# Uniquely monopolar-partitionable block graphs

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# *Uniquely monopolar-partitionable block graphs*

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As a common generalization of bipartite and split graphs, monopolar graphs are defined in terms of the existence of certain vertex partitions. It has been shown that to determine whether a graph has such a partition is an NP-complete problem for general graphs, and is polynomial time solvable for several classes of graphs. In this paper, we investigate graphs that admit a unique such partition and call them uniquely monopolar-partitionable graphs. By employing a tree trimming technique, we obtain a characterization of uniquely monopolar-partitionable block graphs. Our characterization implies a polynomial time algorithm for determining whether a block graph is uniquely monopolar-partitionable.

**Keywords:** Monopolar graph, monopolar partition, uniquely monopolar-partitionable grap, block graph, characterization, polynomial time algorithm

## 1 Introduction

Given a graph $G$, a *monopolar partition* of $G$ is a partition $(A, B)$ of its vertex set where $A$ is an independent set and $B$ induces a disjoint union of cliques in $G$. A graph which admits a monopolar partition is called *monopolar* or *monopolar-partitionable*.

Monopolar graphs were introduced in [17] as a common generalization of bipartite graphs and split graphs. Every bipartition of a bipartite graph is a monopolar partition. Graphs which admit monopolar partitions $(A, B)$ where $B$ induces a single clique are precisely split graphs [12, 14].

A monopolar graph is called *uniquely monopolar-partitionable* if it has exactly one monopolar partition, that is, if $(A, B)$ and $(A', B')$ are both monopolar partitions of $G$ then $A = A'$ (and $B = B'$). Since each complete graph has two monopolar partitions $(A, B)$ and $(A', B')$ where $A$ and $A'$ are the empty set and singleton set respectively, no complete graph is uniquely monopolar-partitionable. On the other hand, the graph obtained from two complete graphs of order at least three by identifying two vertices, one from each, is uniquely monopolar-partitionable.

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Unlike bipartite graphs and split graphs which are easy to recognize, recognizing monopolar graphs in general is an NP-complete problem (cf. [2] and [11]). It is currently unknown whether uniquely monopolar-partitionable graphs are recognizable in polynomial time.

In this paper, we consider the uniqueness of monopolar partitionability of block graphs. A *block* of a graph $G$ is a maximal subgraph of $G$ without cut-vertices. A graph is a *block* graph if every block is a clique (cf. [1, 15]). We shall give a structural characterzation of uniquely monopolar-partitionable block graphs by using a tree trimming technique. As a by-product, we obtain a polynomial time algorithm for determining whether a block graph is uniquely monopolar-partitionable. We note that such an algorithm can be extracted from [8].

# 2 Basic definitions

We follow the standard definition and terminology from [18] and consider only simple graphs. Let $G = (V, E)$ be a graph. The *neighbourhood* $N(v)$ of a vertex $v$ in $G$ consists of all vertices adjacent to $v$. The size of $N(v)$ is the *degree* of $v$ and denoted by $d(v)$. If $d(v) = 1$, then vertex $v$ is called a *leaf* of $G$. The *closed neighhood* $N[v]$ of $v$ is $N(v) \cup \{v\}$. For any $S \subseteq V$, the subgraph of $G$ induced by $S$ is denoted by $G[S]$. For convenience we write $G - S = G[V - S]$. For any two vertices $u, v \in V$, the *distance* $d_G(u, v)$ between $u$ and $v$ in $G$ is the length of a shortest $u, v$-path in $G$, and the *diameter* $diam(G)$ of $G$ is the maximum distance of any two vertices in $G$. We shall use $P_n, C_n$ and $K_n$ to denote the path, cycle and clique with order $n$, respectively.

Let $H$ be a subgraph of a graph $G$. By *contracting* $H$ in $G$ we mean to obtain a new graph $G'$ from $G - V(H)$ by adding a new vertex $w$ adjacent to all vertices which have at least one neighbour in $H$.

Let $G_1, G_2, \cdots, G_t$ be the components of $G - v$. For any $1 \leq i_1 < i_2 < \cdots < i_s \leq t$, $R_v^s = G[V(\bigcup_{1 \leq j \leq s} G_{i_j}) \cup \{v\}]$ is called a *tangent subgraph* of $G$. If a star with order at least two is a component of $G - v$ and only one center of the star is adjacent to $v$, then the star is called an *end star* of $G$, and $v$ is said to be *adjacent to* the end star. The vertex of the star that is adjacent to $v$ is taken as the center of the end star.

Let $G$ be a block graph with at least two blocks and let $Q$ be a block of $G$. A vertex $v$ of $G$ is said to be *adjacent to* $Q$ if $v$ is not in $Q$ but adjacent to a vertex of $Q$. If the order of $Q$ is at least 3, then $Q$ is called a *big block* of $G$. If a big block $Q$ contains a unique cut vertex and $G - Q$ is connected, then $Q$ is called a *terminal block*; if the big block $Q$ contains a unique cut vertex and $G - Q$ is disconnected, then $Q$ is called a *suspending block* of $G$. If two big blocks have no common vertex but contain adjacent vertices, then the two blocks are called *adjacent*. If two big blocks $Q_1$ and $Q_2$ have a common vertex $z$. then we call the subgraph $G[V(Q_1 \cup Q_2)]$ a *bowtie* of $G$ and the common vertex $z$ is the *center* of the bowtie. The bowtie is called a *terminal bowtie* if $G - Q_1 - Q_2$ is connected and its center is adjacent to exactly one vertex $w$ of $V(G) - V(Q_1 \cup Q_2)$. Vertex $w$ is said to be *adjacent to* the terminal bowtie. If the center of an end star or a terminal bowtie is adjacent to a big block, then the end star or the terminal bowtie is said to be *adjacent* to the big block. Two terminal bowties are *adjacent* if their centers are adjacent. A big block is said to be *adhered to a vertex* $v$ if $v$ is identified with a vertex from the block. Adhering a bowtie to a vertex $v$ means adhering two big blocks to the vertex. If a block graph $G$ is induced by $t \geq 3$ big blocks with a common vertex, then $G$ is called a *flower*.

Let $G'$ be an induced subgraph of $G$. Suppose that $(A', B')$ is a monopolar partition of $G'$. If there is a monopolar partition $(A, B)$ of $G$ such that $A \supseteq A'$ and $B \supseteq B'$, then we say that the monopolar partition $(A', B')$ can be *extended to* a monopolar partition of $G$. If there is a unique monopolar partition $(A, B)$

of $G$ such that $A \supseteq A'$ and $B \supseteq B'$, then we say that the monopolar partition $(A', B')$ can be *extended to exactly* one monopolar partition of $G$.

# 3 Basic properties

Suppose that $G$ is a uniquely monopolar-partitionable graph and $(A, B)$ is the unique monopolar partition of $G$. For any vertex $v \in V(G)$, if $G[N(v)]$ has an induced $P_3$, then $v \in B$. If $G[N(v)]$ has no induced $P_3$, then $G[N(v)]$ is a disjoint union of cliques. If $G[N(v)]$ contains two cliques $Q_1$ and $Q_2$ such that each has at least two vertices, then $v \in A$.

**Proposition 3.1** *Let $T$ be a tree with order $n \geq 2$. For any edge $uv \in E(T)$, there exists a monopolar partition $(A, B)$ such that $u, v \in B$.*

**Proof:** Let $T'$ be obtained from $T$ by contracting edge $uv$ and let $w$ denote the new vertex of $T'$. Then $T'$ is a tree. Let $(A', B')$ be the bipartition of $T'$. Say $w \in B'$. Let $(A, B) = (A', (B' \setminus \{w\}) \cup \{u, v\})$. Then $(A, B)$ is a monopolar partition of $T$ such that $u, v \in B$. $\square$

**Corollary 3.2** *No tree is uniquely monopolar-partitionable.*

**Proposition 3.3** *Let $G$ be a uniquely monopolar-partitionable graph and $v$ be a vertex of $G$. Suppose that $C$ is a component of $G - v$ and $V(C) \cup \{v\}$ induces a tree in $G$. Then $V(C) \cup \{v\}$ induces a star in $G$.*

**Proof:** Since $V(C) \cup \{v\}$ induces a tree, $v$ has a unique neighbour in $C$ which we denoted by $x$. We show that $x$ is adjacent to every other vertex in $C$. Suppose not. Then there exist vertices $y, z$ such that $xyz$ is a path in $C$. Since $G$ is a monopolar graph and $G - C$ is a subgraph of $G$, $G - C$ is a monopolar graph. Let $(A', B')$ be a monopolar partition of $G - C$.

Assume first that $v \in A'$. By Proposition 3.1, $C$ has a monopolar partition $(A_1, B_1)$ such that $x, y \in B_1$. Let $(A_2, B_2)$ be a bipartition of $C$ where $x \in B_2$ and $y \in A_2$. Then $(A' \cup A_1, B' \cup B_1)$ and $(A' \cup A_2, B' \cup B_2)$ are different monopolar partitions of $G$, which is a contradiction.

Assume now that $v \in B'$. By Proposition 3.1, $C$ has a monopolar partition $(A_1, B_1)$ such that $y, z \in B_1$. Let $(A_2, B_2)$ be a bipartition of $C$ where $x, z \in A_2$ and $y \in B_2$. Then $(A' \cup A_1, B' \cup B_1)$ and $(A' \cup A_2, B' \cup B_2)$ are different monopolar partitions of $G$, which is a contradiction. $\square$

Suppose that $C$ is a component of $G - v$ of order at least two and $G[V(C) \cup \{v\}]$ is a tree. By Proposition 3.3, if $G$ is a uniquely monopolar-partitionable graph, then $C$ is an end star of $G$.

For any monopolar partition $(A, B)$ of $G$, the center of a bowtie must belong to $A$. Hence, we have the following.

**Proposition 3.4** *Let $Q_i$ be a big block of block graph $G$ for $i = 1, 2, 3$. If $G[V(Q_1 \cup Q_2)]$ and $G[V(Q_2 \cup Q_3)]$ are two bowties of $G$ with different centers, then $G$ has no monopolar partition.* $\square$

**Proposition 3.5** *Let $Q_1, \cdots, Q_t$ be big blocks of block graph $G$ containing vertex $u$ and $t \geq 2$. Let $\widehat{G} = G - V(\bigcup_{1 \leq j \leq t} Q_j)$, $S_1 = N(u) \cap V(\widehat{G})$, and $S_2 = N(V(\bigcup_{1 \leq j \leq t} Q_j) - u) \cap V(\widehat{G})$. Assume $S_1 \cup S_2 \neq \emptyset$. Let $G'$ be obtained from $\widehat{G}$ by the following two operations:*

- *For every $w \in S_1$, adding a bowtie and joining its center to $w$;*

- *For every $w \in S_2$, adhering a bowtie to $w$.*

*Then $G$ is uniquely monopolar-partitionable if and only if $G'$ is.*

**Proof:** Suppose that $G$ is uniquely monopolar-partitionable. Since $\widehat{G}$ is a subgraph of $G$, $\widehat{G}$ is a monopolar graph. The monopolar partition of $G$, when restricted to $\widehat{G}$, can be extended to a monopolar partition of $G'$. Hence $G'$ is a monopolar graph. Assume that $G'$ has at least two different monopolar partitions. For any monopolar partition $(A', B')$ of $G'$, it is obvious that $S_1 \subseteq B'$ and $S_2 \subseteq A'$. So, $\widehat{G}$ has at least two different monopolar partitions. Furthermore, each monopolar partition of $\widehat{G}$ can be extended to a monopolar partition of $G$. Hence $G$ has at least two different monopolar partitions, which is a contradiction. Hence $G'$ is uniquely monopolar-partitionable.

Suppose that $G'$ is uniquely monopolar-partitionable. It is obvious that $G$ is a monopolar graph. For any monopolar partition $(A, B)$ of $G$, $S_1 \subseteq B$ and $S_2 \subseteq A$. If $G$ has at least two different monopolar partitions, then $\widehat{G}$ has at least two different monopolar partitions. Since each monopolar partition of $G$, when restricted to $\widehat{G}$, can be extended to a monopolar partition of $G'$, $G'$ has at least two different monopolar partitions, which is a contradiction. Hence $G$ is uniquely monopolar-partitionable. □

By Proposition 3.5, we can assume that block graph $G$ has no three big blocks with a common vertex. Moreover, each bowtie of $G$ is a terminal bowtie. A proof similar to that of Proposition 3.5 yields the following.

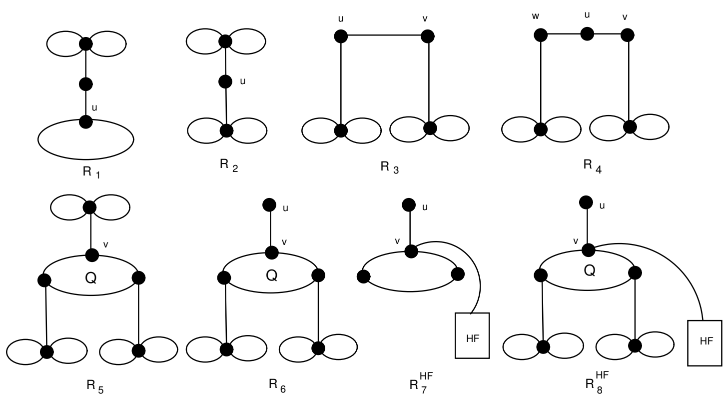


Fig. 1. Each ellipse is a big block, each vertex of $Q \setminus \{v\}$ in $R_5 \cup R_6$ is adjacent to exactly one terminal bowtie, and $HF$ is a subgraph of $G$.

**Proposition 3.6** *Let $G$ be a block graph and let $G'$ be defined as follows:*

- *Suppose that $G$ contains induced subgraph $R_1$. Let $G'$ be obtained from $G$ by deleting the terminal bowtie of $R_1$ and adhering a big block to vertex $u$;*
- *Suppose that $G$ contains induced subgraph $R_2$. Let $G'$ be obtained from $G$ by deleting a terminal bowtie of $R_2$;*

- *Suppose that $G$ contains induced subgraph $R_3$, where $V(G) - V(R_3) \neq \emptyset$ and $N(u) \cap N(v) = \emptyset$. Let $G'$ be obtained from $G$ by deleting $R_3$ and adhering a bowtie to each vertex $w \in (N(u) \cup N(v)) \setminus V(R_3)$;*
- *Suppose that $G$ contains induced subgraph $R_4$. Let $G'$ be obtained from $G$ by deleting the two terminal bowties of $R_4$ and adhering a bowtie to vertex $u$;*
- *Suppose that $G$ contains induced subgraph $R_5$ and $V(G) - V(R_5) \neq \emptyset$. Let $G'$ be obtained from $G$ by deleting $R_5$ and adhering a bowtie to each vertex of $N(Q) \setminus V(R_5)$.*

*Then $G$ is uniquely monopolar-partitionable if and only if $G'$ is.* □

**Proposition 3.7** *Suppose that $(A, B)$ is the unique monopolar partition of block graph $G$.*

*(1) If $Q$ is either a terminal block or a suspending block, then the cut vertex $v$ of $Q$ belongs to $A$;*

*(2) The center $x$ of each end star belongs to $A$.*

**Proof:** (1) Suppose that $v \in B$. If $A \cap V(Q) = \emptyset$, say $u \in V(Q) \setminus \{v\}$, then $(A \cup \{u\}, B \setminus \{u\})$ is a monopolar partition of $G$. If $A \cap V(Q) \neq \emptyset$, say $u \in A \cap V(Q)$, then $(A \setminus \{u\}, B \cup \{u\})$ is a monopolar partition of $G$. Hence, if $v \in B$, then $G$ has a monopolar partition different from $(A, B)$, which is a contradiction. So, $v \in A$.

(2) Assume that $vx \in E(G)$ and $v$ does not belong to the end star. Suppose that $x \in B$. If $v \in B$, then $((A - N(x)) \cup \{x\}, (B \setminus \{x\}) \cup N(x))$ is a monopolar partition of $G$. Suppose that $v \in A$. If $B \cap N(x) = \emptyset$, say $w \in N(x) \setminus \{u\}$, then $(A \setminus \{w\}, B \cup \{w\})$ is a monopolar partition of $G$. If $B \cap N(x) \neq \emptyset$, say $w \in N(x) \cap B$, then $(A \cup \{w\}, B \setminus \{w\})$ is a monopolar partition of $G$. Hence, if $x \in B$, then $G$ has a monopolar partition different from $(A, B)$, which is a contradiction. So, $x \in A$. □

**Corollary 3.8** *Let $G$ be a uniquely monopolar-partitionable block graph. Then no suspending block of $G$ is adjacent to a terminal block, an end star, or a terminal bowtie.* □

**Proposition 3.9** *Let $G'$ be an induced subgraph of block graph $G$. Suppose that each monopolar partition of $G'$ can be extended to at least one monopolar partition of $G$. Moreover, suppose that if $G'$ has a unique monopolar partition, then it can be extended to exactly one monopolar partition of $G$. Then $G$ is uniquely monopolar-partitionable if and only if $G'$ is.*

**Proof:** Suppose that $G$ is uniquely monopolar-partitionable. Since $G'$ is an induced subgraph of $G$, it follows that $G'$ is a monopolar graph. If $G'$ has two different monopolar partitions, then these monopolar partitions can be extended to two different monopolar partitions of $G$, which is a contradiction. Hence, $G'$ is uniquely monopolar-partitionable.

Suppose that $G'$ is uniquely monopolar-partitionable. It is obvious that $G$ is uniquely monopolar-partitionable.

□

By Proposition 3.7 and Proposition 3.9, we have the following corollary.

**Corollary 3.10** *Let $G$ be a block graph and let $G'$ be defined as follows:*

- *if $G$ contains a leaf $w$ adjacent to a block, then $G' = G - w$;*
- *if a vertex $v$ is adjacent to two terminal blocks $Q_1$ and $Q_2$, then $G' = G - Q_2$;*
- *if $G$ contains the tangent subgraph $R_6 = R_u^1$, then $G'$ is obtained from $G$ by deleting $V(R_6) \setminus \{u\}$.*

*Then $G$ is uniquely monopolar-partitionable if and only if $G'$ is.* □

# 4 Reductions of block graphs

Let $G$ be a block graph. In view of Propositions 3.4, 3.5 and 3.6, we may assume that $G$ satisfies the following two conditions:

- Each bowtie of $G$ is a terminal bowtie and $G$ has no two adjacent terminal bowties.
- $G$ has no induced subgraph $R_i$ for $i = 1, 2, 3, 4, 5$, where $u$ and $v$ in $R_3$ do not belong to the same big block of $G$.

Let $T$ be a tree obtained from $G$ by contracting each terminal bowtie, end star, and big block, respectively. Let $v_0v_1 \cdots v_d$ be a longest path of $T$. Let $V_i = \{u \in V(T) | d(u, v_0) = i\}$ for $i = 0, 1, 2, \cdots, d$. Note that $(V_0, V_1, \cdots, V_d)$ is a vertex partition of $T$. From the vertex partition of $T$, we obtain a vertex partition $(V_G^0, V_G^1, \cdots, V_G^d)$ of $G$ as follows: $u \in V_i$ if and only if $u \in V_G^i$ or all the vertices of the corresponding terminal bowtie, end star or big block belong to $V_G^i$.

For each big block $Q$ of $G$, if the block belongs to $V_G^i$ of $G$, then $Q$ is called an *$i^{th}$ level big block*. Suppose that $Q$ is the $i^{th}$ level big block. If $v \in V(Q)$ is adjacent to a vertex in $V_G^{i-1}$, then $v$ is called the *upper vertex* of $Q$, the other vertices are called *down vertices* of $Q$. For any vertex $v \in V_G^i$, if there exists a vertex $u \in V_G^{i-1}$ such that $vu \in E(G)$, then $u$ is called the *parent* of $v$. Both $Q$ and $v$ are called *children* of $u$.

In the section, a family of some special graphs $\{H_i, F_j, Y_k | 1 \le i \le 4, 1 \le j \le 5, k = 2, 5\}$ is given in Fig. 2, Fig. 3 and Fig. 4. Say $d \ge 4$. The basic idea in the section is as follows: By employing a tree trimming techniques, we delete all the vertices of $V_G^d$. Firstly, we consider each component $C$ in the induced subgraph $G[V_G^{d-1} \cup V_G^d]$ having nonempty intersection with $V_G^d$. By local structure of $C$, if $C$ is not isomorphic to $H_i$ for $i \in \{1, 2, 3, 4\}$, either we can determine that $G$ is not uniquely monopolar-partitionable or some blocks of $C$ are deleted. Secondly, we consider each component $C'$ in the induced subgraph $G[V_G^{d-2} \cup V_G^{d-1} \cup V_G^d]$ containing $H_i$ as a tangent subgraph, where $i \in \{1, 2, 3, 4\}$. By local structure of $C'$, if $C'$ is not isomorphic to $F_i$ for $i \in \{1, 2, 3, 4, 5\}$, either we can determine that $G$ is not uniquely monopolar-partitionable or some blocks of $C'$ are deleted. Thirdly, we consider each component $C''$ in the induced subgraph $G[V_G^{d-3} \cup V_G^{d-2} \cup V_G^{d-1} \cup V_G^d]$ containing $F_j$ as a tangent subgraph, where $j \in \{1, 2, 3, 4, 5\}$. By local structure of $C''$, if $C''$ is not isomorphic to $Y_k$ for $k \in \{2, 5\}$, either we can determine that $G$ is not uniquely monopolar-partitionable or some blocks of $C''$ are deleted. Finally, we consider each component $C'''$ in the induced subgraph $G[V_G^{d-4} \cup V_G^{d-3} \cup V_G^{d-2} \cup V_G^{d-1} \cup V_G^d]$ containing $Y_j$ as a tangent subgraph, where $j \in \{2, 5\}$. By local structure of $C'''$, either we can determine that $G$ is not uniquely monopolar-partitionable or all blocks of $C'''$ belonging to $V_G^d$ are deleted. Then we obtain a new block graph whose associated tree has diameter less than $d$.

**Proposition 4.1** *Let $Q$ be a big block of $G$, and let $G_1, G_2, \cdots, G_t$ be the components of $G - Q$. Assume that the upper vertex $v$ of $Q$ is adjacent to $G_1, \cdots, G_s$. Suppose that $G_j$ is a terminal bowtie, a terminal block, an end star or an isolated vertex for $j = s + 1, \cdots, t$.*

*(1) Suppose that there exists a down vertex $w$ of $Q$ such that $w$ is not adjacent to a terminal bowtie. Let $G' = G[V(Q) \cup V(\bigcup_{1\leq i\leq s} G_i)]$. Then $G$ is uniquely monopolar-partitionable if and only if $G'$ is.*

*(2) Suppose that each down vertex of $Q$ is adjacent to a terminal bowtie. Let $G'$ be obtained from $G$ by deleting all the $G_k$ except exactly one terminal bowtie for each down vertex, where $k \in \{s+1, \cdots, t\}$. Then $G$ is uniquely monopolar-partitionable if and only if $G'$ is.*

**Proof:** (1) Suppose that $G$ is uniquely monopolar-partitionable. Since $G'$ is an induced subgraph of $G$, $G'$ is a monopolar graph. For any monopolar partition $(A', B')$ of $G'$, $v \in A'$. Otherwise, $G'$ has at least two monopolar partitions. One has $V(Q) \subseteq B'$, the other has $V(Q) \setminus \{w\} \subseteq B'$ and $w \in A'$. Both of them can be extended to a monopolar partition of $G$, which is a contradiction. Since $v \in A'$, any monopolar partition $(A', B')$ of $G[V(Q) \cup V(\bigcup_{1\leq i\leq s} G_i)]$ can be extended to exactly one monopolar partition of $G$. So $G'$ is uniquely monopolar-partitionable.

Suppose that $G'$ is uniquely monopolar-partitionable. Let $(A', B')$ be its monopolar partition. By Proposition 3.7, $v \in A'$. Since $v \in A'$, $(A', B')$ can be extended to exactly one monopolar partition of $G$. By Proposition 3.9, $G$ is uniquely monopolar-partitionable.

(2) By Proposition 3.9, $G$ is uniquely monopolar-partitionable if and only if $G'$ is. □

**Proposition 4.2** *Let $v$ be a vertex of $G$ not belonging to any big block, and let $t$ denote the parent of $v$. Suppose $v$ is not adjacent to a terminal bowtie and every child of $v$ is a leaf, a terminal block or end star.*

*(1) If $G$ is uniquely monopolar-partitionable, then $v$ is not adjacent to an end star.*

*(2) If $G$ is uniquely monopolar-partitionable, then $v$ is not adjacent to both a leaf and a terminal block.*

**Proof:** (1) Suppose $v$ is adjacent to an end star. Let $u$ and $w$ be the center and a leaf of the end star, respectively. Let $(A, B)$ be the unique monopolar partition of $G$. By Proposition 3.7, $u \in A$ and $v \in B$. If $t \in A$, then let $u \in B$ and $N(u) \cup N(v) \setminus \{u, v\} \subseteq A$. So there exists a monopolar partition such that $u \in B$, which is a contradiction. If $t \in B$, let $N(v) \subseteq B$ and $N(u) \subseteq A$, then there exists a monopolar partition such that $u \in B$, which is a contradiction.

(2) Suppose $v$ is adjacent to both a leaf $s$ and a terminal block. Let $u$ and $w$ be the upper vertex and a down vertex of the terminal block, respectively. Let $(A, B)$ be the unique monopolar partition of $G$. By Proposition 3.7, $u \in A$ and $v \in B$. If $t \in A$, then $s \in A$ or $s \in B$. Then $G$ has two different monopolar partitions, which is a contradiction. If $t \in B$, let $v \in A$ and $N[u] \setminus \{v\} \subseteq B$, then there exists a monopolar partition such that $u \in B$, which is a contradiction. □

By Proposition 4.1, Proposition 4.2, Corollary 3.8, Corollary 3.10 and the fact that $G$ has no induced subgraph $R_1$ and $R_5$, if $d \geq 2$, then we can assume that each component of $G[V_G^{d-1} \cup V_G^d]$, having nonempty intersection with $V_G^d$, is isomorphic to $H_i$ for $i \in \{1, 2, 3, 4\}$.

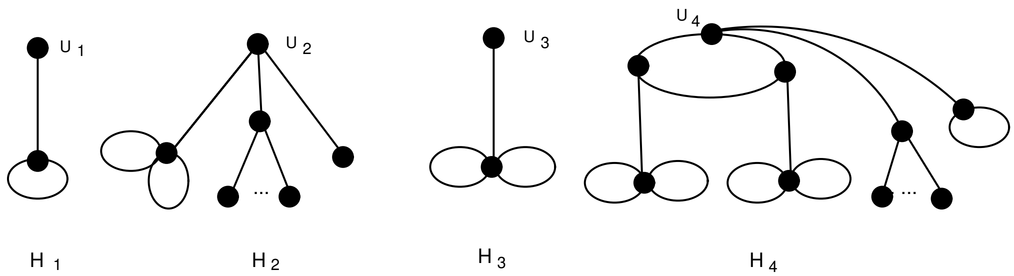


Fig. 2. Each ellipse is a big block, $u_2$ is adjacent to at least one leaf or end star,

$u_4$ is adjacent to at least one end star or terminal block.

For each $H_i$, let $w_i$ denote the parent of $u_i$. For $i = 1, \cdots, 4$, let $F_i = G[V(H_i) \cup \{w_i\}]$. If $w_i = w_j$, then let $F_{ij} = G[V(F_i) \cup V(F_j)]$. That is $F_{ij} = G[V(H_i) \cup V(H_j) \cup \{w_i\}]$. For $k = 1, 2, 3, 4$, let $F_j^k$ denote the graph obtained from $F_j$ by joining $w_j$ to a terminal bowtie, an end star, a terminal block and a leaf, respectively.

By a proof similar to that of Proposition 3.7, we have the following.

**Proposition 4.3** *Suppose that $(A, B)$ is the unique monopolar partition of block graph $G$. If $G$ contains the tangent subgraph $F_i = R^1_{w_i}$ for $i = 1, 4$, then $w_i \in A$. If $G$ contains the tangent subgraph $F_2 = R^1_{w_2}$, then $w_2 \in B$.* □

By Proposition 4.3 and Proposition 3.7, we have the following corollary.

**Corollary 4.4** *Let $G$ be uniquely monopolar-partitionable. Then $G$ does not contain tangent subgraph $F_{12}$, $F_{24}$, $F_1^1$, $F_1^2$, $F_1^3$, $F_4^1$, $F_4^2$ and $F_4^3$.* □

**Proposition 4.5** *If $G$ contains the tangent subgraph $F_{ij}$ for $i, j \in \{1, 4\}$, then $G$ is uniquely monopolar-partitionable if and only if $G - H_j$ is.*

**Proof:** It is obvious that $G - H_j$ is an induced subgraph of $G$. For any monopolar partition of $G - H_j$, it can be extended to at least a monopolar partition of $G$. Suppose that $G - H_j$ is uniquely monopolar-partitionable. Let $(A', B')$ be the unique monopolar partition of $G - H_j$. By Proposition 4.3, $w_i \in A'$. Then $(A', B')$ can be extended to exactly one monopolar partition of $G$. By Proposition 3.9, $G$ is uniquely monopolar-partitionable if and only if $G - H_j$ is. □

By a proof similar to that of Proposition 4.5, we have the following.

**Proposition 4.6** *(1) For any $i \in \{1, 2, 3, 4\}$, if $G$ contains the tangent subgraph $F_i^4$, then $G$ is uniquely monopolar-partitionable if and only if $G - t$ is, where $t \in V(F_i^4)$ is the leaf and is adjacent to $w_i$.*

*(2) Suppose that $G$ contains the tangent subgraph $F_2^i$ for $i = 2, 3$. Let $G'$ be the graph obtained from $G$ by deleting the end star and the terminal block that are adjacent to $w_2$. Then $G$ is uniquely monopolar-partitionable if and only if $G'$ is.* □

**Proposition 4.7** *Suppose that $G$ contains the tangent subgraph $F_{3j}$ for $j \in \{1,4\}$. Let $G'$ be obtained from $G$ by deleting $F_{3j} \setminus \{w_3\}$ and adhering a big block to $w_3$. Then $G$ is uniquely monopolar-partitionable if and only if $G^{'}$ is.*

**Proof:** Let $G'' = G - F_{3j} \setminus \{w_3\}$. Suppose that $G$ is uniquely monopolar-partitionable. Let $(A, B)$ be the unique monopolar partition of $G$. By Proposition 4.3, $w_3 \in A$. Then $(A \cap V(G''), B \cap V(G''))$ can be extended to a monopolar partition of $G'$. Hence, $G'$ is a monopolar graph. For any monopolar partition $(A', B')$ of $G'$, $w_3 \in A'$. Otherwise, $(A' \cap V(G''), B' \cap V(G''))$ can be extended to two different monopolar partitions of $G$, which is a contradiction. Since $w_3 \in A'$, $(A' \cap V(G''), B' \cap V(G''))$ can be extended to exactly one monopolar partition of $G$. Hence, $G^{'}$ is uniquely monopolar-partitionable.

Suppose that $G'$ is uniquely monopolar-partitionable. Let $(A', B')$ be the unique monopolar partition of $G'$. By Proposition 3.7, $w_3 \in A'$. Then $(A' \cap V(G''), B' \cap V(G''))$ can be extended to a monopolar partition of $G$. Hence, $G$ is a monopolar graph. For any monopolar partition $(A, B)$ of $G$, $w_3 \in A$. Otherwise, $G'$ has two different monopolar partitions, which is a contradiction. If $G$ has two different monopolar partitions, then $G''$ has two different monopolar partitions. So $G'$ has two different monopolar partitions, which is a contradiction. Hence, $G$ is uniquely monopolar-partitionable. $\square$

By Corollary 4.4, Proposition 4.5, Proposition 4.6, Proposition 4.7 and the fact that $G$ has no induced subgraph $R_3$ and $R_4$, we can assume that the subgraph induced by $w_i$ and its descendant, having nonempty intersection with $V_G^d$, is isomorphic to $F_i$ in Fig 3 for $i \in \{1, \cdots, 5\}$.

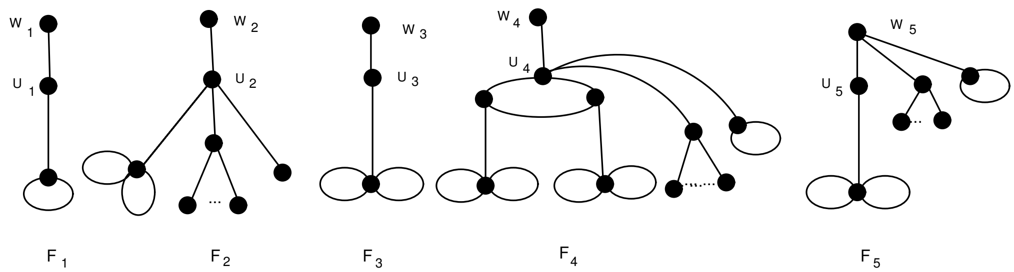


Fig. 3. $w_5$ is adjacent to at least one terminal block or an end star

Since $G$ has no induced subgraph $R_1$, $w_2, w_3, w_5$ do not belong to a big block. Let $R_7^{H_i}$ and $R_8^{H_i}$ be defined in Fig. 1 for $i \in \{1, 4\}$. For each $F_i$, $i = 1, 4$, if $w_i$ belongs to a big block of $G$, then we have the following:

**Proposition 4.8** *Let $G$ be a block graph.*

*(1) Suppose that $G$ contains the tangent subgraph $F_i = R_{w_i}^1$, where $i \in \{1, 4\}$. If $G$ is uniquely monopolar-partitionable, then $w_i$ is not a down vertex of a big block .*

*(2)Suppose that $G$ contains the tangent subgraph $R_7^{H_i} = R_u^1$, where $G[V(H_i) \cup \{v\}] = F_i$ and $i \in \{1, 4\}$. Then $G$ is uniquely monopolar-partitionable if and only if $G - H_i$ is.*

*(3)Suppose that $G$ contains the tangent subgraph $R_8^{H_i} = R_u^1$, where $G[V(H_i) \cup \{v\}] = F_i$ and $i \in \{1, 4\}$. Let $G'$ be obtained from $G$ by deleting $H_i$ and all the children of the big block $Q$. Then $G$ is uniquely monopolar-partitionable if and only if $G'$ is.*

**Proof:** (1) Suppose that $w_i$ is a down vertex of a big block. Let $(A, B)$ be the unique monopolar partition of $G$. By Proposition 4.3, $w_i \in A$. Let $G' = G - H_i$. Then $(A \cap V(G'), B \cap V(G'))$ is a monopolar partition of $G'$. So $(A \cap (V(G') - w_i), (B \cap V(G')) \cup \{w_i\})$ is also a monopolar partition of $G'$ and it can be extended to a monopolar partition of $G$. Hence, $G$ has two different monopolar partitions, which is a contradiction.

(2) It is obvious that $G - H_i$ is an induced subgraph of $G$. For any monopolar partition $(A', B')$ of $G - H_i$, it can be extended to at least a monopolar partitions of $G$. Suppose that $G - H_i$ is uniquely monopolar-partitionable. Let $(A', B')$ be the unique monopolar partition. By Proposition 3.7, $v \in A'$. Then $(A', B')$ can extend to exactly one monopolar partition of $G$. By Proposition 3.9, $G$ is uniquely monopolar-partitionable if and only if $G - H_i$ is.

(3) A proof similar to that of Case 1 in Proposition 4.1 shows that $G$ is uniquely monopolar-partitionable if and only if $G'$ is. $\square$

Let $m_i$ denote the parent of $w_i$ and $Y_i = G[V(F_i) \cup \{m_i\}]$ for $i = 1, 2, \cdots, 5$. If $m_i = w_j$, let $YF_{ij} = G[V(Y_i) \cup V(F_j)]$ for $i = 2, 5$ and $j = 1, 2, \cdots, 5$. If $m_i = m_j$, let $Y_{ij} = G[V(Y_i) \cup V(Y_j)]$ for $i, j = 2, 5$. Let $Y_i^j$ be the graph obtained from $Y_i$ by joining $m_i$ to a terminal bowtie, an end star, a terminal block or an isolated vertex, respectively, for $i = 2, 5$ and $j = 1, 2, 3, 4$. By Proposition 4.8, we can assume that $Y_i = R^1_{m_i}$ is a tangent subgraph of $G$ for $i \in \{1, \cdots, 5\}$.

**Proposition 4.9** *Suppose that $G$ contains the tangent subgraph $Y_i$, where $i \in \{1, 4\}$. Let $H_i$ be the tangent subgraph of $Y_i$. Then $G$ is uniquely monopolar-partitionable if and only if $G - H_i \setminus \{u_i\}$ is.*

**Proof:** It is obvious that $G - H_i \setminus \{u_i\}$ is an induced subgraph of $G$. For any monopolar partition of $G - H_i \setminus \{u_i\}$ can be extended to at least a monopolar partition of $G$. Suppose that $G - H_i \setminus \{u_i\}$ is uniquely monopolar-partitionable. Let $(A', B')$ be the unique monopolar partition. By Proposition 3.7, $w_i \in A'$ and $u_i \in B'$. Then $(A', B')$ can extend to exactly one monopolar partition of $G$. By Proposition 3.9, $G$ is uniquely monopolar-partitionable if and only if $G - H_i \setminus \{u_i\}$ is. $\square$

**Proposition 4.10** *Suppose that $G$ contains the tangent subgraph $Y_3$. Let $F_3$ be the tangent subgraph of $Y_3$. Then $G$ is uniquely monopolar-partitionable if and only if $G - F_3$ is.*

**Proof:** By Proposition 3.9, $G$ is uniquely monopolar-partitionable if and only if $G - F_3$ is. $\square$

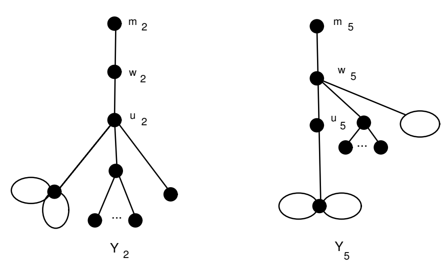


Fig. 4. Subgraphs $Y_2$ and $Y_5$

By Proposition 3.7 and Proposition 4.3, we have the following corollary.

**Corollary 4.11** *Suppose that block graph $G$ has unique monopolar partition $(A, B)$. If $G$ contains the tangent subgraph $Y_i$ for $i = 2, 5$, then $m_i \in A$.* □

By Propositions 4.3 and Corollary 4.11, we have the following corollary.

**Corollary 4.12** *Let $G$ be uniquely monopolar-partitionable block graph. Then $G$ does not contain tangent subgraph $YF_{22}$, $YF_{25}$, $YF_{52}$, $YF_{55}$, $Y_2^j$ and $Y_5^j$ for $j = 1, 2, 3$.* □

By a proof similar to that of Proposition 4.5, we have the following.

**Proposition 4.13** *(1) For $i = 2, 5$ and $j = 1, 4$, suppose that $G$ contains the tangent subgraph $YF_{ij}$. Let $H_j$ be the tangent subgraph of $F_j$. Then $G$ is uniquely monopolar-partitionable if and only if $G - H_j$ is.*

*(2) For $i, j = 2, 5$, suppose that $G$ contains the tangent subgraph $Y_{ij}$. Let $F_j$ be the tangent subgraph of $Y_i$. Then $G$ is uniquely monopolar-partitionable if and only if $G - F_j$ is.*

*(3) For $i = 2, 5$, if $G$ contains the tangent subgraph $Y_i^4$, then $G$ is uniquely monopolar-partitionable if and only if $G - t$ is, where $t \in V(Y_i^4)$ is a leaf and is adjacent to $m_i$.*

By a proof similar to that of Proposition 4.7, we have the following.

**Proposition 4.14** *Suppose that $G$ contains the tangent subgraph $YF_{i3}$ for $i \in \{2, 5\}$. Let $G'$ be obtained from $G$ by deleting $YF_{i3} \setminus \{w_3\}$ and adhering a big block to $w_3$. Then $G$ is uniquely monopolar-partitionable if and only if $G'$ is.*

Let $R_7^{F_i}$ and $R_8^{F_i}$ be defined in Fig. 1 for $i \in \{2, 5\}$. By a proof similar to that of Proposition 4.8, we have the following.

**Proposition 4.15** *Let $G$ be a block graph.*

*(1) Suppose that $G$ contains the tangent subgraph $Y_i$. If $G$ is uniquely monopolar-partitionable block graph, then $m_i$ is not a down vertex of a big block, where $i \in \{2, 5\}$.*

*(2)Suppose that $G$ contains the tangent subgraph $R_7^{F_i} = R_u^1$, where $G[V(F_i) \cup \{v\}] = Y_i$ and $i \in \{2, 5\}$. Then $G$ is uniquely monopolar-partitionable if and only if $G - F_i$ is.*

*(3)Suppose that $G$ contains the tangent subgraph $R_8^{F_i} = R_u^1$, where $G[V(F_i) \cup \{v\}] = Y_i$ and $i \in \{2, 5\}$. Let $G'$ be obtained from $G$ by deleting $F_i$ and all the children of the big block $Q$. Then $G$ is uniquely monopolar-partitionable if and only if $G'$ is.*

Let $t_i$ denote the parent of $m_i$ and $X_i = G[V(Y_i) \cup \{t_i\}]$ for $i = 2, 5$. By Corollary 4.12, Proposition 4.13, Proposition 4.14 and Proposition 4.15, we can assume that $X_i = R_{t_i}^1$ is a tangent subgraph of $G$ for $i \in \{2, 5\}$.

By a proof similar to that of Proposition 4.1, we have the following.

**Proposition 4.16** *For $i = 2, 5$, suppose that $G$ contains the tangent subgraph $X_i$. Let $F_i$ be the tangent subgraph of $X_i$. Then $G$ is uniquely monopolar-partitionable if and only if $G - F_i \setminus \{w_i\}$ is.*

**Remark.** By above Propositions, we have deleted all of vertices of $V_G^d$. So we obtain a new block graph whose associated tree has diameter less than $d$.

# 5 Uniquely monopolar-partitionable block graphs

In order to determine whether a given block graph has a unique monopolar partition, we first define a family of block graphs. Let $\Phi$ be the family of block graphs $G$ satisfying the following conditions:

(1) Either $G$ is a bowtie or each bowtie of $G$ is a terminal bowtie.

(2) $G$ has no induced subgraph $R_i$ for $i = 1, 2, 3, 4, 5$, where $u$ and $v$ in $R_3$ do not belong to the same big block of $G$.

(3) $G$ has no two adjacent terminal bowties.

Let $G$ be a block graph. By Proposition 3.4, if $G$ has two bowties $G[V(Q_1 \cup Q_2)]$ and $G[V(Q_2 \cup Q_3)]$ with different centers, then $G$ is not uniquely monopolar-partitionable. Obviously, if $G$ has two adjacent terminal bowties, then $G$ is not uniquely monopolar-partitionable. Without loss of generality, we can assume that $G$ has neither two bowties $G[V(Q_1 \cup Q_2)]$ and $G[V(Q_2 \cup Q_3)]$ with different centers nor two adjacent terminal bowties. By repeatedly applying Proposition 3.5 and Proposition 3.6, we obtain block graph $G_1, \cdots, G_t$ such that $G_i \in \Phi$ or $G_i \in \{$ a flower, $R_3, R_5$ $\}$ for $1 \le i \le t$. We have the following.

**Theorem 5.1** *Let $G$ be a block graph. Suppose that $G$ has neither two bowties $G[V(Q_1 \cup Q_2)]$ and $G[V(Q_2 \cup Q_3)]$ with different centers nor two adjacent terminal bowties. Then $G$ is uniquely monopolar-partitionable if and only if $G_i$ is uniquely monopolar-partitionable for $1 \le i \le t$, where $G_i$ is defined as above.* □

If $G_i$ is a flower or $G_i \in \{R_3, R_5\}$, then $G_i$ is uniquely monopolar-partitionable. Now we determine whether or not a given block graph in $\Phi$ is uniquely monopolar-partitionable. Suppose that $G$ is a block graph and $G \in \Phi$. Let $T$ denote the tree structure of $G$. Now we define some operations on $G$ as follows:

**Operation $\tau_1$:** If a block is adjacent to a leaf, delete the leaf; If a vertex $v$ is adjacent to two terminal blocks, delete one terminal block; If $G$ contains the tangent subgraph $R_6 = R_u^1$, delete $V(R_6) \setminus \{u\}$.

**Operation $\tau_2$:** Suppose that each down vertex of a big block is only adjacent to a leaf, a terminal bowtie, a terminal block or an end star. If there exists a down vertex such that it is not adjacent to a terminal bowtie, then delete all the children of each down vertex of the block; otherwise, delete all the children of each down vertex except one terminal bowtie.

**Operation $\tau_3$:** Suppose that $i = 1, 4$. If $G$ contains the tangent subgraph $F_{ij}$ for $j = 1, 4$, delete $H_j$; If $G$ contains tangent subgraph $F_j^4$ for $j = 1, 2, 3, 4$, delete the leaf; If $G$ contains tangent subgraph $F_{3i}$, delete $H_i \cup H_3$ and adhere a big block to $w_3$; If $G$ contains tangent subgraph $F_2^j$ for $j = 2, 3$, delete the end star and the terminal block; If $G$ contains the tangent subgraph $Y_i$, delete $V(H_i) \setminus \{u_i\}$; If $G$ contains the tangent subgraph $Y_3$, delete $F_3$.

**Operation $\tau_4$:** Suppose that $i = 2, 5$. If $G$ contains tangent the subgraph $YF_{ij}$ for $j = 1, 4$, delete $H_j$; If $G$ contains tangent subgraph $YF_{3i}$, delete $F_i \cup H_3$ and adhere a big block to $m_3$; If $G$ contains the tangent subgraph $Y_{ij}$ for $j = 2, 5$, delete $F_j$; If $G$ contains the tangent subgraph $Y_i^4$, delete the leaf; If $G$ contains the tangent subgraph $X_i$, delete $V(F_i) \setminus \{w_i\}$.

**Operation $\tau_5$:** For $i \in \{1, 4\}$, if $G$ contains the tangent subgraph $R_7^{H_i} = R_u^1$, delete $H_i$; if $G$ contains the tangent subgraph $R_8^{H_i} = R_u^1$, delete all the children of the big block of $R_8^{H_i}$. For $i \in \{2, 5\}$, if $G$ contains the tangent subgraph $R_7^{F_i} = R_u^1$, delete $F_i$; if $G$ contains the tangent subgraph $R_8^{F_i} = R_u^1$, delete all the children of the big block of $R_8^{F_i}$.

By Propositions in Section 3 and Section 4, we have the following.

**Theorem 5.2** *Let $G'$ be the graph obtained from $G \in \Phi$ by some operation $\tau_i$ for $i \in \{1, \cdots, 5\}$. Then $G$ is uniquely monopolar-partitionable if and only if $G'$ is.* □

Let $G^*$ be the graph obtained from $G \in \Phi$ by a series of operations $\tau_i$, where $i \in \{1, \cdots, 5\}$. It is obvious that $G^* \in \Phi$.

**Theorem 5.3** *Let $G \in \Phi$ with $diam(T) \leq 1$, where $T$ denotes the tree structure of $G$. Then $G$ is uniquely monopolar-partitionable if and only if $G$ is isomorphic to a bowtie or $H_3$.* □

To present our algorithm for recognizing uniquely monopolar-partitionable block graphs, we introduce the following five properties $P_i$ and two sets $\mathcal{G}_i$ of graphs:

$P_1$ : there exists a vertex $v$ and a component $C$ of $G - v$ such that $G[V(C) \cup \{v\}]$ is a tree and is not a star;

$P_2$: some suspending block is adjacent to a terminal block, an end star or a terminal bowtie;

$P_3$: there exists a vertex $v$ such that $v$ neither belongs to a big block nor adjacent to a terminal bowtie, but $v$ is adjacent to either an end star or adjacent to both a leaf and a terminal block;

$P_4$: for $i \in \{1, 4\}$, $F_i$ is a tangent subgraph of $G$ and $w_i$ is a down vertex of a big block;

$P_5$: for $j \in \{2, 5\}$, $Y_j$ is a tangent subgraph of $G$ and $m_j$ is a down vertex of a big block.

$\mathcal{G}_1 = \{F_{12}, F_{24}, YF_{22}, YF_{25}, YF_{52}, YF_{55}, F_i^j, Y_k^j \,|\, i = 1, 4\ k = 2, 5,\ j = 1, 2, 3\}$

$\mathcal{G}_2 = \{F_1, F_2, F_3, H_4, Y_2, Y_5\}$

**Algorithm**

**Input:** A connected block graph $G \in \Phi$. Let $T$ denote the tree structure of $G$, and let $v_0v_1 \cdots v_d$ be a longest path of $T$ and $(V_G^0, V_G^1, \cdots, V_G^d)$ be a vertex partition of $G$ according to $T$.

**Output:** Determine whether or not $G$ is uniquely monopolar-partitionable.

Repeatedly apply operation $\tau_i$ for $i = 1, \cdots, 5$, until one of the following occurs

- $G$ has property $P_i$ for $i \in \{1, \cdots, 5\}$ (// $G$ is not uniquely monopolar-partitionable);
- $G$ contains a graph in $\mathcal{G}_1$ as a tangent subgraph (// $G$ is not uniquely monopolar-partitionable);
- the reduced graph is in $\mathcal{G}_2$ (// $G$ is not uniquely monopolar-partitionable);
- $diam(T) \leq 1$ (// If $diam(T) \leq 1$ and $G$ is isomorphic to a bowtie or to $H_3$, then $G$ is uniquely monopolar-partitionable; otherwise $G$ is not uniquely monopolar-partitionable).

We now discuss the correctness of the algorithm. In applying operations $\tau_i$, if any property $P_i$ occurs, then $G$ is not uniquely monopolar-partitionable according to Proposition 3.3, Corollary 3.8, and Propositions 4.2, 4.8 and 4.15; if some graph in $\mathcal{G}_1$ is a tangent subgraph of $G$, then by Corollaries 4.4 and 4.12, $G$ is not uniquely monopolar-partitionable. Suppose that none of properties $P_i$ occurs and $G$ does not contain any graph of $\mathcal{G}_1$ as a tangent subgraph. Then the operations applied to $G$ yield either a graph in $\mathcal{G}_2$ or a graph whose associated $T$ has diameter at most one. If the reduced graph is in $\mathcal{G}_2$, then it is obvious that $G$ is not uniquely monopolar-partitionable. When $diam(T) \leq 1$, by Theorem 5.3, $G$ is uniquely monopolar-partitionable if it is isomorphic to a bowtie or to $H_3$; otherwise $G$ is not uniquely monopolar-partitionable. Moreover all these steps can be implemented in polynomial time. Therefore we have the following:

**Theorem 5.4** *There is a polynomial time algorithm to decide if an input block graph is uniquely monopolar-partitionable.* □

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