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# Uniquely monopolar-partitionable block graphs

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As a common generalization of bipartite and split graphs, monopolar graphs are defined in terms of the existence of certain vertex partitions. It has been shown that to determine whether a graph has such a partition is an NP-complete problem for general graphs, and is polynomial time solvable for several classes of graphs. In this paper, we investigate graphs that admit a unique such partition and call them uniquely monopolar-partitionable graphs. By employing a tree trimming technique, we obtain a characterization of uniquely monopolar-partitionable block graphs. Our characterization implies a polynomial time algorithm for determining whether a block graph is uniquely monopolar-partitionable.

**Keywords:** Monopolar graph, monopolar partition, uniquely monopolar-partitionable graph, block graph, characterization, polynomial time algorithm

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## 1 Introduction

Given a graph  $G$ , a *monopolar partition* of  $G$  is a partition  $(A, B)$  of its vertex set where  $A$  is an independent set and  $B$  induces a disjoint union of cliques in  $G$ . A graph which admits a monopolar partition is called *monopolar* or *monopolar-partitionable*.

Monopolar graphs were introduced in [17] as a common generalization of bipartite graphs and split graphs. Every bipartition of a bipartite graph is a monopolar partition. Graphs which admit monopolar partitions  $(A, B)$  where  $B$  induces a single clique are precisely split graphs [12, 14].

A monopolar graph is called *uniquely monopolar-partitionable* if it has exactly one monopolar partition, that is, if  $(A, B)$  and  $(A', B')$  are both monopolar partitions of  $G$  then  $A = A'$  (and  $B = B'$ ). Since each complete graph has two monopolar partitions  $(A, B)$  and  $(A', B')$  where  $A$  and  $A'$  are the empty set and singleton set respectively, no complete graph is uniquely monopolar-partitionable. On the other hand, the graph obtained from two complete graphs of order at least three by identifying two vertices, one from each, is uniquely monopolar-partitionable.

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Unlike bipartite graphs and split graphs which are easy to recognize, recognizing monopolar graphs in general is an NP-complete problem (cf. [2] and [11]). It is currently unknown whether uniquely monopolar-partitionable graphs are recognizable in polynomial time.

In this paper, we consider the uniqueness of monopolar partitionability of block graphs. A *block* of a graph  $G$  is a maximal subgraph of  $G$  without cut-vertices. A graph is a *block graph* if every block is a clique (cf. [1, 15]). We shall give a structural characterization of uniquely monopolar-partitionable block graphs by using a tree trimming technique. As a by-product, we obtain a polynomial time algorithm for determining whether a block graph is uniquely monopolar-partitionable. We note that such an algorithm can be extracted from [8].

## 2 Basic definitions

We follow the standard definition and terminology from [18] and consider only simple graphs. Let  $G = (V, E)$  be a graph. The *neighbourhood*  $N(v)$  of a vertex  $v$  in  $G$  consists of all vertices adjacent to  $v$ . The size of  $N(v)$  is the *degree* of  $v$  and denoted by  $d(v)$ . If  $d(v) = 1$ , then vertex  $v$  is called a *leaf* of  $G$ . The *closed neighbourhood*  $N[v]$  of  $v$  is  $N(v) \cup \{v\}$ . For any  $S \subseteq V$ , the subgraph of  $G$  induced by  $S$  is denoted by  $G[S]$ . For convenience we write  $G - S = G[V - S]$ . For any two vertices  $u, v \in V$ , the *distance*  $d_G(u, v)$  between  $u$  and  $v$  in  $G$  is the length of a shortest  $u, v$ -path in  $G$ , and the *diameter*  $diam(G)$  of  $G$  is the maximum distance of any two vertices in  $G$ . We shall use  $P_n, C_n$  and  $K_n$  to denote the path, cycle and clique with order  $n$ , respectively.

Let  $H$  be a subgraph of a graph  $G$ . By *contracting*  $H$  in  $G$  we mean to obtain a new graph  $G'$  from  $G - V(H)$  by adding a new vertex  $w$  adjacent to all vertices which have at least one neighbour in  $H$ .

Let  $G_1, G_2, \dots, G_t$  be the components of  $G - v$ . For any  $1 \leq i_1 < i_2 < \dots < i_s \leq t$ ,  $R_v^s = G[V(\bigcup_{1 \leq j \leq s} G_{i_j}) \cup \{v\}]$  is called a *tangent subgraph* of  $G$ . If a star with order at least two is a component of  $G - v$  and only one center of the star is adjacent to  $v$ , then the star is called an *end star* of  $G$ , and  $v$  is said to be *adjacent to* the end star. The vertex of the star that is adjacent to  $v$  is taken as the center of the end star.

Let  $G$  be a block graph with at least two blocks and let  $Q$  be a block of  $G$ . A vertex  $v$  of  $G$  is said to be *adjacent to*  $Q$  if  $v$  is not in  $Q$  but adjacent to a vertex of  $Q$ . If the order of  $Q$  is at least 3, then  $Q$  is called a *big block* of  $G$ . If a big block  $Q$  contains a unique cut vertex and  $G - Q$  is connected, then  $Q$  is called a *terminal block*; if the big block  $Q$  contains a unique cut vertex and  $G - Q$  is disconnected, then  $Q$  is called a *suspending block* of  $G$ . If two big blocks have no common vertex but contain adjacent vertices, then the two blocks are called *adjacent*. If two big blocks  $Q_1$  and  $Q_2$  have a common vertex  $z$ , then we call the subgraph  $G[V(Q_1 \cup Q_2)]$  a *bowtie* of  $G$  and the common vertex  $z$  is the *center* of the bowtie. The bowtie is called a *terminal bowtie* if  $G - Q_1 - Q_2$  is connected and its center is adjacent to exactly one vertex  $w$  of  $V(G) - V(Q_1 \cup Q_2)$ . Vertex  $w$  is said to be *adjacent to* the terminal bowtie. If the center of an end star or a terminal bowtie is adjacent to a big block, then the end star or the terminal bowtie is said to be *adjacent to* the big block. Two terminal bowties are *adjacent* if their centers are adjacent. A big block is said to be *adhered to a vertex*  $v$  if  $v$  is identified with a vertex from the block. Adhering a bowtie to a vertex  $v$  means adhering two big blocks to the vertex. If a block graph  $G$  is induced by  $t \geq 3$  big blocks with a common vertex, then  $G$  is called a *flower*.

Let  $G'$  be an induced subgraph of  $G$ . Suppose that  $(A', B')$  is a monopolar partition of  $G'$ . If there is a monopolar partition  $(A, B)$  of  $G$  such that  $A \supseteq A'$  and  $B \supseteq B'$ , then we say that the monopolar partition  $(A', B')$  can be *extended to* a monopolar partition of  $G$ . If there is a unique monopolar partition  $(A, B)$

of  $G$  such that  $A \supseteq A'$  and  $B \supseteq B'$ , then we say that the monopolar partition  $(A', B')$  can be *extended to exactly* one monopolar partition of  $G$ .

### 3 Basic properties

Suppose that  $G$  is a uniquely monopolar-partitionable graph and  $(A, B)$  is the unique monopolar partition of  $G$ . For any vertex  $v \in V(G)$ , if  $G[N(v)]$  has an induced  $P_3$ , then  $v \in B$ . If  $G[N(v)]$  has no induced  $P_3$ , then  $G[N(v)]$  is a disjoint union of cliques. If  $G[N(v)]$  contains two cliques  $Q_1$  and  $Q_2$  such that each has at least two vertices, then  $v \in A$ .

**Proposition 3.1** *Let  $T$  be a tree with order  $n \geq 2$ . For any edge  $uv \in E(T)$ , there exists a monopolar partition  $(A, B)$  such that  $u, v \in B$ .*

**Proof:** Let  $T'$  be obtained from  $T$  by contracting edge  $uv$  and let  $w$  denote the new vertex of  $T'$ . Then  $T'$  is a tree. Let  $(A', B')$  be the bipartition of  $T'$ . Say  $w \in B'$ . Let  $(A, B) = (A', (B' \setminus \{w\}) \cup \{u, v\})$ . Then  $(A, B)$  is a monopolar partition of  $T$  such that  $u, v \in B$ .  $\square$

**Corollary 3.2** *No tree is uniquely monopolar-partitionable.*

**Proposition 3.3** *Let  $G$  be a uniquely monopolar-partitionable graph and  $v$  be a vertex of  $G$ . Suppose that  $C$  is a component of  $G - v$  and  $V(C) \cup \{v\}$  induces a tree in  $G$ . Then  $V(C) \cup \{v\}$  induces a star in  $G$ .*

**Proof:** Since  $V(C) \cup \{v\}$  induces a tree,  $v$  has a unique neighbour in  $C$  which we denoted by  $x$ . We show that  $x$  is adjacent to every other vertex in  $C$ . Suppose not. Then there exist vertices  $y, z$  such that  $xyz$  is a path in  $C$ . Since  $G$  is a monopolar graph and  $G - C$  is a subgraph of  $G$ ,  $G - C$  is a monopolar graph. Let  $(A', B')$  be a monopolar partition of  $G - C$ .

Assume first that  $v \in A'$ . By Proposition 3.1,  $C$  has a monopolar partition  $(A_1, B_1)$  such that  $x, y \in B_1$ . Let  $(A_2, B_2)$  be a bipartition of  $C$  where  $x \in B_2$  and  $y \in A_2$ . Then  $(A' \cup A_1, B' \cup B_1)$  and  $(A' \cup A_2, B' \cup B_2)$  are different monopolar partitions of  $G$ , which is a contradiction.

Assume now that  $v \in B'$ . By Proposition 3.1,  $C$  has a monopolar partition  $(A_1, B_1)$  such that  $y, z \in B_1$ . Let  $(A_2, B_2)$  be a bipartition of  $C$  where  $x, z \in A_2$  and  $y \in B_2$ . Then  $(A' \cup A_1, B' \cup B_1)$  and  $(A' \cup A_2, B' \cup B_2)$  are different monopolar partitions of  $G$ , which is a contradiction.  $\square$

Suppose that  $C$  is a component of  $G - v$  of order at least two and  $G[V(C) \cup \{v\}]$  is a tree. By Proposition 3.3, if  $G$  is a uniquely monopolar-partitionable graph, then  $C$  is an end star of  $G$ .

For any monopolar partition  $(A, B)$  of  $G$ , the center of a bowtie must belong to  $A$ . Hence, we have the following.

**Proposition 3.4** *Let  $Q_i$  be a big block of block graph  $G$  for  $i = 1, 2, 3$ . If  $G[V(Q_1 \cup Q_2)]$  and  $G[V(Q_2 \cup Q_3)]$  are two bowties of  $G$  with different centers, then  $G$  has no monopolar partition.*  $\square$

**Proposition 3.5** *Let  $Q_1, \dots, Q_t$  be big blocks of block graph  $G$  containing vertex  $u$  and  $t \geq 2$ . Let  $\widehat{G} = G - V(\bigcup_{1 \leq j \leq t} Q_j)$ ,  $S_1 = N(u) \cap V(\widehat{G})$ , and  $S_2 = N(V(\bigcup_{1 \leq j \leq t} Q_j) - u) \cap V(\widehat{G})$ . Assume  $S_1 \cup S_2 \neq \emptyset$ . Let  $G'$  be obtained from  $\widehat{G}$  by the following two operations:*

- For every  $w \in S_1$ , adding a bowtie and joining its center to  $w$ ;

- For every  $w \in S_2$ , adhering a bowtie to  $w$ .

Then  $G$  is uniquely monopolar-partitionable if and only if  $G'$  is.

**Proof:** Suppose that  $G$  is uniquely monopolar-partitionable. Since  $\widehat{G}$  is a subgraph of  $G$ ,  $\widehat{G}$  is a monopolar graph. The monopolar partition of  $G$ , when restricted to  $\widehat{G}$ , can be extended to a monopolar partition of  $G'$ . Hence  $G'$  is a monopolar graph. Assume that  $G'$  has at least two different monopolar partitions. For any monopolar partition  $(A', B')$  of  $G'$ , it is obvious that  $S_1 \subseteq B'$  and  $S_2 \subseteq A'$ . So,  $\widehat{G}$  has at least two different monopolar partitions. Furthermore, each monopolar partition of  $\widehat{G}$  can be extended to a monopolar partition of  $G$ . Hence  $G$  has at least two different monopolar partitions, which is a contradiction. Hence  $G'$  is uniquely monopolar-partitionable.

Suppose that  $G'$  is uniquely monopolar-partitionable. It is obvious that  $G$  is a monopolar graph. For any monopolar partition  $(A, B)$  of  $G$ ,  $S_1 \subseteq B$  and  $S_2 \subseteq A$ . If  $G$  has at least two different monopolar partitions, then  $\widehat{G}$  has at least two different monopolar partitions. Since each monopolar partition of  $G$ , when restricted to  $\widehat{G}$ , can be extended to a monopolar partition of  $G'$ ,  $G'$  has at least two different monopolar partitions, which is a contradiction. Hence  $G$  is uniquely monopolar-partitionable.  $\square$

By Proposition 3.5, we can assume that block graph  $G$  has no three big blocks with a common vertex. Moreover, each bowtie of  $G$  is a terminal bowtie. A proof similar to that of Proposition 3.5 yields the following.

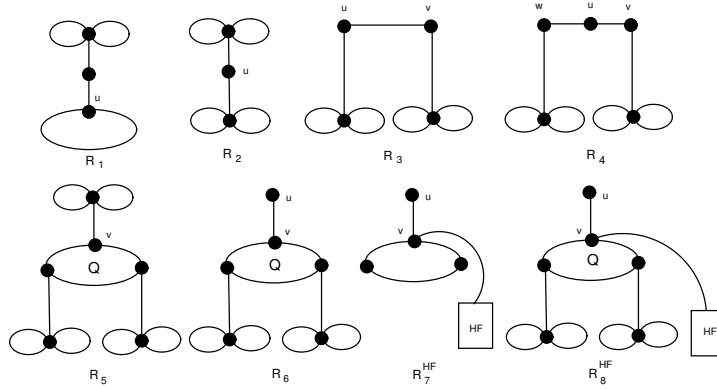


Fig. 1. Each ellipse is a big block, each vertex of  $Q \setminus \{v\}$  in  $R_5 \cup R_6$  is adjacent to exactly one terminal bowtie, and  $HF$  is a subgraph of  $G$ .

**Proposition 3.6** Let  $G$  be a block graph and let  $G'$  be defined as follows:

- Suppose that  $G$  contains induced subgraph  $R_1$ . Let  $G'$  be obtained from  $G$  by deleting the terminal bowtie of  $R_1$  and adhering a big block to vertex  $u$ ;
- Suppose that  $G$  contains induced subgraph  $R_2$ . Let  $G'$  be obtained from  $G$  by deleting a terminal bowtie of  $R_2$ ;

- Suppose that  $G$  contains induced subgraph  $R_3$ , where  $V(G) - V(R_3) \neq \emptyset$  and  $N(u) \cap N(v) = \emptyset$ . Let  $G'$  be obtained from  $G$  by deleting  $R_3$  and adhering a bowtie to each vertex  $w \in (N(u) \cup N(v)) \setminus V(R_3)$ ;
- Suppose that  $G$  contains induced subgraph  $R_4$ . Let  $G'$  be obtained from  $G$  by deleting the two terminal bowties of  $R_4$  and adhering a bowtie to vertex  $u$ ;
- Suppose that  $G$  contains induced subgraph  $R_5$  and  $V(G) - V(R_5) \neq \emptyset$ . Let  $G'$  be obtained from  $G$  by deleting  $R_5$  and adhering a bowtie to each vertex of  $N(Q) \setminus V(R_5)$ .

Then  $G$  is uniquely monopolar-partitionable if and only if  $G'$  is. □

**Proposition 3.7** Suppose that  $(A, B)$  is the unique monopolar partition of block graph  $G$ .

- (1) If  $Q$  is either a terminal block or a suspending block, then the cut vertex  $v$  of  $Q$  belongs to  $A$ ;
- (2) The center  $x$  of each end star belongs to  $A$ .

**Proof:** (1) Suppose that  $v \in B$ . If  $A \cap V(Q) = \emptyset$ , say  $u \in V(Q) \setminus \{v\}$ , then  $(A \cup \{u\}, B \setminus \{u\})$  is a monopolar partition of  $G$ . If  $A \cap V(Q) \neq \emptyset$ , say  $u \in A \cap V(Q)$ , then  $(A \setminus \{u\}, B \cup \{u\})$  is a monopolar partition of  $G$ . Hence, if  $v \in B$ , then  $G$  has a monopolar partition different from  $(A, B)$ , which is a contradiction. So,  $v \in A$ .

(2) Assume that  $vx \in E(G)$  and  $v$  does not belong to the end star. Suppose that  $x \in B$ . If  $v \in B$ , then  $((A - N(x)) \cup \{x\}, (B \setminus \{x\}) \cup N(x))$  is a monopolar partition of  $G$ . Suppose that  $v \in A$ . If  $B \cap N(x) = \emptyset$ , say  $w \in N(x) \setminus \{u\}$ , then  $(A \setminus \{w\}, B \cup \{w\})$  is a monopolar partition of  $G$ . If  $B \cap N(x) \neq \emptyset$ , say  $w \in N(x) \cap B$ , then  $(A \cup \{w\}, B \setminus \{w\})$  is a monopolar partition of  $G$ . Hence, if  $x \in B$ , then  $G$  has a monopolar partition different from  $(A, B)$ , which is a contradiction. So,  $x \in A$ . □

**Corollary 3.8** Let  $G$  be a uniquely monopolar-partitionable block graph. Then no suspending block of  $G$  is adjacent to a terminal block, an end star, or a terminal bowtie. □

**Proposition 3.9** Let  $G'$  be an induced subgraph of block graph  $G$ . Suppose that each monopolar partition of  $G'$  can be extended to at least one monopolar partition of  $G$ . Moreover, suppose that if  $G'$  has a unique monopolar partition, then it can be extended to exactly one monopolar partition of  $G$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G'$  is.

**Proof:** Suppose that  $G$  is uniquely monopolar-partitionable. Since  $G'$  is an induced subgraph of  $G$ , it follows that  $G'$  is a monopolar graph. If  $G'$  has two different monopolar partitions, then these monopolar partitions can be extended to two different monopolar partitions of  $G$ , which is a contradiction. Hence,  $G'$  is uniquely monopolar-partitionable.

Suppose that  $G'$  is uniquely monopolar-partitionable. It is obvious that  $G$  is uniquely monopolar-partitionable. □

By Proposition 3.7 and Proposition 3.9, we have the following corollary.

**Corollary 3.10** Let  $G$  be a block graph and let  $G'$  be defined as follows:

- if  $G$  contains a leaf  $w$  adjacent to a block, then  $G' = G - w$ ;
- if a vertex  $v$  is adjacent to two terminal blocks  $Q_1$  and  $Q_2$ , then  $G' = G - Q_2$ ;
- if  $G$  contains the tangent subgraph  $R_6 = R_w^1$ , then  $G'$  is obtained from  $G$  by deleting  $V(R_6) \setminus \{u\}$ .

Then  $G$  is uniquely monopolar-partitionable if and only if  $G'$  is.  $\square$

## 4 Reductions of block graphs

Let  $G$  be a block graph. In view of Propositions 3.4, 3.5 and 3.6, we may assume that  $G$  satisfies the following two conditions:

- Each bowtie of  $G$  is a terminal bowtie and  $G$  has no two adjacent terminal bowties.
- $G$  has no induced subgraph  $R_i$  for  $i = 1, 2, 3, 4, 5$ , where  $u$  and  $v$  in  $R_3$  do not belong to the same big block of  $G$ .

Let  $T$  be a tree obtained from  $G$  by contracting each terminal bowtie, end star, and big block, respectively. Let  $v_0 v_1 \cdots v_d$  be a longest path of  $T$ . Let  $V_i = \{u \in V(T) \mid d(u, v_0) = i\}$  for  $i = 0, 1, 2, \dots, d$ . Note that  $(V_0, V_1, \dots, V_d)$  is a vertex partition of  $T$ . From the vertex partition of  $T$ , we obtain a vertex partition  $(V_G^0, V_G^1, \dots, V_G^d)$  of  $G$  as follows:  $u \in V_i$  if and only if  $u \in V_G^i$  or all the vertices of the corresponding terminal bowtie, end star or big block belong to  $V_G^i$ .

For each big block  $Q$  of  $G$ , if the block belongs to  $V_G^i$  of  $G$ , then  $Q$  is called an  $i^{\text{th}}$  level big block. Suppose that  $Q$  is the  $i^{\text{th}}$  level big block. If  $v \in V(Q)$  is adjacent to a vertex in  $V_G^{i-1}$ , then  $v$  is called the *upper vertex* of  $Q$ , the other vertices are called *down vertices* of  $Q$ . For any vertex  $v \in V_G^i$ , if there exists a vertex  $u \in V_G^{i-1}$  such that  $vu \in E(G)$ , then  $u$  is called the *parent* of  $v$ . Both  $Q$  and  $v$  are called *children* of  $u$ .

In the section, a family of some special graphs  $\{H_i, F_j, Y_k \mid 1 \leq i \leq 4, 1 \leq j \leq 5, k = 2, 5\}$  is given in Fig. 2, Fig. 3 and Fig. 4. Say  $d \geq 4$ . The basic idea in the section is as follows: By employing a tree trimming techniques, we delete all the vertices of  $V_G^d$ . Firstly, we consider each component  $C$  in the induced subgraph  $G[V_G^{d-1} \cup V_G^d]$  having nonempty intersection with  $V_G^d$ . By local structure of  $C$ , if  $C$  is not isomorphic to  $H_i$  for  $i \in \{1, 2, 3, 4\}$ , either we can determine that  $G$  is not uniquely monopolar-partitionable or some blocks of  $C$  are deleted. Secondly, we consider each component  $C'$  in the induced subgraph  $G[V_G^{d-2} \cup V_G^{d-1} \cup V_G^d]$  containing  $H_i$  as a tangent subgraph, where  $i \in \{1, 2, 3, 4\}$ . By local structure of  $C'$ , if  $C'$  is not isomorphic to  $F_i$  for  $i \in \{1, 2, 3, 4, 5\}$ , either we can determine that  $G$  is not uniquely monopolar-partitionable or some blocks of  $C'$  are deleted. Thirdly, we consider each component  $C''$  in the induced subgraph  $G[V_G^{d-3} \cup V_G^{d-2} \cup V_G^{d-1} \cup V_G^d]$  containing  $F_j$  as a tangent subgraph, where  $j \in \{1, 2, 3, 4, 5\}$ . By local structure of  $C''$ , if  $C''$  is not isomorphic to  $Y_k$  for  $k \in \{2, 5\}$ , either we can determine that  $G$  is not uniquely monopolar-partitionable or some blocks of  $C''$  are deleted. Finally, we consider each component  $C'''$  in the induced subgraph  $G[V_G^{d-4} \cup V_G^{d-3} \cup V_G^{d-2} \cup V_G^{d-1} \cup V_G^d]$  containing  $Y_j$  as a tangent subgraph, where  $j \in \{2, 5\}$ . By local structure of  $C'''$ , either we can determine that  $G$  is not uniquely monopolar-partitionable or all blocks of  $C'''$  belonging to  $V_G^d$  are deleted. Then we obtain a new block graph whose associated tree has diameter less than  $d$ .

**Proposition 4.1** *Let  $Q$  be a big block of  $G$ , and let  $G_1, G_2, \dots, G_t$  be the components of  $G - Q$ . Assume that the upper vertex  $v$  of  $Q$  is adjacent to  $G_1, \dots, G_s$ . Suppose that  $G_j$  is a terminal bowtie, a terminal block, an end star or an isolated vertex for  $j = s + 1, \dots, t$ .*

- (1) Suppose that there exists a down vertex  $w$  of  $Q$  such that  $w$  is not adjacent to a terminal bowtie. Let  $G' = G[V(Q) \cup V(\bigcup_{1 \leq i \leq s} G_i)]$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G'$  is.
- (2) Suppose that each down vertex of  $Q$  is adjacent to a terminal bowtie. Let  $G'$  be obtained from  $G$  by deleting all the  $G_k$  except exactly one terminal bowtie for each down vertex, where  $k \in \{s+1, \dots, t\}$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G'$  is.

**Proof:** (1) Suppose that  $G$  is uniquely monopolar-partitionable. Since  $G'$  is an induced subgraph of  $G$ ,  $G'$  is a monopolar graph. For any monopolar partition  $(A', B')$  of  $G'$ ,  $v \in A'$ . Otherwise,  $G'$  has at least two monopolar partitions. One has  $V(Q) \subseteq B'$ , the other has  $V(Q) \setminus \{w\} \subseteq B'$  and  $w \in A'$ . Both of them can be extended to a monopolar partition of  $G$ , which is a contradiction. Since  $v \in A'$ , any monopolar partition  $(A', B')$  of  $G[V(Q) \cup V(\bigcup_{1 \leq i \leq s} G_i)]$  can be extended to exactly one monopolar partition of  $G$ . So  $G'$  is uniquely monopolar-partitionable.

Suppose that  $G'$  is uniquely monopolar-partitionable. Let  $(A', B')$  be its monopolar partition. By Proposition 3.7,  $v \in A'$ . Since  $v \in A'$ ,  $(A', B')$  can be extended to exactly one monopolar partition of  $G$ . By Proposition 3.9,  $G$  is uniquely monopolar-partitionable.

- (2) By Proposition 3.9,  $G$  is uniquely monopolar-partitionable if and only if  $G'$  is.  $\square$

**Proposition 4.2** *Let  $v$  be a vertex of  $G$  not belonging to any big block, and let  $t$  denote the parent of  $v$ . Suppose  $v$  is not adjacent to a terminal bowtie and every child of  $v$  is a leaf, a terminal block or end star.*

- (1) *If  $G$  is uniquely monopolar-partitionable, then  $v$  is not adjacent to an end star.*
- (2) *If  $G$  is uniquely monopolar-partitionable, then  $v$  is not adjacent to both a leaf and a terminal block.*

**Proof:** (1) Suppose  $v$  is adjacent to an end star. Let  $u$  and  $w$  be the center and a leaf of the end star, respectively. Let  $(A, B)$  be the unique monopolar partition of  $G$ . By Proposition 3.7,  $u \in A$  and  $v \in B$ . If  $t \in A$ , then let  $u \in B$  and  $N(u) \cup N(v) \setminus \{u, v\} \subseteq A$ . So there exists a monopolar partition such that  $u \in B$ , which is a contradiction. If  $t \in B$ , let  $N(v) \subseteq B$  and  $N(u) \subseteq A$ , then there exists a monopolar partition such that  $u \in B$ , which is a contradiction.

(2) Suppose  $v$  is adjacent to both a leaf  $s$  and a terminal block. Let  $u$  and  $w$  be the upper vertex and a down vertex of the terminal block, respectively. Let  $(A, B)$  be the unique monopolar partition of  $G$ . By Proposition 3.7,  $u \in A$  and  $v \in B$ . If  $t \in A$ , then  $s \in A$  or  $s \in B$ . Then  $G$  has two different monopolar partitions, which is a contradiction. If  $t \in B$ , let  $v \in A$  and  $N[u] \setminus \{v\} \subseteq B$ , then there exists a monopolar partition such that  $u \in B$ , which is a contradiction.  $\square$

By Proposition 4.1, Proposition 4.2, Corollary 3.8, Corollary 3.10 and the fact that  $G$  has no induced subgraph  $R_1$  and  $R_5$ , if  $d \geq 2$ , then we can assume that each component of  $G[V_G^{d-1} \cup V_G^d]$ , having nonempty intersection with  $V_G^d$ , is isomorphic to  $H_i$  for  $i \in \{1, 2, 3, 4\}$ .



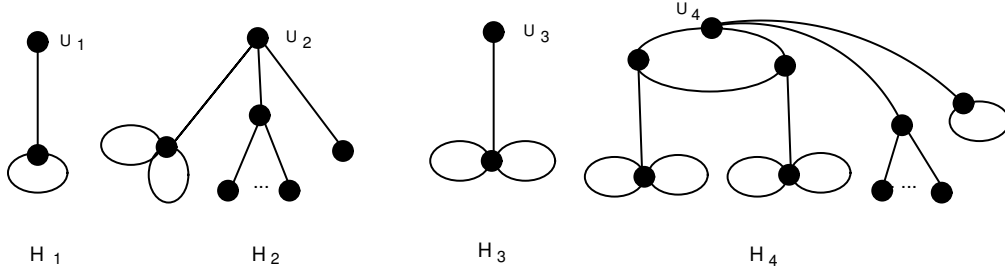


Fig. 2. Each ellipse is a big block,  $u_2$  is adjacent to at least one leaf or end star,

$u_4$  is adjacent to at least one end star or terminal block.

For each  $H_i$ , let  $w_i$  denote the parent of  $u_i$ . For  $i = 1, \dots, 4$ , let  $F_i = G[V(H_i) \cup \{w_i\}]$ . If  $w_i = w_j$ , then let  $F_{ij} = G[V(F_i) \cup V(F_j)]$ . That is  $F_{ij} = G[V(H_i) \cup V(H_j) \cup \{w_i\}]$ . For  $k = 1, 2, 3, 4$ , let  $F_j^k$  denote the graph obtained from  $F_j$  by joining  $w_j$  to a terminal bowtie, an end star, a terminal block and a leaf, respectively.

By a proof similar to that of Proposition 3.7, we have the following.

**Proposition 4.3** *Suppose that  $(A, B)$  is the unique monopolar partition of block graph  $G$ . If  $G$  contains the tangent subgraph  $F_i = R_{w_i}^1$  for  $i = 1, 4$ , then  $w_i \in A$ . If  $G$  contains the tangent subgraph  $F_2 = R_{w_2}^1$ , then  $w_2 \in B$ .  $\square$*

By Proposition 4.3 and Proposition 3.7, we have the following corollary.

**Corollary 4.4** *Let  $G$  be uniquely monopolar-partitionable. Then  $G$  does not contain tangent subgraph  $F_{12}, F_{24}, F_1^1, F_1^2, F_1^3, F_4^1, F_4^2$  and  $F_4^3$ .  $\square$*

**Proposition 4.5** *If  $G$  contains the tangent subgraph  $F_{ij}$  for  $i, j \in \{1, 4\}$ , then  $G$  is uniquely monopolar-partitionable if and only if  $G - H_j$  is.*

**Proof:** It is obvious that  $G - H_j$  is an induced subgraph of  $G$ . For any monopolar partition of  $G - H_j$ , it can be extended to at least a monopolar partition of  $G$ . Suppose that  $G - H_j$  is uniquely monopolar-partitionable. Let  $(A', B')$  be the unique monopolar partition of  $G - H_j$ . By Proposition 4.3,  $w_i \in A'$ . Then  $(A', B')$  can be extended to exactly one monopolar partition of  $G$ . By Proposition 3.9,  $G$  is uniquely monopolar-partitionable if and only if  $G - H_j$  is.  $\square$

By a proof similar to that of Proposition 4.5, we have the following.

**Proposition 4.6** (1) *For any  $i \in \{1, 2, 3, 4\}$ , if  $G$  contains the tangent subgraph  $F_i^4$ , then  $G$  is uniquely monopolar-partitionable if and only if  $G - t$  is, where  $t \in V(F_i^4)$  is the leaf and is adjacent to  $w_i$ .*

(2) *Suppose that  $G$  contains the tangent subgraph  $F_2^i$  for  $i = 2, 3$ . Let  $G'$  be the graph obtained from  $G$  by deleting the end star and the terminal block that are adjacent to  $w_2$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G'$  is.  $\square$*

**Proposition 4.7** *Suppose that  $G$  contains the tangent subgraph  $F_{3j}$  for  $j \in \{1, 4\}$ . Let  $G'$  be obtained from  $G$  by deleting  $F_{3j} \setminus \{w_3\}$  and adhering a big block to  $w_3$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G'$  is.*

**Proof:** Let  $G'' = G - F_{3j} \setminus \{w_3\}$ . Suppose that  $G$  is uniquely monopolar-partitionable. Let  $(A, B)$  be the unique monopolar partition of  $G$ . By Proposition 4.3,  $w_3 \in A$ . Then  $(A \cap V(G''), B \cap V(G''))$  can be extended to a monopolar partition of  $G'$ . Hence,  $G'$  is a monopolar graph. For any monopolar partition  $(A', B')$  of  $G'$ ,  $w_3 \in A'$ . Otherwise,  $(A' \cap V(G''), B' \cap V(G''))$  can be extended to two different monopolar partitions of  $G$ , which is a contradiction. Since  $w_3 \in A'$ ,  $(A' \cap V(G''), B' \cap V(G''))$  can be extended to exactly one monopolar partition of  $G$ . Hence,  $G'$  is uniquely monopolar-partitionable.

Suppose that  $G'$  is uniquely monopolar-partitionable. Let  $(A', B')$  be the unique monopolar partition of  $G'$ . By Proposition 3.7,  $w_3 \in A'$ . Then  $(A' \cap V(G''), B' \cap V(G''))$  can be extended to a monopolar partition of  $G$ . Hence,  $G$  is a monopolar graph. For any monopolar partition  $(A, B)$  of  $G$ ,  $w_3 \in A$ . Otherwise,  $G'$  has two different monopolar partitions, which is a contradiction. If  $G$  has two different monopolar partitions, then  $G''$  has two different monopolar partitions. So  $G'$  has two different monopolar partitions, which is a contradiction. Hence,  $G$  is uniquely monopolar-partitionable.  $\square$

By Corollary 4.4, Proposition 4.5, Proposition 4.6, Proposition 4.7 and the fact that  $G$  has no induced subgraph  $R_3$  and  $R_4$ , we can assume that the subgraph induced by  $w_i$  and its descendant, having nonempty intersection with  $V_G^d$ , is isomorphic to  $F_i$  in Fig 3 for  $i \in \{1, \dots, 5\}$ .

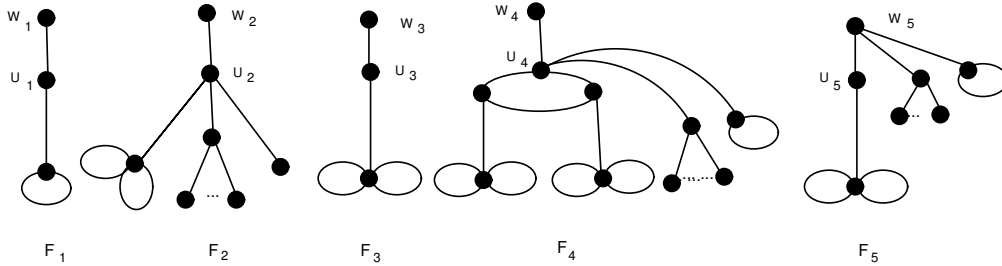


Fig. 3.  $w_5$  is adjacent to at least one terminal block or an end star

Since  $G$  has no induced subgraph  $R_1$ ,  $w_2, w_3, w_5$  do not belong to a big block. Let  $R_7^{H_i}$  and  $R_8^{H_i}$  be defined in Fig. 1 for  $i \in \{1, 4\}$ . For each  $F_i$ ,  $i = 1, 4$ , if  $w_i$  belongs to a big block of  $G$ , then we have the following:

**Proposition 4.8** *Let  $G$  be a block graph.*

(1) *Suppose that  $G$  contains the tangent subgraph  $F_i = R_{w_i}^1$ , where  $i \in \{1, 4\}$ . If  $G$  is uniquely monopolar-partitionable, then  $w_i$  is not a down vertex of a big block.*

(2) *Suppose that  $G$  contains the tangent subgraph  $R_7^{H_i} = R_w^1$ , where  $G[V(H_i) \cup \{v\}] = F_i$  and  $i \in \{1, 4\}$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G - H_i$  is.*

(3) *Suppose that  $G$  contains the tangent subgraph  $R_8^{H_i} = R_w^1$ , where  $G[V(H_i) \cup \{v\}] = F_i$  and  $i \in \{1, 4\}$ . Let  $G'$  be obtained from  $G$  by deleting  $H_i$  and all the children of the big block  $Q$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G'$  is.*

**Proof:** (1) Suppose that  $w_i$  is a down vertex of a big block. Let  $(A, B)$  be the unique monopolar partition of  $G$ . By Proposition 4.3,  $w_i \in A$ . Let  $G' = G - H_i$ . Then  $(A \cap V(G'), B \cap V(G'))$  is a monopolar partition of  $G'$ . So  $(A \cap (V(G') - w_i), (B \cap V(G')) \cup \{w_i\})$  is also a monopolar partition of  $G'$  and it can be extended to a monopolar partition of  $G$ . Hence,  $G$  has two different monopolar partitions, which is a contradiction.

(2) It is obvious that  $G - H_i$  is an induced subgraph of  $G$ . For any monopolar partition  $(A', B')$  of  $G - H_i$ , it can be extended to at least a monopolar partitions of  $G$ . Suppose that  $G - H_i$  is uniquely monopolar-partitionable. Let  $(A', B')$  be the unique monopolar partition. By Proposition 3.7,  $v \in A'$ . Then  $(A', B')$  can extend to exactly one monopolar partition of  $G$ . By Proposition 3.9,  $G$  is uniquely monopolar-partitionable if and only if  $G - H_i$  is.

(3) A proof similar to that of Case 1 in Proposition 4.1 shows that  $G$  is uniquely monopolar-partitionable if and only if  $G'$  is.  $\square$

Let  $m_i$  denote the parent of  $w_i$  and  $Y_i = G[V(F_i) \cup \{m_i\}]$  for  $i = 1, 2, \dots, 5$ . If  $m_i = w_j$ , let  $Y_{ij} = G[V(Y_i) \cup V(F_j)]$  for  $i = 2, 5$  and  $j = 1, 2, \dots, 5$ . If  $m_i = m_j$ , let  $Y_{ij} = G[V(Y_i) \cup V(Y_j)]$  for  $i, j = 2, 5$ . Let  $Y_i^j$  be the graph obtained from  $Y_i$  by joining  $m_i$  to a terminal bowtie, an end star, a terminal block or an isolated vertex, respectively, for  $i = 2, 5$  and  $j = 1, 2, 3, 4$ . By Proposition 4.8, we can assume that  $Y_i = R_{m_i}^1$  is a tangent subgraph of  $G$  for  $i \in \{1, \dots, 5\}$ .

**Proposition 4.9** Suppose that  $G$  contains the tangent subgraph  $Y_i$ , where  $i \in \{1, 4\}$ . Let  $H_i$  be the tangent subgraph of  $Y_i$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G - H_i \setminus \{u_i\}$  is.

**Proof:** It is obvious that  $G - H_i \setminus \{u_i\}$  is an induced subgraph of  $G$ . For any monopolar partition  $(A', B')$  of  $G - H_i \setminus \{u_i\}$ , it can be extended to at least a monopolar partition of  $G$ . Suppose that  $G - H_i \setminus \{u_i\}$  is uniquely monopolar-partitionable. Let  $(A', B')$  be the unique monopolar partition. By Proposition 3.7,  $w_i \in A'$  and  $u_i \in B'$ . Then  $(A', B')$  can extend to exactly one monopolar partition of  $G$ . By Proposition 3.9,  $G$  is uniquely monopolar-partitionable if and only if  $G - H_i \setminus \{u_i\}$  is.  $\square$

**Proposition 4.10** Suppose that  $G$  contains the tangent subgraph  $Y_3$ . Let  $F_3$  be the tangent subgraph of  $Y_3$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G - F_3$  is.

**Proof:** By Proposition 3.9,  $G$  is uniquely monopolar-partitionable if and only if  $G - F_3$  is.  $\square$

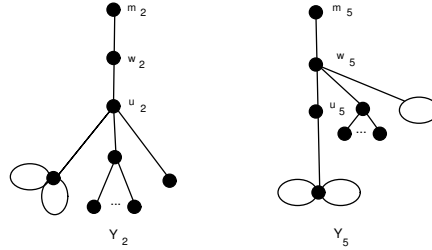


Fig. 4. Subgraphs  $Y_2$  and  $Y_5$

By Proposition 3.7 and Proposition 4.3, we have the following corollary.

**Corollary 4.11** *Suppose that block graph  $G$  has unique monopolar partition  $(A, B)$ . If  $G$  contains the tangent subgraph  $Y_i$  for  $i = 2, 5$ , then  $m_i \in A$ .  $\square$*

By Propositions 4.3 and Corollary 4.11, we have the following corollary.

**Corollary 4.12** *Let  $G$  be uniquely monopolar-partitionable block graph. Then  $G$  does not contain tangent subgraph  $YF_{22}, YF_{25}, YF_{52}, YF_{55}, Y_2^j$  and  $Y_5^j$  for  $j = 1, 2, 3$ .  $\square$*

By a proof similar to that of Proposition 4.5, we have the following.

**Proposition 4.13** (1) *For  $i = 2, 5$  and  $j = 1, 4$ , suppose that  $G$  contains the tangent subgraph  $YF_{ij}$ . Let  $H_j$  be the tangent subgraph of  $F_j$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G - H_j$  is.*

(2) *For  $i, j = 2, 5$ , suppose that  $G$  contains the tangent subgraph  $Y_{ij}$ . Let  $F_j$  be the tangent subgraph of  $Y_i$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G - F_j$  is.*

(3) *For  $i = 2, 5$ , if  $G$  contains the tangent subgraph  $Y_i^A$ , then  $G$  is uniquely monopolar-partitionable if and only if  $G - t$  is, where  $t \in V(Y_i^A)$  is a leaf and is adjacent to  $m_i$ .*

By a proof similar to that of Proposition 4.7, we have the following.

**Proposition 4.14** *Suppose that  $G$  contains the tangent subgraph  $YF_{i3}$  for  $i \in \{2, 5\}$ . Let  $G'$  be obtained from  $G$  by deleting  $YF_{i3} \setminus \{w_3\}$  and adhering a big block to  $w_3$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G'$  is.*

Let  $R_7^{F_i}$  and  $R_8^{F_i}$  be defined in Fig. 1 for  $i \in \{2, 5\}$ . By a proof similar to that of Proposition 4.8, we have the following.

**Proposition 4.15** *Let  $G$  be a block graph.*

(1) *Suppose that  $G$  contains the tangent subgraph  $Y_i$ . If  $G$  is uniquely monopolar-partitionable block graph, then  $m_i$  is not a down vertex of a big block, where  $i \in \{2, 5\}$ .*

(2) *Suppose that  $G$  contains the tangent subgraph  $R_7^{F_i} = R_w^1$ , where  $G[V(F_i) \cup \{v\}] = Y_i$  and  $i \in \{2, 5\}$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G - F_i$  is.*

(3) *Suppose that  $G$  contains the tangent subgraph  $R_8^{F_i} = R_w^1$ , where  $G[V(F_i) \cup \{v\}] = Y_i$  and  $i \in \{2, 5\}$ . Let  $G'$  be obtained from  $G$  by deleting  $F_i$  and all the children of the big block  $Q$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G'$  is.*

Let  $t_i$  denote the parent of  $m_i$  and  $X_i = G[V(Y_i) \cup \{t_i\}]$  for  $i = 2, 5$ . By Corollary 4.12, Proposition 4.13, Proposition 4.14 and Proposition 4.15, we can assume that  $X_i = R_{t_i}^1$  is a tangent subgraph of  $G$  for  $i \in \{2, 5\}$ .

By a proof similar to that of Proposition 4.1, we have the following.

**Proposition 4.16** *For  $i = 2, 5$ , suppose that  $G$  contains the tangent subgraph  $X_i$ . Let  $F_i$  be the tangent subgraph of  $X_i$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G - F_i \setminus \{w_i\}$  is.*

**Remark.** By above Propositions, we have deleted all of vertices of  $V_G^d$ . So we obtain a new block graph whose associated tree has diameter less than  $d$ .

## 5 Uniquely monopolar-partitionable block graphs

In order to determine whether a given block graph has a unique monopolar partition, we first define a family of block graphs. Let  $\Phi$  be the family of block graphs  $G$  satisfying the following conditions:

- (1) Either  $G$  is a bowtie or each bowtie of  $G$  is a terminal bowtie.
- (2)  $G$  has no induced subgraph  $R_i$  for  $i = 1, 2, 3, 4, 5$ , where  $u$  and  $v$  in  $R_3$  do not belong to the same big block of  $G$ .
- (3)  $G$  has no two adjacent terminal bowties.

Let  $G$  be a block graph. By Proposition 3.4, if  $G$  has two bowties  $G[V(Q_1 \cup Q_2)]$  and  $G[V(Q_2 \cup Q_3)]$  with different centers, then  $G$  is not uniquely monopolar-partitionable. Obviously, if  $G$  has two adjacent terminal bowties, then  $G$  is not uniquely monopolar-partitionable. Without loss of generality, we can assume that  $G$  has neither two bowties  $G[V(Q_1 \cup Q_2)]$  and  $G[V(Q_2 \cup Q_3)]$  with different centers nor two adjacent terminal bowties. By repeatedly applying Proposition 3.5 and Proposition 3.6, we obtain block graph  $G_1, \dots, G_t$  such that  $G_i \in \Phi$  or  $G_i \in \{ \text{a flower}, R_3, R_5 \}$  for  $1 \leq i \leq t$ . We have the following.

**Theorem 5.1** *Let  $G$  be a block graph. Suppose that  $G$  has neither two bowties  $G[V(Q_1 \cup Q_2)]$  and  $G[V(Q_2 \cup Q_3)]$  with different centers nor two adjacent terminal bowties. Then  $G$  is uniquely monopolar-partitionable if and only if  $G_i$  is uniquely monopolar-partitionable for  $1 \leq i \leq t$ , where  $G_i$  is defined as above.  $\square$*

If  $G_i$  is a flower or  $G_i \in \{R_3, R_5\}$ , then  $G_i$  is uniquely monopolar-partitionable. Now we determine whether or not a given block graph in  $\Phi$  is uniquely monopolar-partitionable. Suppose that  $G$  is a block graph and  $G \in \Phi$ . Let  $T$  denote the tree structure of  $G$ . Now we define some operations on  $G$  as follows:

**Operation  $\tau_1$ :** If a block is adjacent to a leaf, delete the leaf; If a vertex  $v$  is adjacent to two terminal blocks, delete one terminal block; If  $G$  contains the tangent subgraph  $R_6 = R_u^1$ , delete  $V(R_6) \setminus \{u\}$ .

**Operation  $\tau_2$ :** Suppose that each down vertex of a big block is only adjacent to a leaf, a terminal bowtie, a terminal block or an end star. If there exists a down vertex such that it is not adjacent to a terminal bowtie, then delete all the children of each down vertex of the block; otherwise, delete all the children of each down vertex except one terminal bowtie.

**Operation  $\tau_3$ :** Suppose that  $i = 1, 4$ . If  $G$  contains the tangent subgraph  $F_{ij}$  for  $j = 1, 4$ , delete  $H_j$ ; If  $G$  contains tangent subgraph  $F_j^A$  for  $j = 1, 2, 3, 4$ , delete the leaf; If  $G$  contains tangent subgraph  $F_{3i}$ , delete  $H_i \cup H_3$  and adhere a big block to  $w_3$ ; If  $G$  contains tangent subgraph  $F_2^j$  for  $j = 2, 3$ , delete the end star and the terminal block; If  $G$  contains the tangent subgraph  $Y_i$ , delete  $V(H_i) \setminus \{u_i\}$ ; If  $G$  contains the tangent subgraph  $Y_3$ , delete  $F_3$ .

**Operation  $\tau_4$ :** Suppose that  $i = 2, 5$ . If  $G$  contains tangent the subgraph  $YF_{ij}$  for  $j = 1, 4$ , delete  $H_j$ ; If  $G$  contains tangent subgraph  $YF_{3i}$ , delete  $F_i \cup H_3$  and adhere a big block to  $m_3$ ; If  $G$  contains the tangent subgraph  $Y_{ij}$  for  $j = 2, 5$ , delete  $F_j$ ; If  $G$  contains the tangent subgraph  $Y_i^A$ , delete the leaf; If  $G$  contains the tangent subgraph  $X_i$ , delete  $V(F_i) \setminus \{w_i\}$ .

**Operation  $\tau_5$ :** For  $i \in \{1, 4\}$ , if  $G$  contains the tangent subgraph  $R_7^{H_i} = R_u^1$ , delete  $H_i$ ; if  $G$  contains the tangent subgraph  $R_8^{H_i} = R_u^1$ , delete all the children of the big block of  $R_8^{H_i}$ . For  $i \in \{2, 5\}$ , if  $G$  contains the tangent subgraph  $R_7^{F_i} = R_u^1$ , delete  $F_i$ ; if  $G$  contains the tangent subgraph  $R_8^{F_i} = R_u^1$ , delete all the children of the big block of  $R_8^{F_i}$ .

By Propositions in Section 3 and Section 4, we have the following.

**Theorem 5.2** *Let  $G'$  be the graph obtained from  $G \in \Phi$  by some operation  $\tau_i$  for  $i \in \{1, \dots, 5\}$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G'$  is.  $\square$*

Let  $G^*$  be the graph obtained from  $G \in \Phi$  by a series of operations  $\tau_i$ , where  $i \in \{1, \dots, 5\}$ . It is obvious that  $G^* \in \Phi$ .

**Theorem 5.3** *Let  $G \in \Phi$  with  $\text{diam}(T) \leq 1$ , where  $T$  denotes the tree structure of  $G$ . Then  $G$  is uniquely monopolar-partitionable if and only if  $G$  is isomorphic to a bowtie or  $H_3$ .  $\square$*

To present our algorithm for recognizing uniquely monopolar-partitionable block graphs, we introduce the following five properties  $P_i$  and two sets  $\mathcal{G}_i$  of graphs:

$P_1$ : there exists a vertex  $v$  and a component  $C$  of  $G - v$  such that  $G[V(C) \cup \{v\}]$  is a tree and is not a star;

$P_2$ : some suspending block is adjacent to a terminal block, an end star or a terminal bowtie;

$P_3$ : there exists a vertex  $v$  such that  $v$  neither belongs to a big block nor adjacent to a terminal bowtie, but  $v$  is adjacent to either an end star or adjacent to both a leaf and a terminal block;

$P_4$ : for  $i \in \{1, 4\}$ ,  $F_i$  is a tangent subgraph of  $G$  and  $w_i$  is a down vertex of a big block;

$P_5$ : for  $j \in \{2, 5\}$ ,  $Y_j$  is a tangent subgraph of  $G$  and  $m_j$  is a down vertex of a big block.

$\mathcal{G}_1 = \{F_{12}, F_{24}, YF_{22}, YF_{25}, YF_{52}, YF_{55}, F_i^j, Y_k^j \mid i = 1, 4, k = 2, 5, j = 1, 2, 3\}$

$\mathcal{G}_2 = \{F_1, F_2, F_3, H_4, Y_2, Y_5\}$

#### Algorithm

**Input:** A connected block graph  $G \in \Phi$ . Let  $T$  denote the tree structure of  $G$ , and let  $v_0v_1 \dots v_d$  be a longest path of  $T$  and  $(V_G^0, V_G^1, \dots, V_G^d)$  be a vertex partition of  $G$  according to  $T$ .

**Output:** Determine whether or not  $G$  is uniquely monopolar-partitionable.

Repeatedly apply operation  $\tau_i$  for  $i = 1, \dots, 5$ , until one of the following occurs

- $G$  has property  $P_i$  for  $i \in \{1, \dots, 5\}$  ( $G$  is not uniquely monopolar-partitionable);
- $G$  contains a graph in  $\mathcal{G}_1$  as a tangent subgraph ( $G$  is not uniquely monopolar-partitionable);
- the reduced graph is in  $\mathcal{G}_2$  ( $G$  is not uniquely monopolar-partitionable);
- $\text{diam}(T) \leq 1$  ( $G$  is uniquely monopolar-partitionable if  $\text{diam}(T) \leq 1$  and  $G$  is isomorphic to a bowtie or to  $H_3$ , then  $G$  is uniquely monopolar-partitionable; otherwise  $G$  is not uniquely monopolar-partitionable).

We now discuss the correctness of the algorithm. In applying operations  $\tau_i$ , if any property  $P_i$  occurs, then  $G$  is not uniquely monopolar-partitionable according to Proposition 3.3, Corollary 3.8, and Propositions 4.2, 4.8 and 4.15; if some graph in  $\mathcal{G}_1$  is a tangent subgraph of  $G$ , then by Corollaries 4.4 and 4.12,  $G$  is not uniquely monopolar-partitionable. Suppose that none of properties  $P_i$  occurs and  $G$  does not contain any graph of  $\mathcal{G}_1$  as a tangent subgraph. Then the operations applied to  $G$  yield either a graph in  $\mathcal{G}_2$  or a graph whose associated  $T$  has diameter at most one. If the reduced graph is in  $\mathcal{G}_2$ , then it is obvious that  $G$  is not uniquely monopolar-partitionable. When  $\text{diam}(T) \leq 1$ , by Theorem 5.3,  $G$  is uniquely monopolar-partitionable if it is isomorphic to a bowtie or to  $H_3$ ; otherwise  $G$  is not uniquely monopolar-partitionable. Moreover all these steps can be implemented in polynomial time. Therefore we have the following:

**Theorem 5.4** *There is a polynomial time algorithm to decide if an input block graph is uniquely monopolar-partitionable.  $\square$*

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