The distribution of the number of small cuts in a random planar triangulation

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The distribution of the number of small cuts in a random planar triangulation

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We enumerate rooted 3-connected (2-connected) planar triangulations with respect to the vertices and 3-cuts (2-cuts). Consequently, we show that the distribution of the number of 3-cuts in a random rooted 3-connected planar triangulation with $n+3$ vertices is asymptotically normal with mean $(10/27)n$ and variance $(320/729)n$, and the distribution of the number of 2-cuts in a random 2-connected planar triangulation with $n+2$ vertices is asymptotically normal with mean $(8/27)n$ and variance $(152/729)n$. We also show that the distribution of the number of 3-connected components in a random 2-connected triangulation with $n+2$ vertices is asymptotically normal with mean $n/3$ and variance $\frac{8}{27}n$.

Keywords: cuts; random triangulation; normal distribution.

1 Introduction

Throughout the paper, a *triangulation* is a connected graph G embedded in the plane with no edge crossings such that every face has degree 3 (bounded by a triangle). Our triangulations may contain multiple edges, but not loops. A triangulation is *rooted* by specifying a vertex (called the *root vertex*), an edge (called the *root edge*) incident with the root vertex, and a face (called the *root face*) incident with the root edge. Two rooted triangulations are considered the same if there is a homeomorphism from the plane to itself which transforms one to the other and preserves the rooting. A triangle is *facial* if it bounds a face; otherwise it is called *non-facial*. Non-facial triangles are also called *separating* triangles.

A k-*cut* in a graph G is a set of k vertices whose deletion disconnects the graph. It is well known that a triangulation is k-connected if and only if it does not contain separating cycles of length less than k. Furthermore, $\{a, b\}$ is a 2-cut in a 2-connected triangulation if and only if there are multiple edges joining vertices a and b, and $\{a, b, c\}$ is a 3-cut in a 3-connected triangulation if and only if a, b and c are vertices of a non-facial triangle. Rooted planar triangulations of connectivities 2,3, and 4 were enumerated in the 1960's by Tutte and his students (see [15] for a survey). Triangulations of connectivity 5 were enumerated much later [8]. The distributions of some interesting parameters of a random rooted planar triangulation were also studied by various authors; see [7] about the number of flippable edges, and [3, 9, 1] for the size

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of maximum components. All those papers build on the original approach of Tutte. Let us indicate that the other two main fruitful approaches for the enumeration of planar maps, matrix integrals and bijections with trees, have not lead, until recently to fundamentally new ingredients for the study of minimal cuts. An exception to this statement is the recent paper [4] where 2-cuts in random planar quadrangulations are studied through bijections with well labeled trees, with the aim of characterizing distances in random simple planar quadrangulations: we refer to this paper for further references on these approaches.

In this paper we enumerate rooted 3-connected (2-connected) planar triangulations with respect to the vertices and 3-cuts (2-cuts). Parametric expressions will be obtained for their generating functions. We then perform singularity analysis and apply a central limit theorem of Bender and Richmond [2] to derive the distribution of the number of cuts in a random rooted triangulation. Throughout this paper, a random triangulation is taken *uniformly* at random from a given set of triangulations.

Our main result is the following

Theorem 1 *(i) Let ζ_n be the number of 3-cuts in a random 3-connected triangulation with $n+3$ vertices. Then ζ_n is asymptotically normal with mean $(10/27)n$ and variance $(320/729)n$.*
(ii) Let η_n be the number of 2-cuts in a random 2-connected triangulation with $n+2$ vertices. Then η_n is asymptotically normal with mean $(8/27)n$ and variance $(152/729)n$.

Besides the fact that it is a natural problem to enumerate triangulations with respect to the number of cuts, our work is also motivated by some important graph theoretical properties related to small cuts. A classical theorem of Whitney [17] says that every 4-connected planar triangulation contains a cycle through all the vertices (such a cycle is called a Hamilton cycle). There have been several generalizations and extensions of this important theorem which are related to 3-cuts. For example, Jackson and Yu [13] showed that if a 3-connected triangulation contains at most three 3-cuts, then it contains a Hamilton cycle. Gao, Richter, and Yu [6] showed that every 3-connected triangulation contains a closed walk W through each vertex once or twice such that every vertex visited twice by W is in a 3-cut. It is also known that the length of a longest cycle (path) in a 3-connected triangulation depends heavily on the number of 3-cuts [12, 14].

Another classical theorem of Whitney [18] states that every 3-connected planar graph has a unique embedding in the plane, and a 2-connected planar graph may have many embeddings in the plane using switchings at the 2-cuts. Hence the number of different embeddings of a 2-connected planar graph depends heavily on the number of 2-cuts. Indeed the recent work of Gimenez and Noy [11] on the enumeration of planar graphs build on the decomposition along 2- and 3-cuts.

Minimum length cuts in planar maps were also considered in the physics literature (see references in [1] and [4].

2 Enumeration of triangulations with respect to small cuts

Let $\Delta_{n,m}$ be the number of rooted 3-connected triangulations with $n+3$ vertices and m 3-cuts. Define the generating function

$$\Delta(x,y) = \sum_{n\geq 1, m\geq 0} \Delta_{n,m} x^n y^m.$$

For 2-connected triangulations we need to distinguish two cases according as whether the end vertices of the root edge form a 2-cut. For 2-connected triangulations in which the end vertices of the root edge don't form a 2-cut, we use $\bar{\Delta}_1(x,y)$ to denote their generating function, where the exponent of x is two less than the number of vertices, and the exponent of y is the number of 2-cuts. Similarly we use $\bar{\Delta}_2(x,y)$ to denote the generating function for 2-connected triangulations in which the end vertices of the root edge form a 2-cut. Let $\bar{\Delta}(x,y) = \bar{\Delta}_1(x,y) + \bar{\Delta}_2(x,y)$.

Our main results in this section are summarized below.

Theorem 2 *Let s and t be unique power series in x and y satisfying*

$$\begin{aligned} x &= \frac{t(1-2t)^2}{(1-t)^3(1+s)^2}, && (1) \\ y &= \frac{s(1-2t)}{(1+s)t(1-3t+t^2)}. && (2) \end{aligned}$$

Then

$$\begin{aligned} \Delta(x,y) &= s/y && (3) \\ &= x + 3yx^2 + (1+12y^2)x^3 + (3+10y+55y^3)x^4 \\ &\quad +(12+36y+78y^2+273y^4)x^5 \\ &\quad +(52+175y+315y^2+560y^3+1428y^5)x^6 \\ &\quad +(241+880y+1768y^2+2448y^3+3876y^4+7752y^6)x^7 \\ &\quad +\cdots \end{aligned}$$

Theorem 3 *Let s and t be unique power series in x and y satisfying*

$$\begin{aligned} x &= \frac{t(1-t)^3}{(1+s)^3}, && (4) \\ y &= \frac{s(1+s)}{t(1-2t)} - s. && (5) \end{aligned}$$

Then

$$\begin{aligned} \bar{\Delta}(x,y) &= t(1-2t) && (6) \\ &= x + (1+3y)x^2 + (3+12y+9y^2)x^3 + (13+57y+78y^2+28y^3)x^4 \\ &\quad +(68+318y+570y^2+68+410y^3+90y^4)x^5 \\ &\quad +(399+1989y+4167y^2+4257y^3+1947y^4+297y^5)x^6 \\ &\quad +(2530+13464y+31500y^2+39816y^3+26985y^4+8736y^5+1001y^6)x^7 \\ &\quad +\cdots \end{aligned}$$

Proof of Theorem 2 Let Δ be a rooted 3-connected triangulation and C be a separating triangle in Δ. Then C separates Δ into two triangulations, the one containing the root face of Δ will be called *exterior*, and the other one is called *interior* (By convention, a rooted triangulation is usually embedded in the plane such that the root face is the unbounded face, and hence the above terminology.). A separating triangle in Δ is called maximal if it is not contained in the interior of another separating triangle. Removing all vertices inside each maximal separating triangle yields a 4-connected triangulation. We can reverse this process to obtain rooted 3-connected triangulations by inserting rooted 3-connected triangulations into each non-root face of a rooted 4-connected triangulation in a unique way. This decomposition was used by Tutte [16] to enumerate rooted 4-connected triangulations from 3-connected triangulations.

Let H_n be the number of rooted 4-connected triangulations with $n+3$ vertices, and define the generating function

$$H(x) = \sum_{n\geq 1} H_n x^n.$$

We note that $\Delta(x,1)$ enumerates all 3-connected triangulations with at least one interior vertex. Let $t = t(x)$ be the (unique) power series in x satisfying

$$x = t(1-t)^3, \tag{7}$$

it is known [16] that

$$\Delta(x,1) = (1-2t)/(1-t)^3 - 1. \tag{8}$$

Noting that a triangulation with $n+3$ vertices has exactly $2n+2$ faces, the decomposition described above gives

$$\Delta(x,y) = (1 + y\Delta(x,y))H\left(x(1+y\Delta(x,y))^2\right). \tag{9}$$

Setting $y=1$, $x = t(1-t)^3$ and $z = x(1+y\Delta(x,y))^2$ in (9), and using (8), we obtain

$$\begin{aligned} z &= t(1-2t)^2/(1-t)^3 \\ H(z) &= t(1-3t+t^2)/(1-2t). \end{aligned}$$

(A similar parametric expression for $H(z)$ was first obtained by Tutte [16].)

Now Theorem 2 follows by setting

$$s = y\Delta(x,y), \;\; x(1+y\Delta(x,y))^2 = t(1-2t)^2/(1-t)^3,$$

and using (9). The first few terms in the power series expansion of $\Delta(x,y)$ are obtained by iteration using (1)-(3) and Maple. Rooted 3-connected triangulations with 4, 5, or 6 vertices are shown in Figure 1.

Proof of Theorem 3 Since each triangulation counted by $\bar{\Delta}_2(x,y)$ is uniquely decomposed into a triangulation counted by $\bar{\Delta}_1(x,y)$ and a triangulation counted by $\bar{\Delta}(x,y)$, we have

$$\bar{\Delta}_2(x,y) = \bar{\Delta}_1(x,y)(y\bar{\Delta}_1(x,y) + \bar{\Delta}_2(x,y)). \tag{10}$$

As in the previous section, a 2-cycle C in a rooted 2-connected triangulation Δ is called maximal if it is not contained in the interior of another 2-cycle. Closing each maximal 2-cycle into an edge yields a 3-connected triangulation. We can reverse this process to obtain rooted 2-connected triangulations counted by $\bar{\Delta}_1(x,y)$ by attaching to each nonroot edge of a rooted 3-connected triangulation a rooted 2-connected triangulation. Let $T(x) = x(\Delta(x,1)+1)$ be the generating function for rooted 3-connected triangulations where x marks the number of vertices minus 2. Then it follows from (8) that

$$T(x) = t(1-2t), \; x = t(1-t)^3. \tag{11}$$

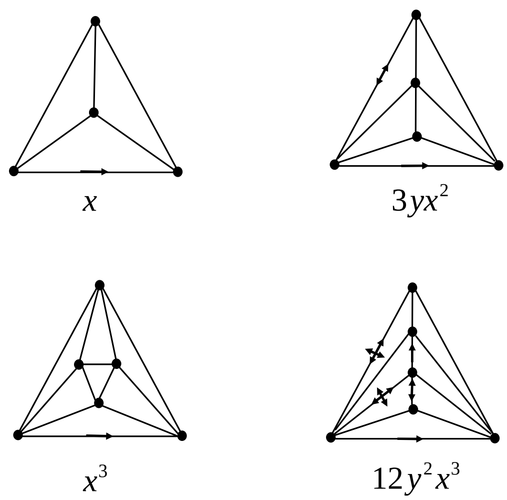

Fig. 1: 3-connected triangulations with 4, 5, or 6 vertices. The arrows on the edges indicate different rootings.

Noting that a triangulation with $n+2$ vertices has exactly $3n$ edges, the decomposition described above gives

$$\bar{\Delta}_1(x,y) = \frac{1}{1+y\bar{\Delta}_1(x,y)+\bar{\Delta}_2(x,y)} T\left(x(1+y\bar{\Delta}_1(x,y)+\bar{\Delta}_2(x,y))^3\right). \tag{12}$$

Setting

$$x(1+y\bar{\Delta}_1(x,y)+\bar{\Delta}_2(x,y))^3 = t(1-t)^3, \tag{13}$$

we obtain from (11) that

$$\bar{\Delta}_1(x,y) = \frac{t(1-2t)}{1+y\bar{\Delta}_1(x,y)+\bar{\Delta}_2(x,y)}. \tag{14}$$

Setting $s = y\bar{\Delta}_1(x,y)+\bar{\Delta}_2(x,y)$, we obtain from (10), (13) and (14) that

$$\bar{\Delta}_1(x,y) = \frac{t(1-2t)}{1+s}, \tag{15}$$

$$\bar{\Delta}_2(x,y) = \frac{st(1-2t)}{1+s}, \tag{16}$$

where s and t are power series in x and y given by (4) and (5). Now Theorem 3 follows immediately.

The first few terms in $\bar{\Delta}(x,y)$ are obtained by iteration using (4)-(6) and Maple. Rooted 2-connected triangulations with 4 or 5 vertices are shown in Figure 2.

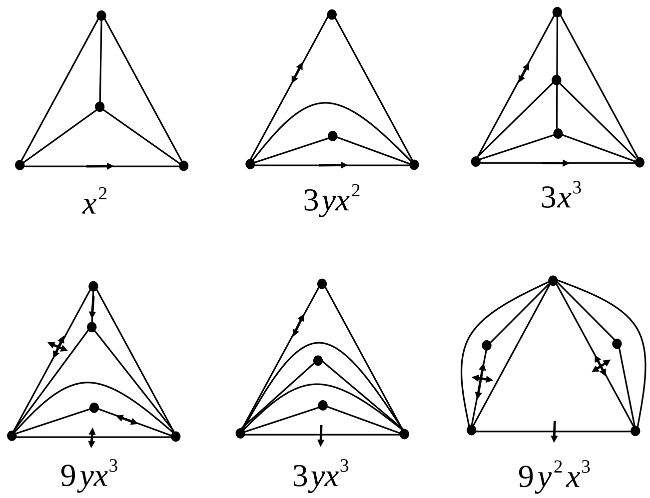

Fig. 2: 2-connected triangulations with 4 or 5 vertices. The arrows on the edges indicate different rootings.

3 Proof of Theorem 1

Proof of Theorem 1(i) We need to determine the singularities of $\Delta(x,y)$ near $y=1$. It is clear that the singularities of $\Delta(x,y)$ arise from inverting (1) and (2), and hence they satisfy

$$\begin{vmatrix} \partial x/\partial t & \partial x/\partial s \\ \partial y/\partial t & \partial y/\partial s \end{vmatrix} = \frac{(1-4t)(1-2t)^2(1-3t+t^2-2ts+6t^2s-6t^3s+2t^4s)}{(1-t)^4(1-3t+t^2)^2(1+ts)^4} = 0.$$

To determine the singularities of $\Delta(x,y)$, we discuss three cases based on the three factors in the numerator of the above determinant.

Case 1: $1-2t=0$. By (1) and (2), this gives $x=0$. We know $x=0$ is not a singularity of $\Delta(x,y)$, hence we can discard this case.

Case 2: $1-3t+t^2-2ts+6t^2s-6t^3s+2t^4s=0$. It follows that

$$\begin{aligned} s &= \frac{1-3t+t^2}{2t(1-t)^3}, \\ x &= \frac{4t(1-t)^3}{(3-3t+t^2)^2}, \\ y &= \frac{1}{t(3-3t+t^2)}. \end{aligned}$$

Hence

$$y-1 = \frac{(1-t)^3}{t(3-3t+t^2)}.$$

At $y = 1$, we have $t = 1$. This implies that $x = 0$ and $\Delta(x, 1) = s/y = \infty$. This contradiction implies that this case should also be discarded.

Case 3: $t = 1/4$. It follows from (1) and (2) that

$$\begin{aligned} x &= \frac{4}{27(1+s/4)^2}, \\ y &= \frac{8s}{5(1+s/4)}, \end{aligned}$$

and hence

$$x = \rho(y) = \frac{1}{6912}(32-5y)^2. \tag{17}$$

We note that $\rho(1) = 27/256$ is indeed the singularity of $\Delta(x, 1)$. Thus $\rho(y)$ is the unique singularity of $\Delta(x, y)$ for each y such that $|y| < 32/5$.

Next we find the singular expansion of $\Delta(x, y)$ at $\rho(y)$. Using (2), we obtain

$$s = \frac{y(1-3t+t^2)}{1-2t-yt+3yt^2-yt^3}. \tag{18}$$

Substituting it into (1) and expanding the right side at $t = 1/4$, we obtain

$$\begin{aligned} x &= \rho(y) - \frac{1}{648}(64-37y)(32-5y)(t-1/4)^2 - \frac{2}{729}(32-59y)(32-5y)(t-1/4)^3 \\ &\quad +O\left((t-1/4)^4\right), \end{aligned}$$

as $t \to 1/4$ uniformly in a small neighborhood of $y = 1$.

Thus

$$t - \frac{1}{4} = -\sqrt{\frac{3(32-5y)}{32(64-37y)}}(1-x/\rho(y))^{1/2} + O(1-x/\rho(y)). \tag{19}$$

Substituting this into (18) and (3), we obtain

$$\begin{aligned} \Delta(x,y) &= \frac{5}{32-5y} - \frac{432}{(32-5y)(64-37y)}(1-x/\rho(y)) \\ &\quad +288\sqrt{6}(32-5y)^{1/2}(64-37y)^{-5/2}(1-x/\rho(y))^{3/2} \\ &\quad +O\left((1-x/\rho(y))^2\right), \end{aligned}$$

as $x \to \rho(y)$ uniformly in a small neighborhood of $y = 1$.

Since $\Delta(x, y)$ is algebraic, we can apply the “transfer theorem" [5, Theorem VI.3] to obtain

$$[x^n]\Delta(x,y) \sim 288\sqrt{6}(32-5y)^{1/2}(64-37y)^{-5/2}\frac{1}{\Gamma(-3/2)}n^{-5/2}\rho(y)^{-n},$$

as $n \to \infty$ uniformly in a small neighborhood of $y = 1$.

Thus

$$\begin{aligned} \mu(y) &= -y(\ln \rho(y))' = \frac{10y}{32-5y}, \\ \sigma^2(y) &= y\mu'(y) = \frac{320y}{(32-5y)^2}. \end{aligned}$$

Now Theorem 2 follows from the central limit theorem of [2] by noting $\mu(1) = 10/27$ and $\sigma^2(1) = 320/729$.

Proof of Theorem 1(ii) As in the proof of Theorem 2, we first determine the singularities of $\bar{\Delta}(x, y)$ near $y = 1$. It is clear that the singularities of $\bar{\Delta}(x, y)$ arise from inverting (4) and (5), and hence they satisfy

$$\begin{vmatrix} \partial x/\partial t & \partial x/\partial s \\ \partial y/\partial t & \partial y/\partial s \end{vmatrix} = \frac{(1-4t)(1-t)^2(1-3t+4t^2-4t^3-s-ts)}{t(1-2t)^2(1+s)^3} = 0.$$

As in the proof of part (i), we discuss three cases based on the three factors in the denominator of the above determinant.

Case 1: $t = 1$. Again this should be discarded as it implies $x = 0$.

Case 2: $1 - 3t + 4t^2 - 4t^3 - s - ts = 0$. It follows that

$$\begin{aligned} s &= \frac{(1-2t)(1-t+2t^2)}{1+t}, \\ x &= \frac{t(1+t)^3}{8(1+2t^2)^3}, \\ y &= \frac{(2-t)(1-t+2t^2)^2}{t(1+t)^2}. \end{aligned}$$

Thus

$$y - 1 = \frac{2(1-t)^3(1+2t^2)}{t(1+t)^2}.$$

At $y = 1$, we have $t = 1$, and hence $x = 0$ and $s = -1$. This contradicts the fact that $s = y\bar{\Delta}_1(x, y) + \bar{\Delta}_2(x, y)$ is a power series in x and y with non-negative coefficients which has value 0 at $x = 0$ and $y = 1$. So we can also discard this case.

Case 3: $t = 1/4$. It follows from (4) and (5) that

$$\begin{aligned} s &= \frac{\sqrt{49+32y}-7}{16}, \\ x &= \bar{\rho}(y) = \frac{432}{(9+\sqrt{49+32y})^3}. \end{aligned}$$

Note that $\bar{\rho}(1) = 2/27$ is indeed the singularity of $\bar{\Delta}(x, 1)$. Hence $\bar{\rho}(y)$ is the unique singularity of $\bar{\Delta}(x, y)$ for each y.

To find the singular expansion of $\bar{\Delta}(x, y)$ at $\bar{\rho}(y)$, we eliminate s from (4) and (5) to express x as a function of t, y, and then expand this function at $t = 1/4$. The result is

$$\begin{aligned} x &= \bar{\rho}(y) - \frac{2304(9\sqrt{49+32y} + 161 - 80y)}{(9+\sqrt{49+32y})^4\sqrt{49+32y}}(t-1/4)^2 \\ &\quad + \frac{8192}{(9+\sqrt{49+32y})^3}(t-1/4)^3 + O\left((t-1/4)^4\right), \end{aligned}$$

as $t \to 1/4$ uniformly in a small neighborhood of $y = 1$.

Thus

$$\begin{aligned} t - \frac{1}{4} &= -\frac{\sqrt{3(9+\sqrt{49+32y})(161-80y+9\sqrt{49+32y})\sqrt{49+32y}}}{4(161-80y+9\sqrt{49+32y})}(1-x/\bar{\rho}(y))^{1/2} \\ &\quad + O(1-x/\bar{\rho}(y)), \end{aligned}$$

as $x \to \bar{\rho}(y)$ uniformly in a small neighborhood of $y = 1$.

Substituting this into (6), we obtain

$$\begin{aligned} \bar{\Delta}(x, y) &= \frac{1}{8} + d_2(1-x/\bar{\rho}(y)) + d_3(1-x/\bar{\rho}(y))^{3/2} \\ &\quad + O\left((1-x/\bar{\rho}(y))^2\right), \end{aligned}$$

where

$$\begin{aligned} d_2 &= -\frac{3(9+\sqrt{49+32y})\sqrt{49+32y}}{8(161-80y+9\sqrt{49+32y})} \\ d_3 &= \frac{(9+\sqrt{49+32y})^2(49+32y)\sqrt{3(9+\sqrt{49+32y})(161-80y+\sqrt{49+32y})\sqrt{49+32y}}}{3(161-80y+9\sqrt{49+32y})^3}. \end{aligned}$$

Since $\bar{\Delta}(x, y)$ is algebraic, we can apply the “transfer theorem" [5, Theorem VI.3] to obtain

$$[x^n]\bar{\Delta}(x, y) \sim \frac{d_3}{\Gamma(-3/2)} n^{-5/2} \bar{\rho}(y)^{-n},$$

as $n \to \infty$ uniformly in a small neighborhood of $y = 1$.

Therefore

$$\begin{aligned} \mu(y) &= -y(\ln \bar{\rho}(y))' = \frac{48y}{(9+\sqrt{49+32y})\sqrt{49+32y}}, \\ \sigma^2(y) &= y\mu'(y) = \frac{48y(441+144y+49\sqrt{49+32y})}{(49+32y)^{3/2}(9+\sqrt{49+32y})^2}. \end{aligned}$$

Now Theorem 2 follows from the central limit theorem [2] by noting $\mu(1) = 8/27$ and $\sigma^2(1) = 152/729$.

4 Concluding remarks

In this paper we studied the distribution of the number of 3-cuts (2-cuts) in a random rooted 3-connected (2-connected) triangulation. Both distributions are shown to be asymptotically normal with mean and variance linear in n. It is not difficult to check that $|\rho(y)| > \rho(|y|)$ and $|\bar{\rho}(y)| > \bar{\rho}(|y|)$ when $y \neq |y|$, and hence one can also apply the local limit theorem to obtain asymptotic expressions for $[x^n y^m]\Delta(x,y)$ and $[x^n y^m]\bar{\Delta}(x,y)$ as $m, n \to \infty$ and m/n lies in some closed subintervals of $(0,1)$.

The approach used in this paper can also be used to study the distribution of the number of k-cuts $(1 \leq k \leq 3)$ in a random k-connected map. In this context, it is convenient to use the well-known correspondence between maps and quadrangulations, and study the distribution of the number of $2k$-cuts in the corresponding quadrangulations. It is also possible to derive the distribution of the number of 4-cuts in a random 4-connected triangulation using the compositional structure described in [8]. We believe that the distributions will also be asymptotically normal with mean and variance proportional to the number of edges of the random map.

One may also study the distribution of the number of 3-connected components in a random 2-connected triangulation (or the number of blocks in a random map). Suppose two vertices a and b are joined by $p \geq 2$ parallel edges in a 2-connected triangulation, then there are p 3-connected components associated with the 2-cut $\{a,b\}$. It turns out that the corresponding equation for the generating function with respect to vertices and 3-connected components is simpler than (12). Let $\tilde{\Delta}(x,y)$ be the generating function for 2-connected triangulations such that x marks the number of vertices minus 2 and y marks the number of 3-connected components. Then we obtain the following analogy of (12):

$$\tilde{\Delta}(x,y) = yT\left(x(1+\tilde{\Delta}(x,y))^3\right). \tag{20}$$

Setting $s = \tilde{\Delta}(x,y)$ and $x(1+s)^3 = t(1-t)^3$, we obtain from (11) that

$$\begin{aligned} x &= \frac{t(1-t)^3}{(1+s)^3}, && (21)\\ y &= \frac{s}{t(1-2t)}, && (22)\\ \tilde{\Delta}(x,y) &= s && (23)\\ &= xy + (y+3y^2)x^2 + (3y+9y^2+12y^3)x^3 \\ &\quad +(13y+42y^2+66y^3+55y^4)x^4 \\ &\quad +(68y+240y^2+420y^3+455y^4+273y^5)x^5 + \cdots \end{aligned}$$

It is easy to see that the dominant singularity of $\tilde{\Delta}(x,y)$, for each fixed y, is given by

$$\tilde{\rho}(y) = \frac{54}{(y+8)^3}.$$

It is easy to compute

$$\mu(y) = -y(\ln\tilde{\rho}(y))' = \frac{3y}{y+8},\ \sigma^2(y) = y\mu'(y) = \frac{24y}{(y+8)^2}.$$

Hence the distribution of the number of 3-connected components in a random 2-connected triangulation with $n+2$ vertices is asymptotically normal with mean $\mu(1)n/3$ and variance $\sigma^2(1)n = \frac{8}{27}n$.

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