

Perfectness of Kirillov–Reshetikhin crystals for nonexceptional types

Ghislain Fourier, Masato Okado, Anne Schilling

► **To cite this version:**

Ghislain Fourier, Masato Okado, Anne Schilling. Perfectness of Kirillov–Reshetikhin crystals for nonexceptional types. Krattenthaler, Christian and Strehl, Volker and Kauers, Manuel. 21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009), 2009, Hagenberg, Austria. Discrete Mathematics and Theoretical Computer Science, DMTCS Proceedings vol. AK, 21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009), pp.421-433, 2009, DMTCS Proceedings. <hal-01185434>

HAL Id: hal-01185434

<https://hal.inria.fr/hal-01185434>

Submitted on 20 Aug 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Perfectness of Kirillov–Reshetikhin crystals for nonexceptional types

Ghislain Fourier^{1†}, Masato Okado^{2‡} and Anne Schilling^{3§}

¹ *Mathematisches Institut der Universität zu Köln, Weyertal 86-90, 50931 Köln, Germany*

² *Department of Mathematical Science, Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka 560-8531, Japan*

³ *Department of Mathematics, University of California, One Shields Avenue, Davis, CA 95616-8633, U.S.A.*

Abstract. For nonexceptional types, we prove a conjecture of Hatayama et al. about the perfectness of Kirillov–Reshetikhin crystals.

Résumé. Pour les types non-exceptionnels, on démontre une conjecture de Hatayama et al. concernant la perfection des cristaux de Kirillov–Reshetikhin.

Keywords: Crystal bases, combinatorial models for Kirillov–Reshetikhin crystals, perfectness

1 Introduction

Kirillov–Reshetikhin (KR) crystals $B^{r,s}$ are crystals corresponding to finite-dimensional $U'_q(\mathfrak{g})$ -modules [3, 4], where \mathfrak{g} is an affine Kac–Moody algebra. Recently, a lot of progress has been made regarding long outstanding problems concerning these crystals which appear in mathematical physics and the path realization of affine highest weight crystals [13]. In [20, 21] the existence of KR crystals was shown. In [5] a major step in understanding these crystals was provided by giving explicit combinatorial realizations for all nonexceptional types. This abstract is based on [5, 6]. We prove a conjecture of Hatayama, Kuniba, Okado, Takagu, and Tsuboi [8, Conjecture 2.1] about the perfectness of these KR crystals.

Conjecture 1.1 [8, Conjecture 2.1] *The Kirillov–Reshetikhin crystal $B^{r,s}$ is perfect if and only if $\frac{s}{c_r}$ is an integer with c_r as in Table 1. If $B^{r,s}$ is perfect, its level is $\frac{s}{c_r}$.*

In [14], this conjecture was proven for all $B^{r,s}$ for type $A_n^{(1)}$, for $B^{1,s}$ for nonexceptional types (except for type $C_n^{(1)}$), for $B^{n-1,s}$, $B^{n,s}$ of type $D_n^{(1)}$, and $B^{n,s}$ for types $C_n^{(1)}$ and $D_{n+1}^{(2)}$. When the highest weight is given by the highest root, level-1 perfect crystals were constructed in [1]. For $1 \leq r \leq n-2$

[†]Supported in part by DARPA and AFOSR through the grant FA9550-07-1-0543 and by the DFG-Projekt “Kombinatorische Beschreibung von Macdonald und Kostka–Foulkes Polynomen”.

[‡]Supported by grant JSPS 20540016.

[§]Supported in part by the NSF grants DMS–0501101, DMS–0652641, and DMS–0652652.

	(c_1, \dots, c_n)
$A_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$	$(1, \dots, 1)$
$B_n^{(1)}$	$(1, \dots, 1, 2)$
$C_n^{(1)}$	$(2, \dots, 2, 1)$

Tab. 1: List of c_r

for type $D_n^{(1)}$, $1 \leq r \leq n-1$ for type $B_n^{(1)}$, and $1 \leq r \leq n$ for type $A_{2n-1}^{(2)}$, the conjecture was proved in [22]. The case $G_2^{(1)}$ and $r=1$ was treated in [24] and the case $D_4^{(3)}$ and $r=1$ was treated in [16]. Naito and Sagaki [18] showed that the conjecture holds for twisted algebras, if it is true for the untwisted simply-laced cases.

In this paper we prove Conjecture 1.1 in general for nonexceptional types.

Theorem 1.2 *If \mathfrak{g} is of nonexceptional type, Conjecture 1.1 is true.*

The paper is organized as follows. In Section 2 we give basic notation and the definition of perfectness in Definition 2.1. In Section 3 we review the realizations of the KR crystals of nonexceptional types as recently provided in [5]. Section 4 is reserved for the proof of Theorem 1.2 and an explicit description of the minimal elements $B_{\min}^{r, c_r, s}$ of the perfect crystals. A long version of this article containing further details and examples is available at [6].

2 Definitions and perfectness

We follow the notation of [12, 5]. Let \mathcal{B} be a $U'_q(\mathfrak{g})$ -crystal [15]. Denote by α_i and Λ_i for $i \in I$ the simple roots and fundamental weights and by c the canonical central element associated to \mathfrak{g} , where I is the index set of the Dynkin diagram of \mathfrak{g} (see Table 2). Let $P = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i$ be the weight lattice of \mathfrak{g} and P^+ the set of dominant weights. For a positive integer ℓ , the set of level- ℓ weights is

$$P_\ell^+ = \{\Lambda \in P^+ \mid \text{lev}(\Lambda) = \ell\},$$

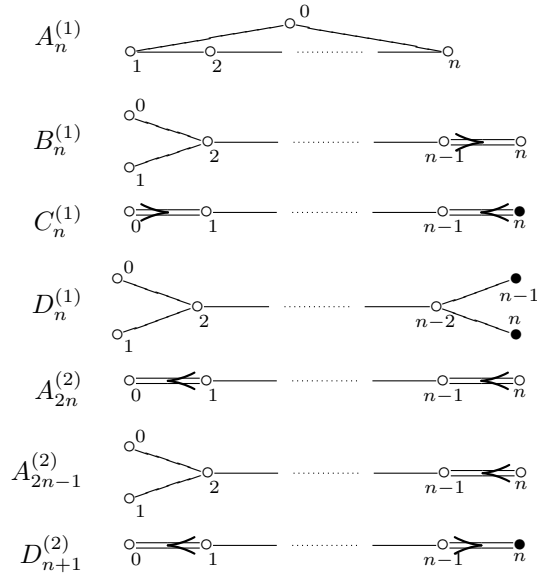
where $\text{lev}(\Lambda) := \Lambda(c)$. The set of level-0 weights is denoted by P_0 . We identify dominant weights with partitions; each Λ_i yields a column of height i (except for spin nodes). For more details, please consult [11].

We denote by $f_i, e_i : \mathcal{B} \rightarrow \mathcal{B} \cup \{\emptyset\}$ for $i \in I$ the Kashiwara operators and by $\text{wt} : \mathcal{B} \rightarrow P$ the weight function on the crystal. For $b \in \mathcal{B}$ we define $\varepsilon_i(b) = \max\{k \mid e_i^k(b) \neq \emptyset\}$, $\varphi_i(b) = \max\{k \mid f_i^k(b) \neq \emptyset\}$, and

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i.$$

Next we define perfect crystals, see for example [11].

Definition 2.1 *For a positive integer $\ell > 0$, a crystal \mathcal{B} is called perfect crystal of level ℓ , if the following conditions are satisfied:*



Tab. 2: Dynkin diagrams

1. \mathcal{B} is isomorphic to the crystal graph of a finite-dimensional $U'_q(\mathfrak{g})$ -module.
2. $\mathcal{B} \otimes \mathcal{B}$ is connected.
3. There exists a $\lambda \in P_0$, such that $\text{wt}(\mathcal{B}) \subset \lambda + \sum_{i \in I \setminus \{0\}} \mathbb{Z}_{\leq 0} \alpha_i$ and there is a unique element in \mathcal{B} of classical weight λ .
4. $\forall b \in \mathcal{B}, \text{lev}(\varepsilon(b)) \geq \ell$.
5. $\forall \Lambda \in P_\ell^+, \text{ there exist unique elements } b_\Lambda, b^\Lambda \in \mathcal{B}, \text{ such that}$

$$\varepsilon(b_\Lambda) = \Lambda = \varphi(b^\Lambda).$$

We denote by \mathcal{B}_{\min} the set of minimal elements in \mathcal{B} , namely

$$\mathcal{B}_{\min} = \{b \in \mathcal{B} \mid \text{lev}(\varepsilon(b)) = \ell\}.$$

Note that condition (5) of Definition 2.1 ensures that $\varepsilon, \varphi : \mathcal{B}_{\min} \rightarrow P_\ell^+$ are bijections. They induce an automorphism $\tau = \varepsilon \circ \varphi^{-1}$ on P_ℓ^+ .

In [22, 5] \pm -diagrams were introduced, which describe the branching $X_n \rightarrow X_{n-1}$ where $X_n = B_n, C_n, D_n$. A \pm -diagram P of shape Λ/λ is a sequence of partitions $\lambda \subset \mu \subset \Lambda$ such that Λ/μ and μ/λ are horizontal strips (i.e. every column contains at most one box). We depict this \pm -diagram by the skew

tableau of shape Λ/λ in which the cells of μ/λ are filled with the symbol $+$ and those of Λ/μ are filled with the symbol $-$. There are further type specific rules which can be found in [5, Section 3.2]. There exists a bijection Φ between \pm -diagrams and the X_{n-1} -highest weight vectors inside the X_n crystal of highest weight Λ .

3 Realization of KR-crystals

Throughout the paper we use the realization of $B^{r,s}$ as given in [5, 21, 22]. In this section we briefly recall the main constructions.

3.1 KR crystals of type $A_n^{(1)}$

Let $\Lambda = \ell_0\Lambda_0 + \ell_1\Lambda_1 + \dots + \ell_n\Lambda_n$ be a dominant weight. Then the level is given by

$$\text{lev}(\Lambda) = \ell_0 + \dots + \ell_n.$$

A combinatorial description of $B^{r,s}$ of type $A_n^{(1)}$ was provided by Shimozono [23]. As a $\{1, 2, \dots, n\}$ -crystal

$$B^{r,s} \cong B(s\Lambda_r).$$

The Dynkin diagram of $A_n^{(1)}$ has a cyclic automorphism $\sigma(i) = i + 1 \pmod{n + 1}$ which extends to the crystal in form of the promotion operator. The action of the affine crystal operators f_0 and e_0 is given by

$$f_0 = \sigma^{-1} \circ f_1 \circ \sigma \quad \text{and} \quad e_0 = \sigma^{-1} \circ e_1 \circ \sigma.$$

3.2 KR crystals of type $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$

Let $\Lambda = \ell_0\Lambda_0 + \ell_1\Lambda_1 + \dots + \ell_n\Lambda_n$ be a dominant weight. Then the level is given by

$$\begin{aligned} \text{lev}(\Lambda) &= \ell_0 + \ell_1 + 2\ell_2 + 2\ell_3 + \dots + 2\ell_{n-2} + \ell_{n-1} + \ell_n && \text{for type } D_n^{(1)} \\ \text{lev}(\Lambda) &= \ell_0 + \ell_1 + 2\ell_2 + 2\ell_3 + \dots + 2\ell_{n-2} + 2\ell_{n-1} + \ell_n && \text{for type } B_n^{(1)} \\ \text{lev}(\Lambda) &= \ell_0 + \ell_1 + 2\ell_2 + 2\ell_3 + \dots + 2\ell_{n-2} + 2\ell_{n-1} + 2\ell_n && \text{for type } A_{2n-1}^{(2)}. \end{aligned} \tag{3.1}$$

We have the following realization of $B^{r,s}$. Let $X_n = D_n, B_n, C_n$ be the classical subalgebra for $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$, respectively.

Definition 3.1 *Let $1 \leq r \leq n - 2$ for type $D_n^{(1)}$, $1 \leq r \leq n - 1$ for type $B_n^{(1)}$, and $1 \leq r \leq n$ for type $A_{2n-1}^{(2)}$. Then $B^{r,s}$ is defined as follows. As an X_n -crystal*

$$B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda), \tag{3.2}$$

where the sum runs over all dominant weights Λ that can be obtained from $s\Lambda_r$ by the removal of vertical dominoes. The affine crystal operators e_0 and f_0 are defined as

$$f_0 = \sigma^{-1} \circ f_1 \circ \sigma \quad \text{and} \quad e_0 = \sigma^{-1} \circ e_1 \circ \sigma, \tag{3.3}$$

where σ is the crystal automorphism defined in [22, Definition 4.2].

Definition 3.2 Let $B_{A_{2n-1}^{(2)}}^{n,s}$ be the $A_{2n-1}^{(2)}$ -KR crystal. Then $B^{n,s}$ of type $B_n^{(1)}$ is defined through the unique injective map $S : B^{n,s} \rightarrow B_{A_{2n-1}^{(2)}}^{n,s}$ such that

$$S(e_i b) = e_i^{m_i} S(b), \quad S(f_i b) = f_i^{m_i} S(b) \quad \text{for } i \in I,$$

where $(m_i)_{0 \leq i \leq n} = (2, 2, \dots, 2, 1)$.

In addition, the \pm -diagrams of $A_{2n-1}^{(2)}$ that occur in the image are precisely those which can be obtained by doubling a \pm -diagram of $B^{n,s}$ (see [5, Lemma 3.5]). S induces an embedding of dominant weights of $B_n^{(1)}$ into dominant weights of $A_{2n-1}^{(2)}$, namely $S(\Lambda_i) = m_i \Lambda_i$. It is easy to see that for any $\Lambda \in P^+$ we have $\text{lev}(S(\Lambda)) = 2 \text{lev}(\Lambda)$ using (3.1).

For the definition of $B^{n,s}$ and $B^{n-1,s}$ of type $D_n^{(1)}$, see for example [5, Section 6.2].

3.3 KR crystal of type $C_n^{(1)}$

The level of a dominant $C_n^{(1)}$ weight $\Lambda = \ell_0 \Lambda_0 + \dots + \ell_n \Lambda_n$ is given by

$$\text{lev}(\Lambda) = \ell_0 + \dots + \ell_n.$$

We use the realization of $B^{r,s}$ as the fixed point set of the automorphism σ [22, Definition 4.2] (see Definition 3.1) inside $B_{A_{2n+1}^{(2)}}^{r,s}$ of [5, Theorem 5.7].

Definition 3.3 For $1 \leq r < n$, the KR crystal $B^{r,s}$ of type $C_n^{(1)}$ is defined to be the fixed point set under σ inside $B_{A_{2n+1}^{(2)}}^{r,s}$ with the operators

$$e_i = \begin{cases} e_0 e_1 & \text{for } i = 0, \\ e_{i+1} & \text{for } 1 \leq i \leq n, \end{cases}$$

where the Kashiwara operators on the right act in $B_{A_{2n+1}^{(2)}}^{r,s}$. Under the crystal embedding $S : B^{r,s} \rightarrow B_{A_{2n+1}^{(2)}}^{r,s}$ we have

$$\Lambda_i \mapsto \begin{cases} \Lambda_0 + \Lambda_1 & \text{for } i = 0, \\ \Lambda_{i+1} & \text{for } 1 \leq i \leq n. \end{cases}$$

Under the embedding S , the level of $\Lambda \in P^+$ doubles, that is $\text{lev}(S(\Lambda)) = 2 \text{lev}(\Lambda)$.

For $B^{n,s}$ of type $C_n^{(1)}$ we refer to [5, Section 6.1].

3.4 KR crystals of type $A_{2n}^{(2)}, D_{n+1}^{(2)}$

Let $\Lambda = \ell_0 \Lambda_0 + \ell_1 \Lambda_1 + \dots + \ell_n \Lambda_n$ be a dominant weight. The level is given by

$$\begin{aligned} \text{lev}(\Lambda) &= \ell_0 + 2\ell_1 + 2\ell_2 + \dots + 2\ell_{n-2} + 2\ell_{n-1} + 2\ell_n && \text{for type } A_{2n}^{(2)} \\ \text{lev}(\Lambda) &= \ell_0 + 2\ell_1 + 2\ell_2 + \dots + 2\ell_{n-2} + 2\ell_{n-1} + \ell_n && \text{for type } D_{n+1}^{(2)}. \end{aligned}$$

Define positive integers m_i for $i \in I$ as follows:

$$(m_0, m_1, \dots, m_{n-1}, m_n) = \begin{cases} (1, 2, \dots, 2, 2) & \text{for } A_{2n}^{(2)}, \\ (1, 2, \dots, 2, 1) & \text{for } D_{n+1}^{(2)}. \end{cases} \quad (3.4)$$

Then $B^{r,s}$ can be realized as follows.

Definition 3.4 For $1 \leq r \leq n$ for $\mathfrak{g} = A_{2n}^{(2)}$, $1 \leq r < n$ for $\mathfrak{g} = D_{n+1}^{(2)}$ and $s \geq 1$, there exists a unique injective map $S : B_{\mathfrak{g}}^{r,s} \rightarrow B_{C_n^{(1)}}^{r,2s}$ such that

$$S(e_i b) = e_i^{m_i} S(b), \quad S(f_i b) = f_i^{m_i} S(b) \quad \text{for } i \in I.$$

The \pm -diagrams of $C_n^{(1)}$ that occur in the image of S are precisely those which can be obtained by doubling a \pm -diagram of $B^{r,s}$ (see [5, Lemma 3.5]). S induces an embedding of dominant weights for $A_{2n}^{(2)}, D_{n+1}^{(2)}$ into dominant weights of type $C_n^{(1)}$, with $S(\Lambda_i) = m_i \Lambda_i$. This map preserves the level of a weight, that is $\text{lev}(S(\Lambda)) = \text{lev}(\Lambda)$.

For the case $r = n$ of type $D_{n+1}^{(2)}$ we refer to [5, Definition 6.2].

4 Proof of Theorem 1.2

For type $A_n^{(1)}$, perfectness of $B^{r,s}$ was proven in [14]. For all other types, in the case that $\frac{s}{c_r}$ is an integer, we need to show that the 5 defining conditions in Definition 2.1 are satisfied:

1. This was recently shown in [21].
2. This follows from [7, Corollary 6.1] under [7, Assumption 1]. Assumption 1 is satisfied except for type $A_{2n}^{(2)}$: The regularity of $B^{r,s}$ is ensured by (1), the existence of an automorphism σ was proven in [5, Section 7], and the unique element $u \in B^{r,s}$ such that $\varepsilon(u) = s\Lambda_0$ and $\varphi(u) = s\Lambda_\nu$ (where $\nu = 1$ for r odd for types $B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}$, $\nu = r$ for $A_n^{(1)}$, and $\nu = 0$ otherwise) is given by the classically highest weight element in the component $B(0)$ for $\nu = 0$, $B(s\Lambda_1)$ for $\nu = 1$, and $B(s\Lambda_r)$ for $\nu = r$. Note that $\Lambda_0 = \tau(\Lambda_\nu)$, where $\tau = \varepsilon \circ \varphi^{-1}$. For type $A_{2n}^{(2)}$, perfectness follows from [18].
3. The statement is true for $\lambda = s(\Lambda_r - \Lambda_r(c)\Lambda_0)$, which follows from the decomposition formulas [2, 9, 10, 19].

Conditions (4) and (5) will be shown in the following subsections using case by case considerations: Section 4.1 for type $A_n^{(1)}$, Sections 4.2, 4.3, and 4.4 for types $B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}$, Sections 4.5 and 4.6 for type $C_n^{(1)}$, Section 4.7 for type $A_{2n}^{(2)}$, and Sections 4.8 and 4.9 for type $D_{n+1}^{(2)}$.

When $\frac{s}{c_r}$ is not an integer, we show in the subsequent sections that the minimum of the level of $\varepsilon(b)$ is the smallest integer exceeding $\frac{s}{c_r}$, and provide examples that contradict condition (5) of Definition 2.1 for each crystal, thereby proving that $B^{r,s}$ is not perfect. In the case that $\frac{s}{c_r}$ is an integer, we provide an explicit construction of the minimal elements of $B^{r,s}$.

4.1 Type $A_n^{(1)}$

It was already proven in [14] that $B^{r,s}$ is perfect. We give below its associated automorphism τ and minimal elements. τ on P is defined by

$$\tau\left(\sum_{i=0}^n k_i \Lambda_i\right) = \sum_{i=0}^n k_i \Lambda_{i-r \bmod n+1}.$$

Recall that $B^{r,s}$ is identified with the set of semistandard tableaux of $r \times s$ rectangular shape over the alphabet $\{1, 2, \dots, n+1\}$. For $b \in B^{r,s}$ let $x_{ij} = x_{ij}(b)$ denote the number of letters j in the i -th row of b for $1 \leq i \leq r, 1 \leq j \leq n+1$. Set $r' = n+1-r$, then

$$x_{ij} = 0 \quad \text{unless} \quad i \leq j \leq i+r'.$$

Let $\Lambda = \sum_{i=0}^n \ell_i \Lambda_i$ be in P_s^+ , that is, $\ell_0, \ell_1, \dots, \ell_n \in \mathbb{Z}_{\geq 0}, \sum_{i=0}^n \ell_i = s$. Then $x_{ij}(b)$ of the minimal element b such that $\varepsilon(b) = \Lambda$ is given by

$$\begin{aligned} x_{ii} &= \ell_0 + \sum_{\alpha=i}^{r-1} \ell_{\alpha+r'}, \\ x_{ij} &= \ell_{j-i} \quad (i < j < i+r'), \\ x_{i,i+r'} &= \sum_{\alpha=0}^{i-1} \ell_{\alpha+r'} \end{aligned} \tag{4.1}$$

for $1 \leq i \leq r$.

4.2 Types $B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}$

Conditions (4) and (5) of Definition 2.1 for $1 \leq r \leq n-2$ for type $D_n^{(1)}$, $1 \leq r \leq n-1$ for type $B_n^{(1)}$, and $1 \leq r \leq n$ for type $A_{2n-1}^{(2)}$ were shown in [22, Section 6]. To a given fundamental weight Λ_k a \pm -diagram $\text{diagram}(\Lambda_k)$ was associated. This map can be extended to any dominant weight $\Lambda = \ell_0 \Lambda_0 + \dots + \ell_n \Lambda_n$ by concatenating the columns of the \pm -diagrams of each piece. To every fundamental weight Λ_k a string of operators $f(\Lambda_k)$ can be associated as in [22, Section 6].

The minimal element b in $B^{r,s}$ that satisfies $\varepsilon(b) = \Lambda$ can now be constructed as follows

$$b = f(\Lambda_n)^{\ell_n} \dots f(\Lambda_2)^{\ell_2} \Phi(\text{diagram}(\Lambda)).$$

For $\Lambda = \sum_{i=0}^n \ell_i \Lambda_i \in P_s^+$, we have

$$\tau(\Lambda) = \begin{cases} \Lambda & \text{if } r \text{ is even,} \\ \ell_0 \Lambda_1 + \ell_1 \Lambda_0 + \sum_{i=2}^n \ell_i \Lambda_i & \text{if } r \text{ is odd,} \\ & \text{types } B_n^{(1)}, A_{2n-1}^{(2)}, \\ \ell_0 \Lambda_1 + \ell_1 \Lambda_0 + \sum_{i=2}^{n-2} \ell_i \Lambda_i + \ell_{n-1} \Lambda_n + \ell_n \Lambda_{n-1} & \text{if } r \text{ is odd, type } D_n^{(1)}. \end{cases}$$

4.3 Type $D_n^{(1)}$ for $r = n - 1, n$

The cases when $r = n, n - 1$ for type $D_n^{(1)}$ were treated in [14]. We refer to [14] or [6, Section 4.3] for an explicit description of the minimal elements.

The automorphism τ is given by

$$\tau\left(\sum_{i=0}^n \ell_i \Lambda_i\right) = \ell_0 \Lambda_{n-1} + \ell_1 \Lambda_n + \sum_{i=2}^{n-2} \ell_i \Lambda_{n-i} + \begin{cases} \ell_{n-1} \Lambda_0 + \ell_n \Lambda_1 & n \text{ even,} \\ \ell_{n-1} \Lambda_1 + \ell_n \Lambda_0 & n \text{ odd.} \end{cases}$$

4.4 Type $B_n^{(1)}$ for $r = n$

In this section we consider the perfectness of $B^{n,s}$ of type $B_n^{(1)}$.

Proposition 4.1 *We have*

$$\begin{aligned} \min\{\text{lev}(\varepsilon(b)) \mid b \in B^{n,2s+1}\} &\geq s + 1, \\ \min\{\text{lev}(\varepsilon(b)) \mid b \in B^{n,2s}\} &\geq s. \end{aligned}$$

Proof: Suppose, there exists an element $b \in B^{n,2s+1}$ with $\text{lev}(\varepsilon(b)) = p < s + 1$. Since $B^{n,2s+1}$ is embedded into $B_{A_{2n-1}^{(2)}}^{n,2s+1}$ by Definition 3.2, this would yield an element $\tilde{b} \in B_{A_{2n-1}^{(2)}}^{n,2s+1}$ with $\text{lev}(\tilde{b}) < 2s + 1$.

But this is not possible, since $B_{A_{2n-1}^{(2)}}^{n,2s+1}$ is a perfect crystal of level $2s + 1$.

Suppose there exists an element $b \in B^{n,2s}$ with $\text{lev}(\varepsilon(b)) = p < s$. By the same argument one obtains a contradiction to the level of $B_{A_{2n-1}^{(2)}}^{n,2s}$. \square

Hence to show that $B^{n,2s+1}$ is not perfect, it is enough to provide two elements $b_1, b_2 \in B_{A_{2n-1}^{(2)}}^{n,2s+1}$ which are in the realization of $B^{r,s}$ under S and satisfy $\varepsilon(b_1) = \varepsilon(b_2) = \Lambda$, where $\text{lev}(\Lambda) = 2s + 2$. We use the notation $f_{\vec{a}} = f_{a_1}^{m_1} \cdots f_{a_k}^{m_k}$ for $\vec{a} = (a_1^{m_1}, \dots, a_k^{m_k})$.

Proposition 4.2 *Define the following elements $b_1, b_2 \in B_{A_{2n-1}^{(2)}}^{n,2s+1}$: For n odd, let P_1 be the \pm -diagram corresponding to one column of height n containing one $+$, and $2s$ columns of height 1 each containing a $-$ sign, and P_2 the analogous \pm -diagram but with a $-$ in the column of height n . Set $\vec{a} = (n, (n-1)^2, n, (n-2)^2, (n-1)^2, n, \dots, 2^2, \dots, (n-1)^2, n)$ and*

$$b_1 = f_{\vec{a}}(\Phi(P_1)) \quad \text{and} \quad b_2 = f_{\vec{a}}(\Phi(P_2)).$$

For n even, replace the columns of height 1 with columns of height 2 and fill them with \pm -pairs. Then $b_1, b_2 \in S(B^{n,2s+1})$ and $\varepsilon(b_1) = \varepsilon(b_2) = 2s\Lambda_1 + \Lambda_n$, which is of level $2s + 2$.

Proof: It is clear from the construction that the \pm -diagrams corresponding to b_1 and b_2 can be obtained by doubling a $B_n^{(1)}$ \pm -diagram (see [5, Lemma 3.5]). Hence $\Phi(P_1), \Phi(P_2) \in S(B^{n,2s+1})$. The sequence \vec{a} can be obtained by doubling a type $B_n^{(1)}$ sequence using $(m_1, m_2, \dots, m_n) = (2, \dots, 2, 1)$, so by Definition 3.2 b_1 and b_2 are in the image of the embedding S that realizes $B^{n,2s+1}$. The claim that $\varepsilon(b_1) = \varepsilon(b_2) = 2s\Lambda_1 + \Lambda_n$ can be checked explicitly. \square

Corollary 4.3 *The KR crystal $B^{n,2s+1}$ of type $B_n^{(1)}$ is not perfect.*

Proof: This follows directly from Proposition 4.2 using the embedding S of Definition 3.2. □

Proposition 4.4 *There exists a bijection, induced by ε , from $B_{\min}^{n,2s}$ to P_s^+ . Hence $B^{n,2s}$ is perfect of level s .*

Proof: Let S be the embedding from Definition 3.2. Then we have an induced embedding of dominant weights Λ of $B_n^{(1)}$ into dominant weights of $A_{2n-1}^{(2)}$ via the map S , that sends $\Lambda_i \mapsto m_i \Lambda_i$.

In [22, Section 6] (see Section 4.2) the minimal elements for $A_{2n-1}^{(2)}$ were constructed by giving a \pm -diagram and a sequence from the $\{2, \dots, n\}$ -highest weight to the minimal element. Since $(m_0, \dots, m_n) = (2, \dots, 2, 1)$ and columns of height n for type $A_{2n-1}^{(2)}$ are doubled, it is clear from the construction that the \pm -diagrams corresponding to weights $S(\Lambda)$ are in the image of S of \pm -diagrams for $B_n^{(1)}$ (see [5, Lemma 3.5]). Also, since under S all weights Λ_i for $1 \leq i < n$ are doubled, it follows that the sequences are “doubled” using the m_i . Hence a minimal element of $B^{n,2s}$ of level s is in one-to-one correspondence with those minimal elements in $B_{A_{2n-1}^{(2)}}^{n,2s}$ that can be obtained from doubling a \pm -diagram of $B^{n,2s}$. This implies that ε defines a bijection between $B_{\min}^{n,2s}$ and P_s^+ . □

The automorphism τ of the perfect KR crystal $B^{n,2s}$ is given by

$$\tau\left(\sum_{i=0}^n \ell_i \Lambda_i\right) = \begin{cases} \sum_{i=0}^n \ell_i \Lambda_i & \text{if } n \text{ is even,} \\ \ell_0 \Lambda_1 + \ell_1 \Lambda_0 + \sum_{i=2}^n \ell_i \Lambda_i & \text{if } n \text{ is odd.} \end{cases}$$

4.5 Type $C_n^{(1)}$

In this section we consider $B^{r,s}$ of type $C_n^{(1)}$ for $r < n$.

Proposition 4.5 *Let $r < n$. Then*

$$\begin{aligned} \min\{\text{lev}(\varepsilon(b)) \mid b \in B^{r,2s+1}\} &\geq s + 1, \\ \min\{\text{lev}(\varepsilon(b)) \mid b \in B^{r,2s}\} &\geq s. \end{aligned}$$

Proof: By Definition 3.3, the crystal $B^{r,s}$ is realized inside $B_{A_{2n+1}^{(2)}}^{r,s}$. The proof is similar to the proof of Proposition 4.1 for type $B_n^{(1)}$. □

Hence to show that $B^{r,2s+1}$ is not perfect, it suffices to give two elements $b_1, b_2 \in B_{A_{2n+1}^{(2)}}^{r,2s+1}$ that are fixed points under σ with $\varepsilon(b_1) = \varepsilon(b_2) = \Lambda$, where $\text{lev}(\Lambda) = 2s + 2$.

Proposition 4.6 *Let $b_1, b_2 \in B_{A_{2n+1}^{(2)}}^{r,2s+1}$, where b_1 consists of s columns of the form read from bottom to top $(1, 2, \dots, r)$, s columns of the form $(\bar{r}, \overline{r-1}, \dots, \bar{1})$, and a column $(\overline{r+1}, \dots, \bar{2})$. In b_2 the last column is replaced by $(r+2, \dots, 2r+2)$ if $2r+2 \leq n$ and $(r+2, \dots, n, \bar{n}, \dots, \bar{k})$ of height n otherwise. Then*

$$\varepsilon(b_1) = \varepsilon(b_2) = \begin{cases} s\Lambda_r + \Lambda_{r+1} & \text{if } r > 1, \\ s(\Lambda_0 + \Lambda_1) + \Lambda_2 & \text{if } r = 1, \end{cases}$$

which is of level $2s + 2$.

Proof: The claim is easy to check explicitly. □

Corollary 4.7 *The KR crystal $B^{n,2s+1}$ of type $C_n^{(1)}$ is not perfect.*

Proof: The $\{2, \dots, n\}$ -highest weight elements in the same component as b_1 and b_2 of Proposition 4.6 correspond to \pm -diagrams that are invariant under σ . Hence, by Definition 3.3, b_1 and b_2 are fixed points under σ . Combining this result with Proposition 4.5 proves that $B^{r,2s+1}$ is not perfect. □

Proposition 4.8 *There exists a bijection, induced by ε , from $B_{\min}^{r,2s}$ to P_s^+ . Hence $B^{r,2s}$ is perfect of level s .*

Proof: By Definition 3.3, $B^{r,s}$ of type $C_n^{(1)}$ is realized inside $B_{A_{2n+1}^{(2)}}^{r,s}$ as the fixed points under σ . Under the embedding S , it is clear that a dominant weight $\Lambda = \ell_0\Lambda_0 + \ell_1\Lambda_1 + \dots + \ell_{n+1}\Lambda_{n+1}$ of type $A_{2n+1}^{(2)}$ is in the image if and only if $\ell_0 = \ell_1$. Hence it is clear from the construction of the minimal elements for $A_{2n+1}^{(2)}$ as described in Section 4.2 that the minimal elements corresponding to Λ with $\ell_0 = \ell_1$ are invariant under σ . By [22, Theorem 6.1] there is a bijection between all dominant weights Λ of type $A_{2n+1}^{(2)}$ with $\ell_0 = \ell_1$ and $\text{lev}(\Lambda) = 2s$ and minimal elements in $B_{A_{2n+1}^{(2)}}^{r,2s}$ that are invariant under σ . Hence using S , there is a bijection between dominant weights in P_s^+ of type $C_n^{(1)}$ and $B_{\min}^{r,2s}$. □

The automorphism τ of the perfect KR crystal $B^{r,2s}$ is given by the identity.

4.6 Type $C_n^{(1)}$ for $r = n$

This case is treated in [14]. For the minimal elements, we follow the construction in Section 4.2. To every fundamental weight Λ_k we associate a column tableau $T(\Lambda_k)$ of height n whose entries are $k + 1, k + 2, \dots, n, \bar{n}, \dots, n - k + 1$ ($1, 2, \dots, n$ for $k = 0$) reading from bottom to top. Let $f(\Lambda_k)$ be defined such that $T(\Lambda_k) = f(\Lambda_k)b_1$, where b_k is the highest weight tableau in $B(k\Lambda_n)$. Then the minimal element b in $B^{n,s}$ such that $\varepsilon(b) = \Lambda = \sum_{i=0}^n \ell_i\Lambda_i \in P_s^+$ is constructed as

$$b = f(\Lambda_n)^{\ell_n} \dots f(\Lambda_1)^{\ell_1} b_s.$$

The automorphism τ is given by

$$\tau\left(\sum_{i=0}^n \ell_i\Lambda_i\right) = \sum_{i=0}^n \ell_i\Lambda_{n-i}.$$

4.7 Type $A_{2n}^{(2)}$

For type $A_{2n}^{(2)}$ one may use the result of Naito and Sagaki [18, Theorem 2.4.1] which states that under their [18, Assumption 2.3.1] (which requires that $B^{r,s}$ for $A_{2n}^{(1)}$ is perfect) all $B^{r,s}$ for $A_{2n}^{(2)}$ are perfect. Here we provide a description of the minimal elements via the embedding S into $B_{C_n^{(1)}}^{r,2s}$.

Proposition 4.9 *The minimal elements of $B^{r,s}$ of level s are precisely those that corresponding to doubled \pm -diagrams in $B_{C_n^{(1)}}^{r,2s}$.*

Proof: In Proposition 4.8 a description of the minimal elements of $B_{C_n^{(1)}}^{r,2s}$ is given. We have the realization of $B^{r,s}$ via the map S from Definition 3.4. In the same way as in the proof of Proposition 4.4 one can show, that the minimal elements of $B_{C_n^{(1)}}^{r,2s}$ that correspond to doubled dominant weights are precisely those in the realization of $B^{r,s}$, hence ε defines a bijection between $B_{\min}^{r,s}$ and P_s^+ . \square

The automorphism τ is given by the identity.

4.8 Type $D_{n+1}^{(2)}$ for $r < n$

Proposition 4.10 *Let $r < n$. There exists a bijection $B_{\min}^{r,s}$ to P_s^+ , defined by ε . Hence $B^{r,s}$ is perfect.*

Proof: This proof is analogous to the proof of Proposition 4.9. \square

The automorphism τ is given by the identity.

4.9 Type $D_{n+1}^{(2)}$ for $r = n$

This case is already treated in [14], which we summarize below. As a B_n -crystal it is isomorphic to $B(s\Lambda_n)$. There is a description of its elements in terms of semistandard tableaux of $n \times s$ rectangular shape with letters from the alphabet $\mathcal{A} = \{1 < 2 < \dots < n < \bar{n} < \dots < \bar{1}\}$. Moreover, each column does not contain both k and \bar{k} . Let c_i be the i th column, then the action of e_i, f_i ($i = 1, \dots, n$) is calculated through that of $c_s \otimes \dots \otimes c_1$ of $B(\Lambda_n)^{\otimes s}$. With this realization the minimal element b_Λ such that $\varepsilon(b_\Lambda) = \Lambda = \sum_{i=0}^n \ell_i \Lambda_i \in P_s^+$ is given as follows. Let x_{ij} ($1 \leq i \leq n, j \in \mathcal{A}$) be the number of j in the i th row. Note that $x_{ij} = 0$ unless $i \leq j \leq \overline{n-i+1}$. The table (x_{ij}) of b_Λ is then given by $x_{ii} = \ell_0 + \dots + \ell_{n-i}$ ($1 \leq i \leq n$), $x_{ij} = \ell_{j-i}$ ($i+1 \leq j \leq n$), $x_{i\bar{j}} = \ell_j + \dots + \ell_n$ ($n-i+1 \leq j \leq n$). The automorphism τ is given by

$$\tau\left(\sum_{i=0}^n \ell_i \Lambda_i\right) = \sum_{i=0}^n \ell_i \Lambda_{n-i}.$$

References

- [1] G. Benkart, I. Frenkel, S-J. Kang, and H. Lee, *Level 1 perfect crystals and path realizations of basic representations at $q = 0$* , Int. Math. Res. Not. 2006, Art. ID 10312, 28 pp.
- [2] V. Chari, *On the fermionic formula and the Kirillov-Reshetikhin conjecture*, Internat. Math. Res. Notices **12** (2001) 629–654.
- [3] V. Chari, A. Pressley, *Quantum affine algebras and their representations*, in Representations of groups, CMS Conf. Proc. **16**, Amer. Math. Soc., Providence, RI (1995) 59–78.
- [4] V. Chari, A. Pressley, *Twisted quantum affine algebras*, Comm. Math. Phys. **196** (1998) 461–476.
- [5] G. Fourier, M. Okado, A. Schilling *Kirillov-Reshetikhin crystals of non-exceptional type*, Advances in Mathematics, to appear (arXiv:0810.5067 [math.RT])
- [6] G. Fourier, M. Okado, A. Schilling *Perfectness of Kirillov-Reshetikhin crystals of non-exceptional type*, Contemp. Math., to appear (arXiv:0811.1604 [math.RT])

- [7] G. Fourier, A. Schilling, M. Shimozono, *Demazure structure inside Kirillov-Reshetikhin crystals*, J. Algebra **309** (2007) 386–404.
- [8] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, Z. Tsuboi, *Paths, crystals and fermionic formulae*, MathPhys Odyssey 2001, 205–272, Prog. Math. Phys. **23**, Birkhäuser Boston, Boston, MA, 2002.
- [9] D. Hernandez, *The Kirillov-Reshetikhin conjecture and solution of T-systems*, J. Reine Angew. Math. **596** (2006) 63–87.
- [10] D. Hernandez, *Kirillov-Reshetikhin conjecture: The general case*, preprint arXiv:0704.2838.
- [11] J. Hong, S.-J. Kang, Introduction to quantum groups and crystal bases, Graduate Studies in Mathematics, **42**, American Mathematical Society, Providence, RI, 2002. xviii+307 pp.
- [12] V. G. Kac, “*Infinite Dimensional Lie Algebras*,” 3rd ed., Cambridge Univ. Press, Cambridge, UK, 1990.
- [13] S.-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima, A. Nakayashiki, *Affine crystals and vertex models*, Int. J. Mod. Phys. A **7** (suppl. 1A) (1992), 449–484.
- [14] S.-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima, A. Nakayashiki, *Perfect crystals of quantum affine Lie algebras*, Duke Math. J. **68** (1992) 499–607.
- [15] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516.
- [16] M. Kashiwara, K. C. Misra, M. Okado, D. Yamada, *Perfect crystals for $U_q(D_4^{(3)})$* , J. Alg. **317** (2007) 392–423.
- [17] M. Kashiwara, T. Nakashima, *Crystal graphs for representations of the q-analogue of classical Lie algebras*, J. Alg. **165** (1994) 295–345.
- [18] S. Naito, D. Sagaki, *Construction of perfect crystals conjecturally corresponding to Kirillov-Reshetikhin modules over twisted quantum affine algebras*, Comm. Math. Phys. **263** (2006), no. 3, 749–787.
- [19] H. Nakajima, *t-analogues of q-characters of Kirillov-Reshetikhin modules of quantum affine algebras*, Represent. Theory **7** (2003) 259–274.
- [20] M. Okado, *Existence of crystal bases for Kirillov-Reshetikhin modules of type D*, Publ. RIMS **43** (2007) 977–1004.
- [21] M. Okado, A. Schilling, *Existence of Kirillov-Reshetikhin crystals for nonexceptional types*, Representation Theory **12** (2008) 186–207.
- [22] A. Schilling, *Combinatorial structure of Kirillov-Reshetikhin crystals of type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$* , J. Algebra **319** (2008) 2938–2962.
- [23] M. Shimozono, *Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties*, J. Algebraic Combin. **15** (2002), no. 2, 151–187.

- [24] S. Yamane, *Perfect crystals of $U_q(G_2^{(1)})$* , J. Algebra **210** (1998), no. 2, 440–486.