The absolute order on the hyperoctahedral group

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HAL Id: hal-01185381

https://hal.inria.fr/hal-01185381

Submitted on 20 Aug 2015

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Abstract. The absolute order on the hyperoctahedral group B_n is investigated. It is shown that every closed interval in this order is shellable, those closed intervals which are lattices are characterized and their zeta polynomials are computed. Moreover, using the notion of strong constructibility, it is proved that the order ideal generated by the Coxeter elements of B_n is homotopy Cohen-Macaulay and the Euler characteristic of the order complex of the proper part of this ideal is computed. Finally, an example of a non Cohen-Macaulay closed interval in the absolute order on the group D_4 is given and the closed intervals of D_n which are lattices are characterized.

Résumé. Nous étudions l'ordre absolu sur le groupe hyperoctahédral B_n. Nous montrons que chaque intervalle fermé de cet ordre est shellable, caractérisons les treillis parmi ces intervalles et calculons les polynômes zêta de ces derniers. De plus, en utilisant la notion de constructibilité forte, nous prouvons que l'idéal engendré par les éléments de Coxeter de B_n est Cohen-Macaulay pour l'homotopie, et nous calculons la caractéristique d'Euler du complexe associé à cet idéal. Pour finir, nous exhibons un exemple d'intervalle fermé non Cohen-Macaulay dans l'ordre absolu du groupe D_4, et caractérisons les intervalles fermés de D_n qui sont des treillis.

Keywords: Coxeter group, hyperoctaherdal group, absolute order, Cohen-Macaulay poset, shellability

1   Introduction and results

Coxeter groups are fundamental combinatorial structures which appear in several areas of mathematics. Partial orders on Coxeter groups often provide an important tool for understanding the questions of interest. Examples of such partial orders are the Bruhat order and the weak order. We refer the reader to [7, 10, 15] for background on Coxeter groups and their orderings.

In this work we study the absolute order. Let W be a finite Coxeter group with respect to the set $\mathcal{T}$ of *all* reflections in W. The absolute order on W is denoted by $\mathrm{Abs}(W)$ and defined as the partial order on W whose Hasse diagram is obtained from the Cayley graph of W with respect to $\mathcal{T}$ by directing its edges away from the identity (see Section 2.1 for a precise definition). The poset $\mathrm{Abs}(W)$ is locally self-dual

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[†]The present research will be part of the author's Doctoral Dissertation at the University of Athens

and graded. It has a minimum element, the identity $e \in W$, but will typically not have a maximum, since every Coxeter element of W is a maximal element of Abs(W). Its rank function is called the absolute length and is denoted by $\ell_\mathcal{T}$. The absolute length and order arise naturally in combinatorics [2], group theory [5, 11], statistics [13] and invariant theory [15]. For instance, $\ell_\mathcal{T}(w)$ can also be defined as the codimension of the fixed space of w, when W acts faithfully as a group generated by orthogonal reflections on a vector space V by its standard geometric representation. In this case, the rank generating polynomial of Abs(W) satisfies

$$\sum_{w \in W} t^{\ell_\mathcal{T}(w)} \; = \; \prod_{i=1}^{\ell} (1 + e_i t),$$

where $e_1, e_2, \ldots, e_\ell$ are the exponents [15, Section 3.20] of W and ℓ is its rank. We refer to [2, Section 2.4] and [4, Section 1] for further discussion of the importance of the absolute order and related historical remarks.

In this paper we will be interested in the combinatorics and topology of Abs(W). These have been studied extensively for the interval $[e, c] := NC(W, c)$ of Abs(W), known as the poset of noncrossing partitions associated to W, where $c \in W$ denotes a Coxeter element. For instance, it was shown in [3] that $NC(W, c)$ is shellable for every finite Coxeter group W. In particular, $NC(W, c)$ is homotopy Cohen-Macaulay and the order complex of $NC(W, c) \setminus \{e, c\}$ has the homotopy type of a wedge of spheres. The problem to determine the topology of the poset Abs(W) $\setminus \{e\}$ and to decide whether Abs(W) is Cohen-Macaulay or shellable, was naturally posed by Athanasiadis (unpublished) and Reiner [1, Problem 3.1], see also [19, Problem 3.3.7]. Computer calculations carried out by Reiner showed that the absolute order is not Cohen-Macaulay for the group D_4. In the case of the symmetric group, it is still not known whether Abs(S_n) is shellable. However, the following result was obtained in [4].

[4, Theorem 1.1] *The poset Abs(S_n) is homotopy Cohen-Macaulay for all $n \geq 1$. In particular, the order complex of Abs(S_n) $\setminus \{e\}$ is homotopy equivalent to a wedge of $(n-2)$-dimensional spheres and Cohen-Macaulay over $\mathbb{Z}$.*

Here we focus on the hyperoctahedral group B_n. Contrary to the case of the symmetric group, not every maximal element of the absolute order on B_n is a Coxeter element. The maximal intervals in Abs(B_n) include the posets $NC^B(n)$ of noncrossing partitions of type B [17] and $NC^B(p, q)$ of annular noncrossing partitions, introduced and studied recently by Nica and Oancea [16]. Our main results are as follows. In Section 3 we prove that every interval of B_n is shellable and present an example of a maximal element x of Abs(D_4) for which the interval $[e, x]$ is not Cohen-Macaulay over any field (Example 3.3). In Section 4 we comment on the proof a B_n-analogue of [4, Theorem 1.1], stating that the order ideal $\mathcal{J}_n$ of Abs(B_n) generated by the set of Coxeter elements of B_n is homotopy Cohen-Macaulay for all $n \geq 2$ (see Theorem 4.1). In particular, the order complex of $\mathcal{J}_n \setminus \{e\}$ is homotopy equivalent to a wedge of $(n-1)$-dimensional spheres and Cohen-Macaulay over $\mathbb{Z}$. The number of such spheres is also computed (see Theorem 4.8). We conjecture that the poset Abs(B_n) is Cohen-Macaulay for every $n \geq 2$ [i]. Finally, in Section 5 we characterize the maximal intervals of Abs(B_n) and Abs(D_n) which are lattices and compute some of their enumerative invariants. We refer the reader to [18, Chapter 3] and [9, 19] for background on partially ordered sets and the topology of simplicial complexes, respectively.

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[i] This conjecture has now been proved by the author

2 Preliminaries

2.1 The absolute length and absolute order

Let W be a finite Coxeter group with set of all reflections $\mathcal{T}$. Given $w \in W$, let $\ell_{\mathcal{T}}(w)$ denote the smallest integer k such that w can be written as a product of k reflections in $\mathcal{T}$. The *absolute order*, or *reflection length order*, is the partial order on W denoted by $\preceq$ and defined by letting

$$u \preceq v \quad \text{if and only if} \quad \ell_{\mathcal{T}}(u) + \ell_{\mathcal{T}}(u^{-1}v) = \ell_{\mathcal{T}}(v)$$

for $u, v \in W$. Equivalently, $\preceq$ is the partial order on W with covering relations $w \to wt$, where $w \in W$ and $t \in \mathcal{T}$ are such that $\ell_{\mathcal{T}}(w) < \ell_{\mathcal{T}}(wt)$. In that case we write $w \xrightarrow{t} wt$. The poset $\mathrm{Abs}(W)$ is graded with rank function $\ell_{\mathcal{T}}$.

2.2 The posets $Abs(B_n)$ and $Abs(D_n)$

We view the hyperoctahedral group B_n as the group of permutations u of the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ such that $u(-i) = -u(i)$ for every $1 \leq i \leq n$. Following [11], the permutation which has cycle form $(a_1\, a_2\, \cdots\, a_k)(-a_1\, -a_2\, \cdots\, -a_k)$ is denoted by $((a_1, a_2, \ldots, a_k))$ and is called a *paired k-cycle*, while the cycle $(a_1\, a_2\, \cdots\, a_k\, -a_1\, -a_2\, \cdots\, -a_k)$ is denoted by $[a_1, a_2, \ldots, a_k]$ and is called a *balanced k-cycle*. Every element $u \in B_n$ can be written (uniquely) as a product of disjoint paired or balanced cycles, called cycles of u. With this notation, the set $\mathcal{T}$ of reflections of B_n is equal to the union

$$\{[i] : 1 \leq i \leq n\} \cup \{((i,j)), ((i,-j)) : 1 \leq i < j \leq n\}. \tag{1}$$

The length $\ell_{\mathcal{T}}(u)$ of $u \in B_n$ is equal to $n - \gamma(u)$, where $\gamma(u)$ denotes the number of paired cycles in the cycle decomposition of u. An element $u \in B_n$ is maximal in $\mathrm{Abs}(B_n)$ if and only if it can be written as a product of disjoint balanced cycles whose lengths sum to n. The Coxeter elements of B_n are precisely the balanced n-cycles. To simplify the notation, we will denote by ℓ the absolute length $\ell_{\mathcal{T}}$. The covering relations $w \xrightarrow{t} wt$ of $\mathrm{Abs}(B_n)$, when w and t are non-disjoint cycles, can be described as follows: for $1 \leq i < j \leq m \leq n$ we have:

(a) $((a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m)) \xrightarrow{((a_{i-1}, a_i))} ((a_1, \ldots, a_m))$

(b) $((a_1, \ldots, a_m)) \xrightarrow{[a_i]} [a_1, \ldots, a_{i-1}, a_i, -a_{i+1}, \ldots, -a_m]$

(c) $((a_1, \ldots, a_m)) \xrightarrow{((a_i, -a_j))} [a_1, \ldots, a_i, -a_{j+1}, \ldots, -a_m][a_{i+1}, \ldots, a_j]$

(d) $[a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m] \xrightarrow{((a_{i-1}, a_i))} [a_1, \ldots, a_m]$

(e) $[a_1, \ldots, a_j]((a_{j+1}, \ldots, a_m)) \xrightarrow{((a_j, a_m))} [a_1, \ldots, a_m]$

where $a_1, \ldots, a_m$ are elements of the set $\{\pm 1, \ldots, \pm n\}$ with pairwise distinct absolute values.

The Coxeter group D_n is the subgroup of index two of the group B_n generated by the set of reflections

$$\{((i,j)), ((i,-j)) : 1 \leq i < j \leq n\}. \tag{2}$$

(these are all reflections in D_n). The absolute length on D_n is the restriction of absolute length of B_n on the set D_n. The number of balanced cycles of any element $u \in D_n$ is even and every Coxeter element of D_n has the form $[a_1, a_2, \ldots, a_{n-1}][a_n]$, where $|a_i| \in \{1, 2, \ldots, n\}$ and $|a_i| \neq |a_j|$ for all $i \neq j$.

3   Shellability

In this section we prove the following theorem.

Theorem 3.1 *Every interval of Abs(B_n) is shellable.*

Proof: (sketch) We show that every closed interval of Abs(B_n) admits an EL-labeling. The result then follows, since EL-shellability implies shellability (we refer to [8] for the definition of EL-labeling and EL-shellability).

Let $C(B_n)$ be the set of covering relations of Abs(B_n) and $(a, b) \in C(B_n)$. Then $a^{-1}b$ is a reflection of B_n, thus either $a^{-1}b = [i]$ for some $i \in \{1, 2, \ldots, n\}$, or there exist $i, j \in \{1, 2, \ldots, n\}$, with $i < j$, such that $a^{-1}b = ((i, j))$ or $a^{-1}b = ((i, -j))$. We define a map $\lambda : C(B_n) \to \{1, 2, \ldots, n\}$ as follows:

$$\lambda(a, b) = \begin{cases} i & \text{if } a^{-1}b = [i], \\ j & \text{if } a^{-1}b = ((i, j)) \text{ or } ((i, -j)). \end{cases}$$

A similar labeling was used by Biane [6] in order to study the maximal chains of the poset $NC^B(n)$ of noncrossing B_n-partitions. Figure 1 illustrates the Hasse diagram of the interval $[e, x]$, for $n = 4$ and $x = [3, -4]((1, 2))$, together with the corresponding labels.

Fig. 1: The interval $[e, x]$ for $x = [3, -4]((1, 2))$

The restricion of the map λ to the interval $[x, y]$ is an EL-labeling for all $x, y \in B_n$ with $x \preceq y$. To prove that, it suffices to show that for every $u \in B_n$ the map $\lambda|_{[e, u]}$ is an EL-labeling. Indeed, let

$x, y \in B_n$ with $x \preceq y$ and define the map $\phi : [x, y] \rightarrow [e, x^{-1}y]$ by $\phi(t) = x^{-1}t$. Clearly, ϕ is a poset isomorphism. Moreover, if $(a, b) \in C([x, y])$, then $\phi(a)^{-1}\phi(b) = (x^{-1}a)^{-1}x^{-1}b = a^{-1}xx^{-1}b = a^{-1}b$, which implies that $\lambda(a, b) = \lambda(\phi(a), \phi(b))$.

Let $u = b_1 b_2 \cdots b_k \, p_1 p_2 \cdots p_l$ be written as a product of disjoint cycles, where $b_i = [b_i^1, \ldots, b_i^{k_i}]$ for $i \leq k$ and $p_j = ((p_j^1, \ldots, p_j^{l_j}))$ with $p_j^1 = \min\{|p_j^m| : 1 \leq m \leq l_j\}$ for $j \leq l$. We consider the sequence of positive integers obtained by placing the numbers $|b_i^h|$ and $|p_j^m|$, for $i, j, h \geq 1$ and $m > 1$, in increasing order. There are $r = \ell(u)$ such integers. To simplify the notation, we denote by $c(u) = (c_1, c_2, \ldots, c_r)$ this sequence and say that c_μ ($\mu = 1, 2, \ldots, r$) belongs to a balanced (respectively paired) cycle if it is equal to some $|b_i^h|$ (respectively $|p_j^m|$). Clearly we have $c_1 < c_2 < \cdots < c_r$ and $\lambda(a, b) \in \{c_1, c_2, \ldots, c_r\}$ for every pair $a, b \in [e, u]$, with $a \rightarrow b$. To the sequence $(c_1, c_2, \ldots, c_r)$ corresponds a unique maximal chain

$$\mathcal{C}_u : \ u_0 = e \xrightarrow{c_1} u_1 \xrightarrow{c_2} u_2 \xrightarrow{c_3} \cdots \xrightarrow{c_r} u_r = u,$$

which can be constructed inductively as follows (here, the integer κ in $a \xrightarrow{\kappa} b$ denotes the label $\lambda(a, b)$). If c_1 belongs to a balanced cycle, then $u_1 = [c_1]$. Otherwise, if c_1 belongs to some p_i, say p_1, then we set u_1 to be either $((p_1^1, c_1))$ or $((p_1^1, -c_1))$, so that $u_1 \preceq p_1$ holds. In both cases $\lambda(e, u_1) = c_1$ and $\lambda(e, u_1) < \lambda(e, x)$ for any other atom $x \in [e, u]$. Suppose now that we have uniquely defined the elements $u_1, u_2, \ldots, u_j$, so that for every $i = 1, 2, \ldots, j$ we have $u_{i-1} \rightarrow u_i$ with $\lambda(u_{i-1}, u_i) = c_i$ and $\lambda(u_{i-1}, u_i) < \lambda(u_{i-1}, x)$ for every $x \in [e, u]$ such that $x \neq u_i$ and $u_{i-1} \rightarrow x$. We consider the number c_{j+1} and distinguish two cases.

Case 1: c_{j+1} belongs to a cycle whose elements have not been used. In this case, if c_{j+1} belongs to a balanced cycle, then we set $u_{j+1} = u_j[c_{j+1}]$, while if c_{j+1} belongs to p_s for some $s \in \{1, 2, \ldots, l\}$, then we set u_{j+1} to be either $u_j((p_s^1, c_{j+1}))$ or $u_j((p_s^1, -c_{j+1}))$, so that $u_j^{-1}u_{j+1} \preceq p_s$ holds.

Case 2: c_{j+1} belongs to a cycle some element of which has been used. Then there exist an $i < j+1$ such that c_i belongs to the same cycle as c_{j+1}. If c_i, c_{j+1} belong to some b_s, then there is a balanced cycle of u_j, say a, that contains c_i. In this case we set u_{j+1} to be the permutation that we obtain from u_j if we add the number c_{j+1} in the cycle a in the same order and with the same sign that it appears in b_s. We proceed similarly if c_i, c_{j+1} belong the same paired cycle.

Using the relations written in Section 2.2, one can show that $\mathcal{C}_u$ is lexicographically first and the unique strictly increasing chain in $[e, u]$. Thus Theorem 3.1 is proved. □

Example 3.2 (i) Let $n = 7$ and $u = [1, -7][3]((2, -6, -5))((4)) \in B_7$. Then $c(u) = (1, 3, 5, 6, 7)$ and
$$\mathcal{C}_u : e \xrightarrow{1} [1] \xrightarrow{3} [1][3] \xrightarrow{5} [1][3]((2, -5)) \xrightarrow{6} [1][3]((2, -6, -5)) \xrightarrow{7} u.$$

(ii) Let $n = 4$ and $v = [3, -4]((1, 2))$. Then $c(v) = (2, 3, 4)$ and $\mathcal{C}_v : e \xrightarrow{2} ((1, 2)) \xrightarrow{3} ((1, 2))[3] \xrightarrow{4} v$.

Example 3.3 Figure 2 illustrates the Hasse diagram of the interval $I = [e, x]$ of $\mathrm{Abs}(D_4)$, where $x = [1][2][3][4]$. Note that the Hasse diagram of the open interval (e, x) is disconnected and, therefore, I is not Cohen-Macaulay. It follows that $\mathrm{Abs}(D_n)$ is neither Cohen-Macaulay nor shellable for $n \geq 4$ [19, Corollary 3.1.9]. This is in accordance with Reiner's computations showing that $\mathrm{Abs}(D_4)$ is not Cohen-Macaulay and answers in the negative a question raised by Athanasiadis (personal communication), asking whether all intervals of the absolute order on a Coxeter group are shellable.
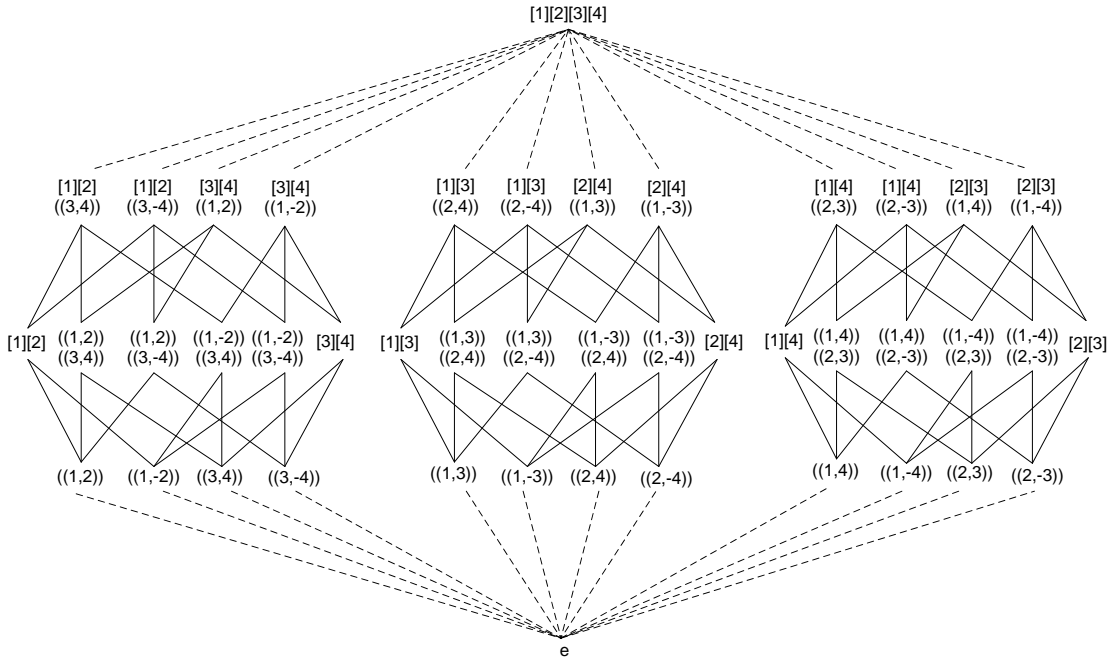
Fig. 2: The interval $[e, [1][2][3][4]]$ in D_4

4 The ideal of Coxeter elements

Recall that the Coxeter elements of B_n are precisely the balanced n-cycles.

Theorem 4.1 *The order ideal $\mathcal{J}_n$ of Abs(B_n) generated by the set of Coxeter elements of B_n is homotopy Cohen-Macaulay for all $n \geq 2$. In particular, the order complex of $\mathcal{J}_n \setminus \{e\}$ is homotopy equivalent to a wedge of $(n-1)$-dimensional spheres and Cohen-Macaulay over $\mathbb{Z}$.*

Since the set of maximal elements of Abs(S_n) coincides with the set of Coxeter elements of S_n, Theorem 4 can be considered as a B_n-analogue of [4, Theorem 1.1]. It is not known whether the order ideal generated by the Coxeter elements is Cohen-Macaulay for every Coxeter group W. To prove Theorem 4.1 we will use the notion of strong constructibility, introduced in [4]. We first review some definitions and results given in [4].

Definition 4.2 *A d-dimensional simplicial complex Δ is constructible if it is a simplex or it can be written as $\Delta = \Delta_1 \cup \Delta_2$, where Δ_1, Δ_2 are d-dimensional constructible simplicial complexes such that $\Delta_1 \cap \Delta_2$ is constructible of dimension at least $d - 1$.*

We do not know whether this notion of constructibility coincides with the classical notion, which differs in that the dimension of the intersection $\Delta_1 \cap \Delta_2$ has to equal to $d - 1$. However, it is proved in [4] that every constructible simplicial complex, in the sense of Definition 4.2, is homotopy Cohen-Macaulay. Figure 3 illustrates two 2-dimensional strongly constructible complexes, Δ_1 and Δ_2, the intersection of which is the 2-dimensional simplex F_3. Thus, the union $\Delta_1 \cup \Delta_2$ is strongly constructible as well.
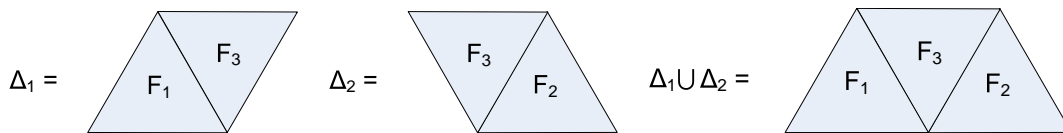
Fig. 3: A 2-dimensional strongly constructible simplicial complex

Definition 4.3 *A finite poset P of rank d with a minimum element is strongly constructible if it is bounded and pure shellable or it can be written as a union $P = I_1 \cup I_2$ of two strongly constructible proper ideals I_1, I_2 of rank d, such that $I_1 \cap I_2$ is strongly constructible of rank at least $d - 1$.*

Proposition 4.4 *The order complex of any strongly constructible poset is constructible.*

Remark 4.5 Every strongly constructible poset is homotopy Cohen-Macaulay.

Proposition 4.6 *The poset $Abs(S_n)$ is strongly constructible for every $n \geq 1$.*

The main idea to prove Proposition 4.6 is to partition the set of maximal elements (n-cycles) of $Abs(S_n)$ by placing x and y in the same part of the partition if $x(1) = y(1)$. This is the partition of S_n into the left cosets of the subgroup which consists of the permutations of the set $\{2, 3, \ldots, n\}$. Then we show that the order ideal generated by each part is strongly constructible and that so is the intersection of two or more of these ideals. We extend this construction to the case of $\mathcal{J}_n \subset B_n$ by defining the following equivalence relation on the set of cycles of B_n.

Definition 4.7 *Given cycles u, v of B_n, we write $u \sim v$ if*

- *u, v are either both paired or both balanced cycles and*

- *$u(i) = \pm v(i)$ for every $i = 1, 2, \ldots, n$.*

We denote by $\bar{u}$ the equivalence class of $u \in B_n$. If $u_1, u_2, \ldots, u_k$ are disjoint cycles of B_n, we set $\bar{u}_1 \bar{u}_2 \cdots \bar{u}_k = \{v_1 v_2 \cdots v_k : v_i \in \bar{u}_i, i = 1, 2, \ldots, k\} \subset B_n$. For example,

$$\overline{((1,2))}\,\overline{[3,4]} = \{((1,2))[3,4], ((1,-2))[3,4], ((1,2))[3,-4], ((1,-2))[3,-4]\}.$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a sequence of distinct positive integers, with $\alpha_i \leq n$ for every $i = 1, 2, \ldots, n$. To the sequence α we associate the permutations $[\alpha] = [\alpha_1, \alpha_2, \ldots, \alpha_k]$ and $((\alpha)) = ((\alpha_1, \alpha_2, \ldots, \alpha_k))$ of B_n and the cycle $(\alpha) = (\alpha_1\, \alpha_2\, \cdots\, \alpha_k)$ of S_n. Let A be a subset of S_n consisting of permutations that have length equal to $n - k$ (i.e. permutations of S_n that have exactly k cycles in their decomposition). To the order ideal $\langle A \rangle$ of $Abs(S_n)$, which has rank $n - k$, we associate the order ideal $\overline{\langle A \rangle}$ of $Abs(B_n)$, which has rank $n - k + 1$ and is defined as:

$$\overline{\langle A \rangle} := \langle x \in \overline{[\alpha_1]}\,\overline{((\alpha_2))}\,\overline{((\alpha_3))} \cdots \overline{((\alpha_k))} : (\alpha_1)(\alpha_2)(\alpha_3) \cdots (\alpha_k) \in A \rangle.$$

Let A_1, A_2 be subsets of S_n as above. Then $\overline{\langle A_1 \rangle} \cup \overline{\langle A_2 \rangle} = \overline{\langle A_1 \cup A_2 \rangle}$ and $\overline{\langle A_1 \rangle} \cap \overline{\langle A_2 \rangle} = \overline{\langle \langle A_1 \rangle \cap \langle A_2 \rangle \rangle}$. This connection allows us to compute intersections of ideals generated by certain equivalence classes, using the intersections of ideals generated by the corresponding cycles in S_n and to adapt the proof of [4, Proposition 4.2].

The next result is a B_n-analogue of [4, Theorem 1.2].

Theorem 4.8 *Let $\mathcal{J}_n$ denote the order ideal of $Abs(B_n)$ generated by the Coxeter elements of B_n and $\bar{\mathcal{J}}_n = \mathcal{J}_n \setminus \{\hat{0}\}$. The reduced Euler characteristic of the order complex $\Delta(\bar{\mathcal{J}}_n)$ satisfies*

$$\sum_{n \geq 2} (-1)^n \tilde{\chi}(\Delta(\bar{\mathcal{J}}_n)) \frac{t^n}{n!} = 1 - \sqrt{C(2t)} \exp\left\{-2tC(2t)\right\} \left(1 + \sum_{n \geq 1} 2^{n-1} \binom{2n-1}{n} \frac{t^n}{n}\right),$$

where $C(t) = \frac{1}{2t}\left(1 - \sqrt{1-4t}\right)$ is the ordinary generating function for the Catalan numbers.

5   Combinatorics of intervals

5.1   Intervals with the lattice property

In this section we characterize the maximal intervals in $Abs(B_n)$ and $Abs(D_n)$ which are lattices. It is known that the interval $[e, c]$ of $Abs(W)$ is a lattice for every finite Coxeter group W and Coxeter element c of W (see [5, Fact 2.3.1], [11, Section 4], [12]). Moreover, it was shown in [16, Theorem 1.6] that $[e, x]$ is a lattice for every maximal element x of $Abs(B_n)$ that is a product of exactly two Coxeter elements, one of which is a reflection.

Theorem 5.1 *Let x be a maximal element of $Abs(B_n)$. The interval $[e, x]$ of $Abs(B_n)$ is a lattice if and only if x has the form*

$$x = [a_1, a_2, \ldots, a_k][a_{k+1}][a_{k+2}] \cdots [a_n]$$

where $k \in \{0, 1, \ldots, n\}$ and the $a_i \in \{\pm 1, \pm 2, \ldots, \pm n\}$ have pairwise disjoint absolute values.

We now consider the absolute order on the group D_n and prove the following theorem.

Theorem 5.2 *Let x be a maximal element of $Abs(D_n)$. The interval $[e, x]$ of $Abs(D_n)$ is a lattice if and only if x is a Coxeter element or $n = 4$ and $x = [1][2][3][4]$.*

Proof: As previously mentioned, the interval $[e, x]$ of $Abs(D_n)$, where x ia a Coxeter element of D_n, is known to be a lattice. Let x be a maximal non-Coxeter element of $Abs(D_n)$ such that the interval $[e, x]$ of $Abs(D_n)$ is a lattice. One can show that in this case at most one cycle of x is not a reflection. Thus we may write $x = [a_1, a_2, \ldots, a_m][b_2] \cdots [b_k]$, where $k > 2$ and $m + k - 1 = n$. Suppose that $m \geq 2$. Then $u = [a_1, a_2][b_2]$ and $v = [a_1, a_2][b_3]$ are elements of $[e, x]$. However, the intersection $[e, u] \cap [e, v] \subset Abs(D_n)$ has two maximal elements, namely the paired reflections $((a_1, a_2))$ and $((a_1, -a_2))$. This implies that the elements u and v do not have a meet in $[e, x]$ and, therefore, the interval $[e, x]$ is not a lattice. Thus we must have $m = 1$, so $k = n$ and $x = [1][2] \cdots [n]$. Suppose that $n \geq 5$. We consider the elements $u = [1][2][3][4]$ and $v = [1][2][3][5]$ of $[e, x]$ and note that the intersection $[e, u] \cap [e, v]$ has three maximal elements, namely $[1][2]$, $[1][3]$ and $[2][3]$. This implies that the interval $[e, x]$ is not a lattice, contradicting our assumption. Thus $n = 4$ and $x = [1][2][3][4]$. Figure 2 shows that the interval $[e, [1][2][3][4]]$ is indeed a lattice and Theorem 5.2 is proved.   □

Remark 5.3 For the remainder of this paper we denote by $L(k, r)$ the order ideal of $Abs(B_n)$ generated by the element $[1, 2, \ldots, k][k+1] \cdots [k+r]$, where k, r are nonnegative integers such that $k + r \leq n$. Moreover, we set $L_n := L(0, n) = [e, [1][2] \cdots [n]]$.

5.2 The lattice L_n

We compute some of the basic enumerative invariants of the lattice L_n, as follows.

Proposition 5.4 For the lattice L_n the following hold:

(i) The number of elements of L_n is equal to

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^{n-k}(2k-1)!!.$$

(ii) The number of elements of L_n of rank r is equal to

$$\sum_{k=0}^{\min\{r,n-r\}} \frac{n!}{k!(r-k)!(n-r-k)!}.$$

(iii) The zeta polynomial of L_n is given by the formula

$$Z_n(m) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} m^{n-k}(m-1)^k(2k-1)!!.$$

(iv) The number of maximal chains of L_n is equal to

$$n! \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}(2k-1)!!.$$

(v) For the Möbius function of L_n we have

$$\mu_n(\hat{0}, \hat{1}) = (-1)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^k(2k-1)!!,$$

where $\hat{0}$ and $\hat{1}$ denotes the minimum and the maximum element of L_n, respectively.

Remark 5.5 By the proof of Theorem 3.1, the lattice L_n is EL-shellable. We describe two more EL-labelings for L_n.

- Let $\Lambda = \{[i] : i = 1, 2, \ldots, n\} \cup \{((i,j)) : 1 \leq i < j \leq n\}$. We linearly order the elements of Λ in the following way. We first order the balanced reflections so that $[i] <_\Lambda [j]$ if and only if $i < j$. Then we order the paired reflections lexicographically. Finally, we set $[n] <_\Lambda ((1,2))$. The map $\lambda_1 : C(B_n) \to \Lambda$ defined as

$$\lambda_1(a, b) = \begin{cases} [i] & \text{if } a^{-1}b = [i], \\ ((i,j)) & \text{if } a^{-1}b = ((i,j)) \text{ or } ((i,-j)) \end{cases}$$

is an EL-labeling for L_n.

- Let $\mathcal{T}$ be the set of reflections of B_n. We define a total order $<_{\mathcal{T}}$ on $\mathcal{T}$ which extends the order $<_{\Lambda}$, by ordering the reflections $((i, -j))$, for $1 \leq i < j \leq n$, lexicographically and letting $((n - 1, n)) <_{\mathcal{T}} ((1, -2))$. Let t_i be the i-th reflection in the order above. We define a map $\lambda_2 : C(B_n) \to \{1, 2, \ldots, n^2\}$ as

$$\lambda_2(a, b) = \min_{1 \leq i \leq n^2} \{i : t_i \vee a = b\},$$

where $t_i \vee a$ denotes the join of t_i and a in the lattice L_n. The map λ_2 is an EL-labeling for L_n.

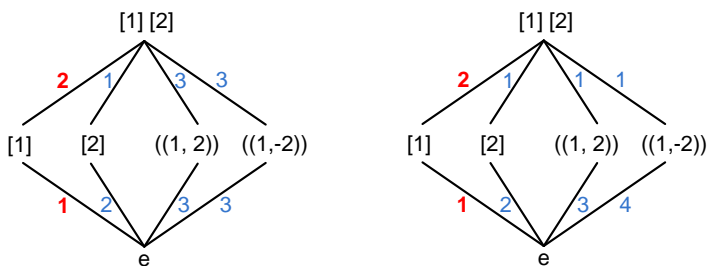See Figure 4 for an example of these two EL-labelings when $n = 2$.

Fig. 4: EL-labelings for L_2

5.3   Enumerative combinatorics of $L(k, r)$

In this section we compute the cardinality, zeta polynomial and Möbius function of $L(k, r)$, where k, r are nonnegative integers with $k + r = n$. The case $k = n - 1$ was treated by Goulden, Nica and Oancea [14]. We will use their results, as well as the formulas for the cardinality and zeta polynomial for $NC^B(n)$ and Proposition 5.4, to find the corresponding formulas for $L(k, r)$.

Proposition 5.6 *Let $\alpha_r = |L_r|$, $\beta_r(m) = Z(L_r, m)$ and $\mu_r = \mu_r(L_r)$, where $\alpha_r = \beta_r(m) = \mu_r = 1$ for $r \in \{0, 1\}$. For fixed nonnegative integers k, r such that $k + r = n$, the cardinality, zeta polynomial and Möbius function of the lattice*

$$L(k, r) = [e, [1, 2, \ldots, k][k + 1] \cdots [k + r]]$$

are given by:

- $\#L(k, r) = \binom{2k}{k} \left(\dfrac{2\, r\, k}{k + 1} \alpha_{r-1} + a_r \right),$

- $Z(L(k, r), m) = \binom{mk}{k} \left(\dfrac{2\, r\, k}{k + 1} (m - 1)\, \beta_{r-1}(m) + \beta_r(m) \right),$

- $\mu(L(k, r)) = (-1)^n \binom{2k - 1}{k} \left(\dfrac{4\, r\, k}{k + 1} |\mu_{r-1}| + |\mu_r| \right).$

Proof: (sketch) We denote by A the subset of $L(k,r)$ which consists of elements x with the following property: every cycle of x that contains at least one of $\pm 1, \pm 2, \ldots, \pm k$ is less than or equal to the element $[1, 2, \ldots, k]$ in $\mathrm{Abs}(B_n)$. Let $x = x_1 x_2 \cdots x_\nu \in A$, written as a product of disjoint cycles. Without loss of generality, we may assume that there is a $t \in \{0, 1, \ldots, \nu\}$ such that $x_1 x_2 \cdots x_t \preceq [1, 2, \ldots, k]$ and $x_{t+1} x_{t+2} \cdots x_\nu \preceq [k+1][k+2] \cdots [k+r]$. Clearly, there exist a poset isomorphism

$$f : A \to NC^B(k) \ \times \ [e, [k+1] \cdots [k+r]]$$
$$x \mapsto (x_1 \cdots x_t \ , \ x_{t+1} \cdots x_\nu),$$

so that

$$A \cong NC^B(k) \times L_r. \tag{3}$$

Let $C := L(k,r) \backslash A$ and $x = x_1 x_2 \cdots x_\nu \in C$, written as a product of disjoint cycles. Then there exists a paired cycle of x, say x_1, and a reflection $((i,j))$ with $|j| \in \{1, 2, \ldots, k\}$, $j \in \{k+1, k+2, \ldots, k+r\}$, such that $((i,j)) \preceq x_1$. Note that the cycle x_1 and the reflection $((i,j))$ are unique with this property. For every $j \in \{k+1, k+2, \ldots, k+r\}$ denote by C_j the set of permutations $x \in L(k,r)$ which have a cycle, say x_1, such that $((i,j)) \preceq x_1$ for some $i \in \{\pm 1, \pm 2, \ldots, \pm k\}$. Thus, for every $x \in C$ there exists an ordering $x_1, x_2, \ldots, x_\nu$ of the cycles of x and a unique index $t \in \{1, 2, \ldots, \nu\}$ such that $x_1 x_2 \cdots x_t \preceq [1, 2, \ldots, k][j]$ and $x_{t+1} x_{t+2} \cdots x_\nu \preceq [k+1][k+2] \cdots [j-1][j+1] \cdots [k+r]$. Let

$$E_j = \{x \in C : x \preceq [1, 2, \ldots, k][j]\}.$$

Clearly, there exist a poset isomorphism

$$g_j : C_j \to \quad E_j \quad \times \ [e, [k+1] \cdots [j-1][j+1] \cdots [k+r]]$$
$$x \mapsto (x_1 \cdots x_t \ , \ x_{t+1} \cdots x_\nu),$$

so that

$$C_j \cong E_l \times L(0, r-1). \tag{4}$$

The results follow by using the poset isomorphisms (3) and (4) and [14, Section 5]. $\qquad\square$

Acknowledgments

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