbb{X};q,t). \tag{14}$$

And classically, the following property holds (see *e.g.* [12, VI.7] for a short proof of this identity),

Proposition 2.2 *Let $\mathbb{X}$ and $\mathbb{Y}$ be two alphabets, one has*

$$Q_\lambda(\mathbb{X}+\mathbb{Y};q,t) = \sum_\mu Q_\mu(\mathbb{X};q,t)Q_{\lambda/\mu}(\mathbb{Y};q,t),$$

or equivalently

$$P_\lambda(\mathbb{X}+\mathbb{Y};q,t) = \sum_\mu P_\mu(\mathbb{X};q,t)P_{\lambda/\mu}(\mathbb{Y};q,t).$$

Equalities (3) and (7) are generalized by identities (15) and (16) as shown in [12, example 6 p.352],

$$\sum_\rho P_{\rho/\lambda}(\mathbb{X};q,t)Q_{\rho/\mu}(\mathbb{Y};q,t) = K_{qt}(\mathbb{X},\mathbb{Y})\sum_\rho P_{\mu/\rho}(\mathbb{X};q,t)Q_{\lambda/\rho}(\mathbb{Y};q,t), \tag{15}$$

$$\sum_\rho Q_{\rho'/\lambda'}(\mathbb{X};t,q)Q_{\rho/\mu}(\mathbb{Y};q,t) = \lambda_1(\mathbb{X}\mathbb{Y})\sum_\rho Q_{\mu'/\rho'}(\mathbb{X},t,q)Q_{\lambda/\rho}(\mathbb{Y};q,t). \tag{16}$$

3 The substitution $x^p \to S^p(\mathbb{Y})$ and the Macdonald polynomials

Let $\mathbb{X} = \{x_1,\dots,x_n\}$ be a finite alphabet and $\mathbb{Y}$ be an other (potentially infinite) alphabet. For simplicity we will denote by $\int_\mathbb{Y}$ the substitution

$$\int_\mathbb{Y}:\ x^p \to S^p(\mathbb{Y}), \tag{17}$$

for each $x\in\mathbb{X}$ and each $p\in\mathbb{Z}$. Let us define the symmetric function

$$\mathfrak{H}^{n,k}_{\lambda/\mu}(\mathbb{Y};q,t) := \frac{1}{n!}\int_\mathbb{Y} P_\lambda(\mathbb{X};q,t)Q_\mu(\mathbb{X}^\vee;q,t)\Delta(\mathbb{X},q,t) \tag{18}$$

where $\mathbb{X}^\vee = \{x_1^{-1},\dots,x_n^{-1}\}$.

Set $\mathbb{Y}^{tq} := \frac{1-t}{1-q}\mathbb{Y}$ and consider the substitution

$$\int_{\mathbb{Y}^{tq}} x^p = S^p\left(\mathbb{Y}^{tq}\right) = Q_p(\mathbb{Y};q,t). \tag{19}$$

The following result shows that $\mathfrak{H}^{n,k}_{\lambda/\mu}$ is a skew Macdonald polynomial on a suitable alphabet.

Theorem 3.1 *Let $\mathbb{X} = \{x_1,\dots,x_n\}$ be an alphabet, $\lambda = (\lambda_1,\dots,\lambda_n)$ be a partition and $\mu\subset\lambda$. The polynomial $\mathfrak{H}^{n,k}_{\lambda/\mu}(\mathbb{Y}^{tq};q,t)$ is the Macdonald polynomial*

$$\mathfrak{H}^{n,k}_{\lambda/\mu}(\mathbb{Y}^{tq};q,t) = \frac{1}{n!}\prod_{(i,j)\in\lambda}\frac{1-q^{i-1}t^{n-j+1}}{1-q^it^{n-j}}\mathrm{C.T.}\{\Delta(\mathbb{X},q,t)\}P_{\lambda/\mu}(\mathbb{Y},q,t) \tag{20}$$

Set $\overline{\mathbb{Y}} = \{-y_1,\dots,-y_m,\dots\}$ if $\mathbb{Y} = \{y_1,\dots,y_m,\dots\}$ and note that the operation $\mathbb{Y}\to\overline{\mathbb{Y}}$ makes sense even for virtual alphabet since it sends any homogeneous symmetric polynomial $P(\mathbb{Y})$ of degree p to $(-1)^pP(\mathbb{Y})$. One observes the following phenomenon which is obtained from Theorem 3.1 by applying the operations of the λ-ring structure.

Corollary 3.2 *Let $\mathbb{X} = \{x_1,\dots,x_n\}$ be an alphabet, $\lambda = (\lambda_1,\dots,\lambda_n)$ be a partition and $\mu\subset\lambda$. One has*

$$\mathfrak{H}^{n,k}_{\lambda/\mu}(-\overline{\mathbb{Y}};q,t) = \frac{1}{n!}\prod_{(i,j)\in\lambda}\frac{1-q^{i-1}t^{n-j+1}}{1-q^it^{n-j}}\mathrm{C.T.}\{\Delta(\mathbb{X},q,t)\}Q_{\lambda'/\mu'}(\mathbb{Y},t,q). \tag{21}$$

Note that in the case of partitions, one has

Corollary 3.3

$$\mathfrak{H}_{\lambda}^{n,k}(-\overline{\mathbb{Y}}, q, t) = \frac{1}{n!} \prod_{(i,j)\in\lambda} \frac{1-q^{i-1}t^{n-j+1}}{1-q^{i}t^{n-j}} \mathrm{C.T.}\{\Delta(\mathbb{X}, q, t)\} Q_{\lambda'}(\mathbb{Y}, t, q) \tag{22}$$

Example 3.4 Consider the following equality

$$\mathfrak{H}_{41/3}^{2,3}(-\overline{\mathbb{Y}}; q, t) = (*)\mathrm{C.T.}\{\Delta(\mathbb{X}, q, t)\} Q_{2111/111}(\mathbb{Y}; t, q).$$

where $\mathbb{X} = \{x_1, x_2\}$. The coefficient $(*)$ is computed as follows. One writes the partition $[41]$ in a rectangle of height 2 and length 4.

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Each $\times$ of coordinates (i, j) is read as the fraction $[i, j] := \frac{1-q^{i-1}t^{3-j}}{1-q^{i}t^{2-j}}$. Hence

$$(*) = [1,2][1,1][2,1][3,1][4,1] = \frac{(1-t)(1-t^2)(1-qt^2)(1-q^2t^2)(1-q^3t^2)}{(1-q)(1-qt)(1-q^2t)(1-q^3t)(1-q^4t)}$$

4 A formula involving the polynomials $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right)$ and $P_\lambda\left(\mathbb{X}; q, q^k\right)$

Now, we suppose that $t = q^k$ with $k \in \mathbb{N}$. In that case, the constant term $\mathrm{C.T.}\{\Delta(\mathbb{X}, q, t)\}$ admits a closed form and Corollary 3.3 gives

Corollary 4.1

$$\mathfrak{H}_{\lambda}^{n,k}(-\overline{\mathbb{Y}}, q, q^k) = \beta_{\lambda}^{n,k}(q) Q_{\lambda'}(\mathbb{Y}; q^k, q). \tag{23}$$

where

$$\beta_{\lambda}^{n,k}(q) = \prod_{i=0}^{n-1} \begin{bmatrix} \lambda_{n-i} - 1 + k(i+1) \\ k-1 \end{bmatrix}_q$$

and $\begin{bmatrix} n \\ p \end{bmatrix}_q = \frac{(1-q^n)\ldots(1-q^{n-p+1})}{(1-q)\ldots(1-q^r)}$ *denotes the q-binomial.*

Example 4.2 Set $k = 2, n = 3$ and consider the polynomial

$$\mathfrak{H}_{[320]}^{3,2}(-\overline{\mathbb{Y}}; q, q^2) = \frac{1}{n!} \int_{-\overline{\mathbb{Y}}} P_{[32]}(x_1 + x_2 + x_3; q, q^2) \prod_{i\neq j} (1 - x_i x_j^{-1})(1 - q x_i x_j^{-1}).$$

One has

$$\mathfrak{H}_{[320]}^{3,2}(-\overline{\mathbb{Y}}; q, q^2) = \frac{\left(1-q^5\right)\left(1-q^8\right)}{\left(1-q\right)^2} Q_{[221]}(\mathbb{Y}; q^2, q).$$

Let

$$\Omega_S := \frac{1}{n!}\int_{\mathbb{X}}\prod_{i\neq j}(1 - x_i x_j^{-1}) \tag{24}$$

and for each $v \in \mathbb{Z}^n$,

$$\tilde{S}_v(\mathbb{X}) = \det\left(x_i^{v_j+n-j}\right)\prod_{i<j}(x_i - x_j)^{-1}.$$

Lemma 4.3 *If v is any vector in $\mathbb{Z}^n$, one has*

$$\Omega_S\tilde{S}_v(\mathbb{X}) = S_v(\mathbb{X}) := \det(S^{v_i-i+j}(\mathbb{X})) \tag{25}$$

In particular, Ω_S leaves invariant any symmetric polynomial. The operator

$$\mathfrak{A}_m := \Omega_S\Lambda^n(\mathbb{X})^{-m} \tag{26}$$

acts on symmetric polynomials by substracting m from each part of the partitions appearing in their expansion in the Schur basis.

Example 4.4 If $\mathbb{X} = \{x_1, x_2, x_3\}$ and $\lambda = [320]$, one has

$$P_{32}(\mathbb{X}; q, t) = S_{32}(\mathbb{X}) + \frac{(-q+t)\,S_{311}(\mathbb{X})}{qt-1} + \frac{(q+1)\left(qt^2-1\right)(-q+t)\,S_{221}(\mathbb{X})}{(qt-1)^2\,(qt+1)}.$$

Hence,

$$\begin{aligned}\mathfrak{A}_1 P_{32}(\mathbb{X}; q, t) &= \tfrac{(-q+t)S_2(\mathbb{X})}{qt-1} + \tfrac{(q+1)\left(qt^2-1\right)(-q+t)S_{11}(\mathbb{X})}{(qt-1)^2(qt+1)}\\ &= \tfrac{(-q+t)(t+1)\left(q^2t-1\right)P_{11}(\mathbb{X};q,t)}{(qt-1)^2(qt+1)} + \tfrac{(-q+t)P_2(\mathbb{X};q,t)}{qt-1}.\end{aligned}$$

Theorem 4.5 *If λ denotes a partition of length at most n, one has*

$$\mathfrak{A}_{(k-1)(n-1)}P_\lambda(\mathbb{X}; q, q^k)\prod_{l=1}^{k-1}\prod_{i\neq j}(x_i - q^l x_j) = \beta_\lambda^{n,k}(q)P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right) \tag{27}$$

Example 4.6 Set $k = 2$, $n = 3$ and $\lambda = [2]$. One has

$$\begin{aligned}&P_{[2]}(x_1 + x_2 + x_3; q, q^2)\prod_{i\neq j}(x_i - qx_j) = -q^3 S_{[6,2]} + q^2\frac{q^3-1}{q-1}S_{[6,1,1]}\\ &+\frac{q^2(q^5-1)}{q^3-1}S_{[5,3]} - \frac{q(q^2+1)(q^5-1)}{q^3-1}S_{[5,2,1]} - \frac{q(q^7-1)}{q^3-1}S_{[4,3,1]} + \frac{q^7-1}{q-1}S_{[4,2,2]}.\end{aligned}$$

And,

$$\mathfrak{A}_2 P_{[2]}(x_1 + x_2 + x_3; q, q^2)\prod_{i\neq j}(x_i - qx_j) = \frac{q^7-1}{q-1}S_{[2]}.$$

Since,

$$P_{[2]}\left(\frac{x_1+x_2+x_3}{1+q}; q, q^2\right) = \frac{q-1}{q^3-1}S_{[2]}$$

one obtains

$$\mathfrak{A}_2 P_{[2]}(x_1+x_2+x_3;q,q^2)\prod_{i\neq j}(x_i-qx_j)=\begin{bmatrix}1\\1\end{bmatrix}_q\begin{bmatrix}3\\1\end{bmatrix}_q\begin{bmatrix}7\\1\end{bmatrix}_q P_{[2]}\left(\frac{x_1+x_2+x_3}{1+q};q,q^2\right).$$

As a consequence, one has

Corollary 4.7 *If* $\lambda=\mu+[((k-1)(n-1))^n]$,

$$P_\mu(\mathbb{X};q,q^k)\prod_{l=1}^{k-1}\prod_{i\neq j}(x_i-q^l x_j)=\beta_\lambda^{n,k}(q)P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X};q,q^k\right).$$

Example 4.8 Set $k=3$, $n=2$ and $\lambda=[5,2]$. One has

$$\begin{aligned}&P_{[5,2]}(x_1+x_2;q,q^3)(x_1-qx_2)(x_1-q^2x_2)(x_2-qx_1)(x_2-q^2x_1)=\\&q^3S_{[9,2]}+\frac{(1-q^7)(1+q^4)}{1-q^5}S_{[7,4]}-\frac{(1-q^2)(1+q)(1+q^2)(1+q^4)}{1-q^5}S_{[8,3]}.\end{aligned}$$

This implies

$$\begin{aligned}&\mathfrak{A}_2P_{[5,2]}(x_1+x_2;q,q^3)(x_1-qx_2)(x_1-q^2x_2)(x_2-qx_1)(x_2-q^2x_1)=\\&(x_1x_2)^{-2}P_{[5,2]}(x_1+x_2;q,q^3)(x_1-qx_2)(x_1-q^2x_2)(x_2-qx_1)(x_2-q^2x_1)=\\&P_{[3]}(x_1+x_2;q,q^3)(x_1-qx_2)(x_1-q^2x_2)(x_2-qx_1)(x_2-q^2x_1).\end{aligned}$$

One verifies that

$$\begin{aligned}&P_{[3]}(x_1+x_2;q,q^3)(x_1-qx_2)(x_1-q^2x_2)(x_2-qx_1)(x_2-q^2x_1)=\\&\begin{bmatrix}4\\2\end{bmatrix}_q\begin{bmatrix}10\\2\end{bmatrix}_q P_{[5,2]}(\frac{x_1+x_2}{1+q+q^2};q,q^3).\end{aligned}$$

Remark 4.9 If μ is the empty partition, Corollary 4.7 gives

$$\prod_{l=1}^{k-1}\prod_{i\neq j}(x_i-q^l x_j)=\beta_\lambda^{n,k}(q)P_{[((k-1)(n-1))^n]}\left(\frac{1-q}{1-q^k}\mathbb{X};q,q^k\right). \tag{28}$$

This equality generalizes an identity given in [1]:

$$\prod_{i<j}(x_i-x_j)^{2(k-1)}=\frac{(-1)^{\frac{((k-1)n(n-1)}{2}}}{n!}\binom{kn}{k,\dots,k}P^{(k)}_{n^{(n-1)(k-1)}}(-\mathbb{X}),$$

where $P_\lambda^{(k)}(\mathbb{X})=\lim_{q\to1}P_\lambda^{(\alpha)}(\mathbb{X};q,q^k)$ denotes a Jack polynomial (see *e.g.* [12]).

The expansion of the powers of the discriminant and their q-deformations in different basis of symmetric functions is a difficult problem having many applications, for example, in the study of Hua-type integrals (see *e.g.* [5, 7]) or in the context of the fractional quantum Hall effect (*e.g.* [3, 6, 8, 14]).

Note that in [2], we gave an expression of an other q-deformation of the powers of the discriminant as staircase Macdonald polynomials. This deformation is also relevant in the study of the expansion of $\prod_{i<j}(x_i-x_j)^{2k}$ in the Schur basis (for example, we generalized in [2] a result of [6]).

5 Macdonald polynomials at $t = q^k$ as eigenfunctions

Let $\mathbb{Y} = \{y_1, \ldots, y_{kn}\}$ be an alphabet of cardinality kn with $y_1 = x_1,\ \ldots,\ y_n = x_n$. One considers the symmetrizer π_ω defined by

$$\pi_\omega f(y_1, \ldots, y_{kn}) = \prod_{i<j}(x_i - x_j)^{-1} \sum_{\sigma \in \mathfrak{S}_{kn}} \mathrm{sign}(\sigma) f(y_{\sigma(1)}, \ldots, y_{\sigma(kn)}) y_{\sigma(1)}^{kn-1} \cdots y_{\sigma(kn-1)}.$$

Note that π_ω is the isobaric divided difference associated to the maximal permutation ω in $\mathfrak{S}_{kn}$.

This operator applied to a symmetric function of the alphabet $\mathbb{X}$ increases the alphabet from $\mathbb{X}$ to $\mathbb{Y}$ in its expansion in the Schur basis, since

$$\pi_\omega S_\lambda(\mathbb{X}) = S_\lambda(\mathbb{Y}). \tag{29}$$

Indeed, the image of the monomial $y_1^{i_1} \ldots y_{kn}^{i_{kn}}$ is the Schur function $S_I(\mathbb{Y})$. Since

$$\pi_\omega S_\lambda(\mathbb{X}) = \pi_\omega x_1^{\lambda_1} \ldots x_n^{\lambda_n} = \pi_\omega y_1^{\lambda_1} \ldots y_n^{\lambda_n} y_{n+1}^0 \ldots y_{kn}^0,$$

one recovers equality (29).

One defines the operator π^{tq} which consists in applying π_ω and specializing the result to the alphabet

$$\mathbb{X}^{tq} := \{x_1, \ldots, x_n, qx_1, \ldots, qx_n, \ldots, q^{k-1}x_1, \ldots, q^{k-1}x_n\}.$$

From equality (29), one has

$$\pi_\omega^{tq} S_\lambda(\mathbb{X}) = S_\lambda\left((1 + q + \ldots + q^{k-1})\mathbb{X}\right), \tag{30}$$

for $l(\lambda) \le n$. Furthermore, the expansion of $S_\lambda\left((1 + q + \ldots + q^{k-1})\mathbb{X}\right)$ in the Schur basis is triangular, so the operator π^{tq} defines an automorphism of the space $Sym_{\le n}$ generated by the Schur functions indexed by partitions whose length are less or equal to n, *i.e.* for each function $f \in Sym_{\le n}$, one has

$$\pi^{tq} f(\mathbb{X}) = f(\mathbb{X}^{tq}). \tag{31}$$

In particular,

Lemma 5.1 *Let λ be a partition such that $l(\lambda) \le n$ then*

$$\pi_\omega^{tq} P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, t = q^k\right) = P_\lambda(\mathbb{X}, q, q^k). \tag{32}$$

Consider the operator $\mathfrak{M} : f \to \mathfrak{M}f$ defined by

$$\mathfrak{M} := (x_1 \ldots x_n)^{(k-1)(1-n)} \pi_\omega^{tq} \prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j).$$

The following theorem shows that the Macdonald polynomials are the eigenfunctions of the operator $\mathfrak{M}$.

Theorem 5.2 *The Macdonald polynomials $P_\lambda(\mathbb{X}; q, q^k)$ are eigenfunctions of $\mathfrak{M}$. The eigenvalue associated to $P_\mu(\mathbb{X}; q, q^k)$ is $\beta^{n,k}_{\mu+((k-1)(n-1))^n}(q)$. Furthermore, if $k > 1$, the dimension of each eigenspace is* 1.

Example 5.3 If $n = 5$, the eigenvalues associated to the partitions of 4 are

$$\begin{array}{lcl}
\beta^{4,k}_{[4k,4k-4,4k-4,4k-4,4k-4]} & = & \begin{bmatrix}5k-5\\k-1\end{bmatrix}_q \begin{bmatrix}6k-5\\k-1\end{bmatrix}_q \begin{bmatrix}7k-5\\k-1\end{bmatrix}_q \begin{bmatrix}8k-5\\k-1\end{bmatrix}_q \begin{bmatrix}9k-1\\k-1\end{bmatrix}_q \ (\lambda = [4,0,0,0,0]),\\
\beta^{4,k}_{[4k-1,4k-3,4k-4,4k-4,4k-4]} & = & \begin{bmatrix}5k-5\\k-1\end{bmatrix}_q \begin{bmatrix}6k-5\\k-1\end{bmatrix}_q \begin{bmatrix}7k-5\\k-1\end{bmatrix}_q \begin{bmatrix}8k-4\\k-1\end{bmatrix}_q \begin{bmatrix}9k-2\\k-1\end{bmatrix}_q \ (\lambda = [3,1,0,0,0]),\\
\beta^{4,k}_{[4k-2,4k-2,4k-4,4k-4,4k-4]} & = & \begin{bmatrix}5k-5\\k-1\end{bmatrix}_q \begin{bmatrix}6k-5\\k-1\end{bmatrix}_q \begin{bmatrix}7k-5\\k-1\end{bmatrix}_q \begin{bmatrix}8k-3\\k-1\end{bmatrix}_q \begin{bmatrix}9k-3\\k-1\end{bmatrix}_q \ (\lambda = [2,2,0,0,0]),\\
\beta^{4,k}_{[4k-2,4k-3,4k-3,4k-4,4k-4]} & = & \begin{bmatrix}5k-5\\k-1\end{bmatrix}_q \begin{bmatrix}6k-5\\k-1\end{bmatrix}_q \begin{bmatrix}7k-4\\k-1\end{bmatrix}_q \begin{bmatrix}8k-4\\k-1\end{bmatrix}_q \begin{bmatrix}9k-3\\k-1\end{bmatrix}_q \ (\lambda = [2,1,1,0,0]),\\
\beta^{4,k}_{[4k-3,4k-3,4k-3,4k-3,4k-4]} & = & \begin{bmatrix}5k-5\\k-1\end{bmatrix}_q \begin{bmatrix}6k-4\\k-1\end{bmatrix}_q \begin{bmatrix}7k-4\\k-1\end{bmatrix}_q \begin{bmatrix}8k-4\\k-1\end{bmatrix}_q \begin{bmatrix}9k-4\\k-1\end{bmatrix}_q \ (\lambda = [1,1,1,1,0]).
\end{array}$$

6 Expression of $\mathfrak{M}$ in terms of the Cherednik elements

In this paragraph, we restate Proposition 5.2 in terms of Cherednik operators. Cherednik's operators $\{\xi_i; i \in \{1,\dots,n\}\} =: \Xi$ are commutative elements of the double affine Hecke algebra. The Macdonald polynomials $P_\lambda(\mathbb{X}; q, t)$ are eigenfunctions of symmetric polynomials $f(\Xi)$ and the eigenvalues are obtained substituting each occurrence of ξ_i in $f(\Xi)$ by $q^{\lambda_i} t^{n-i}$ (see [11] for more details).
Suppose that $k > 1$ and consider the operator $\tilde{\mathfrak{M}} : f \to \tilde{\mathfrak{M}} f$ defined by

$$\tilde{\mathfrak{M}} := \prod_{i=1}^{k-1} (1 - q^i)^n \mathfrak{M}. \tag{33}$$

From Proposition 5.2, one has

$$\tilde{\mathfrak{M}} P_\lambda(\mathbb{X}; q, q^k) = \prod_{i=0}^{n-1} \prod_{j=1}^{k-1} (1 - q^{\lambda_{n-i} + k(i+1) - j}) P_\lambda(\mathbb{X}; q, q^k). \tag{34}$$

The following proposition shows that $\tilde{\mathfrak{M}}$ admits a closed expression in terms of the Cherednick elements.

Proposition 6.1 *One supposes that $k > 1$. For any symmetric function f, one has*

$$\tilde{\mathfrak{M}} f(\mathbb{X}) = \prod_{l=1}^{k-1} \prod_{i=1}^{n} (1 - q^{l+k} \xi_i) f(\mathbb{X}). \tag{35}$$

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References

[1] H. Belbachir, A. Boussicault, J.-G. Luque, Hankel hyperdeterminants, rectangular Jack polynomials and even power of the Vandermonde, Journal of Algebra **320** (2008) 3911-3925.

[2] A. Boussicault, J.G. Luque, Staircase Macdonald polynomials and the q-Discriminant Formal Power Series and Algebraic Combinatorics June 23-27, 2008 Valparaiso Viña del Mar - Chile, preprint arXiv:0801.2443.

[3] P. Di Francesco, M. Gaudin, C. Itzykson, F. Lesage, Laughlin's wave functions, Coulomb gases and expansions of the discriminant, Int. J. Mod. Phys. A **9** (1994) 4257-4351.

[4] B. Feigin, M. Jimbo, T. Miwa, E. Mukhin, Symmetric polynomials vanishing on the shifted diagonals and Macdonald polynomials, Int Math Res Notices. (2003) 1015-1034.

[5] J. Kaneko, Selberg integrals and hypergeometric functions associated with Jack polynomials, S.I.A.M. Journal Math. Analysis, **24** (1993) 1086-1110.

[6] R.C. King, F. Toumazet, B.G. Wybourne, The square of the Vandermonde determinant and its q-generalization, J. Phys. A: Math. Gen. **37** (2004) 735-767.

[7] A. Korányi, Hua-type integrals, hypergeomatric functions and symmetric polynomials, Proceeding of a Conference in Memory of L K Hua, Beijin (1998).

[8] R. B. Laughlin, Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations , Phys. Rev. Lett. **50** (1983) 1395-1398

[9] L. Lapointe, A. Lascoux, J. Morse, Determinantal expressions for Macdonald polynomials, International Mathematics Research Notices 18 (1998).

[10] A Lascoux, Symmetric function and combinatorial operators on polynomials, CBMS 99, American Mathematical Society 2001.

[11] A. Lascoux, Yang-Baxter graphs, Jack and Macdonald polynomials, Ann. Comb., 5(3-4) (2001) 397-424. Dedicated to the memory of Gian-Carlo Rota (Tianjin, 1999)

[12] I. G. Macdonald, Symetric functions and Hall polynomials, second edition, Oxford University Press Inc., New York 1995.

[13] I. G. Macdonald, Affine Hecke algebra and orthogonal polynomials, Cambridge University Press, Cambridge, 2003.

[14] T. Scharf, J.-Y. Thibon, B.G. Wybourne, Powers of the Vandermonde determinant and the quantum Hall effect, J. Phys. A.: Math. Gen. **27** (1994) 4211-4219.

[15] D. Zeilberger, D. Bressoud, A proof of Andrew's q-Dyson conjecture, Discrete Math. **54** (1985) 201-224.