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Uniquely monopolar-partitionable block graphs

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As a common generalization of bipartite and split graphs, monopolar graphs are defined in terms of the existence of certain vertex partitions. It has been shown that to determine whether a graph has such a partition is an NP-complete problem for general graphs, and is polynomial time solvable for several classes of graphs. In this paper, we investigate graphs that admit a unique such partition and call them uniquely monopolar-partitionable graphs. By employing a tree trimming technique, we obtain a characterization of uniquely monopolar-partitionable block graphs. Our characterization implies a polynomial time algorithm for determining whether a block graph is uniquely monopolar-partitionable.

Keywords: Monopolar graph, monopolar partition, uniquely monopolar-partitionable graph, block graph, characterization, polynomial time algorithm

1 Introduction

Given a graph G , a *monopolar partition* of G is a partition (A, B) of its vertex set where A is an independent set and B induces a disjoint union of cliques in G . A graph which admits a monopolar partition is called *monopolar* or *monopolar-partitionable*.

Monopolar graphs were introduced in [17] as a common generalization of bipartite graphs and split graphs. Every bipartition of a bipartite graph is a monopolar partition. Graphs which admit monopolar partitions (A, B) where B induces a single clique are precisely split graphs [12, 14].

A monopolar graph is called *uniquely monopolar-partitionable* if it has exactly one monopolar partition, that is, if (A, B) and (A', B') are both monopolar partitions of G then $A = A'$ (and $B = B'$). Since each complete graph has two monopolar partitions (A, B) and (A', B') where A and A' are the empty set and singleton set respectively, no complete graph is uniquely monopolar-partitionable. On the other hand, the graph obtained from two complete graphs of order at least three by identifying two vertices, one from each, is uniquely monopolar-partitionable.

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Unlike bipartite graphs and split graphs which are easy to recognize, recognizing monopolar graphs in general is an NP-complete problem (cf. [2] and [11]). It is currently unknown whether uniquely monopolar-partitionable graphs are recognizable in polynomial time.

In this paper, we consider the uniqueness of monopolar partitionability of block graphs. A *block* of a graph G is a maximal subgraph of G without cut-vertices. A graph is a *block graph* if every block is a clique (cf. [1, 15]). We shall give a structural characterization of uniquely monopolar-partitionable block graphs by using a tree trimming technique. As a by-product, we obtain a polynomial time algorithm for determining whether a block graph is uniquely monopolar-partitionable. We note that such an algorithm can be extracted from [8].

2 Basic definitions

We follow the standard definition and terminology from [18] and consider only simple graphs. Let $G = (V, E)$ be a graph. The *neighbourhood* $N(v)$ of a vertex v in G consists of all vertices adjacent to v . The size of $N(v)$ is the *degree* of v and denoted by $d(v)$. If $d(v) = 1$, then vertex v is called a *leaf* of G . The *closed neighbourhood* $N[v]$ of v is $N(v) \cup \{v\}$. For any $S \subseteq V$, the subgraph of G induced by S is denoted by $G[S]$. For convenience we write $G - S = G[V - S]$. For any two vertices $u, v \in V$, the *distance* $d_G(u, v)$ between u and v in G is the length of a shortest u, v -path in G , and the *diameter* $diam(G)$ of G is the maximum distance of any two vertices in G . We shall use P_n, C_n and K_n to denote the path, cycle and clique with order n , respectively.

Let H be a subgraph of a graph G . By *contracting* H in G we mean to obtain a new graph G' from $G - V(H)$ by adding a new vertex w adjacent to all vertices which have at least one neighbour in H .

Let G_1, G_2, \dots, G_t be the components of $G - v$. For any $1 \leq i_1 < i_2 < \dots < i_s \leq t$, $R_v^s = G[V(\bigcup_{1 \leq j \leq s} G_{i_j}) \cup \{v\}]$ is called a *tangent subgraph* of G . If a star with order at least two is a component of $G - v$ and only one center of the star is adjacent to v , then the star is called an *end star* of G , and v is said to be *adjacent to* the end star. The vertex of the star that is adjacent to v is taken as the center of the end star.

Let G be a block graph with at least two blocks and let Q be a block of G . A vertex v of G is said to be *adjacent to* Q if v is not in Q but adjacent to a vertex of Q . If the order of Q is at least 3, then Q is called a *big block* of G . If a big block Q contains a unique cut vertex and $G - Q$ is connected, then Q is called a *terminal block*; if the big block Q contains a unique cut vertex and $G - Q$ is disconnected, then Q is called a *suspending block* of G . If two big blocks have no common vertex but contain adjacent vertices, then the two blocks are called *adjacent*. If two big blocks Q_1 and Q_2 have a common vertex z , then we call the subgraph $G[V(Q_1 \cup Q_2)]$ a *bowtie* of G and the common vertex z is the *center* of the bowtie. The bowtie is called a *terminal bowtie* if $G - Q_1 - Q_2$ is connected and its center is adjacent to exactly one vertex w of $V(G) - V(Q_1 \cup Q_2)$. Vertex w is said to be *adjacent to* the terminal bowtie. If the center of an end star or a terminal bowtie is adjacent to a big block, then the end star or the terminal bowtie is said to be *adjacent to* the big block. Two terminal bowties are *adjacent* if their centers are adjacent. A big block is said to be *adhered to a vertex* v if v is identified with a vertex from the block. Adhering a bowtie to a vertex v means adhering two big blocks to the vertex. If a block graph G is induced by $t \geq 3$ big blocks with a common vertex, then G is called a *flower*.

Let G' be an induced subgraph of G . Suppose that (A', B') is a monopolar partition of G' . If there is a monopolar partition (A, B) of G such that $A \supseteq A'$ and $B \supseteq B'$, then we say that the monopolar partition (A', B') can be *extended to* a monopolar partition of G . If there is a unique monopolar partition (A, B)

of G such that $A \supseteq A'$ and $B \supseteq B'$, then we say that the monopolar partition (A', B') can be *extended to exactly* one monopolar partition of G .

3 Basic properties

Suppose that G is a uniquely monopolar-partitionable graph and (A, B) is the unique monopolar partition of G . For any vertex $v \in V(G)$, if $G[N(v)]$ has an induced P_3 , then $v \in B$. If $G[N(v)]$ has no induced P_3 , then $G[N(v)]$ is a disjoint union of cliques. If $G[N(v)]$ contains two cliques Q_1 and Q_2 such that each has at least two vertices, then $v \in A$.

Proposition 3.1 *Let T be a tree with order $n \geq 2$. For any edge $uv \in E(T)$, there exists a monopolar partition (A, B) such that $u, v \in B$.*

Proof: Let T' be obtained from T by contracting edge uv and let w denote the new vertex of T' . Then T' is a tree. Let (A', B') be the bipartition of T' . Say $w \in B'$. Let $(A, B) = (A', (B' \setminus \{w\}) \cup \{u, v\})$. Then (A, B) is a monopolar partition of T such that $u, v \in B$. \square

Corollary 3.2 *No tree is uniquely monopolar-partitionable.*

Proposition 3.3 *Let G be a uniquely monopolar-partitionable graph and v be a vertex of G . Suppose that C is a component of $G - v$ and $V(C) \cup \{v\}$ induces a tree in G . Then $V(C) \cup \{v\}$ induces a star in G .*

Proof: Since $V(C) \cup \{v\}$ induces a tree, v has a unique neighbour in C which we denoted by x . We show that x is adjacent to every other vertex in C . Suppose not. Then there exist vertices y, z such that xyz is a path in C . Since G is a monopolar graph and $G - C$ is a subgraph of G , $G - C$ is a monopolar graph. Let (A', B') be a monopolar partition of $G - C$.

Assume first that $v \in A'$. By Proposition 3.1, C has a monopolar partition (A_1, B_1) such that $x, y \in B_1$. Let (A_2, B_2) be a bipartition of C where $x \in B_2$ and $y \in A_2$. Then $(A' \cup A_1, B' \cup B_1)$ and $(A' \cup A_2, B' \cup B_2)$ are different monopolar partitions of G , which is a contradiction.

Assume now that $v \in B'$. By Proposition 3.1, C has a monopolar partition (A_1, B_1) such that $y, z \in B_1$. Let (A_2, B_2) be a bipartition of C where $x, z \in A_2$ and $y \in B_2$. Then $(A' \cup A_1, B' \cup B_1)$ and $(A' \cup A_2, B' \cup B_2)$ are different monopolar partitions of G , which is a contradiction. \square

Suppose that C is a component of $G - v$ of order at least two and $G[V(C) \cup \{v\}]$ is a tree. By Proposition 3.3, if G is a uniquely monopolar-partitionable graph, then C is an end star of G .

For any monopolar partition (A, B) of G , the center of a bowtie must belong to A . Hence, we have the following.

Proposition 3.4 *Let Q_i be a big block of block graph G for $i = 1, 2, 3$. If $G[V(Q_1 \cup Q_2)]$ and $G[V(Q_2 \cup Q_3)]$ are two bowties of G with different centers, then G has no monopolar partition.* \square

Proposition 3.5 *Let Q_1, \dots, Q_t be big blocks of block graph G containing vertex u and $t \geq 2$. Let $\widehat{G} = G - V(\bigcup_{1 \leq j \leq t} Q_j)$, $S_1 = N(u) \cap V(\widehat{G})$, and $S_2 = N(V(\bigcup_{1 \leq j \leq t} Q_j) - u) \cap V(\widehat{G})$. Assume $S_1 \cup S_2 \neq \emptyset$. Let G' be obtained from \widehat{G} by the following two operations:*

- For every $w \in S_1$, adding a bowtie and joining its center to w ;

- For every $w \in S_2$, adhering a bowtie to w .

Then G is uniquely monopolar-partitionable if and only if G' is.

Proof: Suppose that G is uniquely monopolar-partitionable. Since \widehat{G} is a subgraph of G , \widehat{G} is a monopolar graph. The monopolar partition of G , when restricted to \widehat{G} , can be extended to a monopolar partition of G' . Hence G' is a monopolar graph. Assume that G' has at least two different monopolar partitions. For any monopolar partition (A', B') of G' , it is obvious that $S_1 \subseteq B'$ and $S_2 \subseteq A'$. So, \widehat{G} has at least two different monopolar partitions. Furthermore, each monopolar partition of \widehat{G} can be extended to a monopolar partition of G . Hence G has at least two different monopolar partitions, which is a contradiction. Hence G' is uniquely monopolar-partitionable.

Suppose that G' is uniquely monopolar-partitionable. It is obvious that G is a monopolar graph. For any monopolar partition (A, B) of G , $S_1 \subseteq B$ and $S_2 \subseteq A$. If G has at least two different monopolar partitions, then \widehat{G} has at least two different monopolar partitions. Since each monopolar partition of G , when restricted to \widehat{G} , can be extended to a monopolar partition of G' , G' has at least two different monopolar partitions, which is a contradiction. Hence G is uniquely monopolar-partitionable. \square

By Proposition 3.5, we can assume that block graph G has no three big blocks with a common vertex. Moreover, each bowtie of G is a terminal bowtie. A proof similar to that of Proposition 3.5 yields the following.

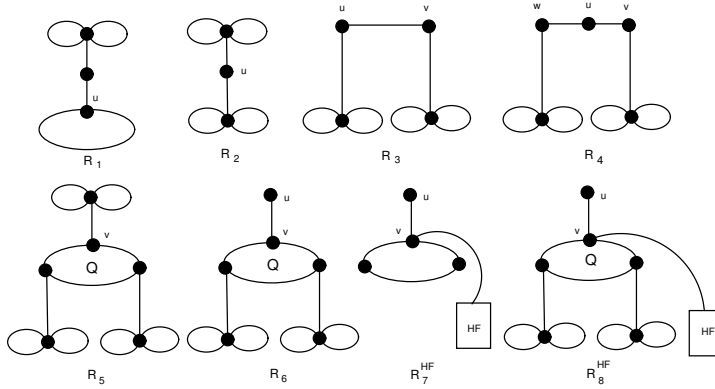


Fig. 1. Each ellipse is a big block, each vertex of $Q \setminus \{v\}$ in $R_5 \cup R_6$ is adjacent to exactly one terminal bowtie, and HF is a subgraph of G .

Proposition 3.6 Let G be a block graph and let G' be defined as follows:

- Suppose that G contains induced subgraph R_1 . Let G' be obtained from G by deleting the terminal bowtie of R_1 and adhering a big block to vertex u ;
- Suppose that G contains induced subgraph R_2 . Let G' be obtained from G by deleting a terminal bowtie of R_2 ;

- Suppose that G contains induced subgraph R_3 , where $V(G) - V(R_3) \neq \emptyset$ and $N(u) \cap N(v) = \emptyset$. Let G' be obtained from G by deleting R_3 and adhering a bowtie to each vertex $w \in (N(u) \cup N(v)) \setminus V(R_3)$;
- Suppose that G contains induced subgraph R_4 . Let G' be obtained from G by deleting the two terminal bowties of R_4 and adhering a bowtie to vertex u ;
- Suppose that G contains induced subgraph R_5 and $V(G) - V(R_5) \neq \emptyset$. Let G' be obtained from G by deleting R_5 and adhering a bowtie to each vertex of $N(Q) \setminus V(R_5)$.

Then G is uniquely monopolar-partitionable if and only if G' is. \square

Proposition 3.7 Suppose that (A, B) is the unique monopolar partition of block graph G .

- (1) If Q is either a terminal block or a suspending block, then the cut vertex v of Q belongs to A ;
- (2) The center x of each end star belongs to A .

Proof: (1) Suppose that $v \in B$. If $A \cap V(Q) = \emptyset$, say $u \in V(Q) \setminus \{v\}$, then $(A \cup \{u\}, B \setminus \{u\})$ is a monopolar partition of G . If $A \cap V(Q) \neq \emptyset$, say $u \in A \cap V(Q)$, then $(A \setminus \{u\}, B \cup \{u\})$ is a monopolar partition of G . Hence, if $v \in B$, then G has a monopolar partition different from (A, B) , which is a contradiction. So, $v \in A$.

(2) Assume that $vx \in E(G)$ and v does not belong to the end star. Suppose that $x \in B$. If $v \in B$, then $((A - N(x)) \cup \{x\}, (B \setminus \{x\}) \cup N(x))$ is a monopolar partition of G . Suppose that $v \in A$. If $B \cap N(x) = \emptyset$, say $w \in N(x) \setminus \{u\}$, then $(A \setminus \{w\}, B \cup \{w\})$ is a monopolar partition of G . If $B \cap N(x) \neq \emptyset$, say $w \in N(x) \cap B$, then $(A \cup \{w\}, B \setminus \{w\})$ is a monopolar partition of G . Hence, if $x \in B$, then G has a monopolar partition different from (A, B) , which is a contradiction. So, $x \in A$. \square

Corollary 3.8 Let G be a uniquely monopolar-partitionable block graph. Then no suspending block of G is adjacent to a terminal block, an end star, or a terminal bowtie. \square

Proposition 3.9 Let G' be an induced subgraph of block graph G . Suppose that each monopolar partition of G' can be extended to at least one monopolar partition of G . Moreover, suppose that if G' has a unique monopolar partition, then it can be extended to exactly one monopolar partition of G . Then G is uniquely monopolar-partitionable if and only if G' is.

Proof: Suppose that G is uniquely monopolar-partitionable. Since G' is an induced subgraph of G , it follows that G' is a monopolar graph. If G' has two different monopolar partitions, then these monopolar partitions can be extended to two different monopolar partitions of G , which is a contradiction. Hence, G' is uniquely monopolar-partitionable.

Suppose that G' is uniquely monopolar-partitionable. It is obvious that G is uniquely monopolar-partitionable. \square

By Proposition 3.7 and Proposition 3.9, we have the following corollary.

Corollary 3.10 Let G be a block graph and let G' be defined as follows:

- if G contains a leaf w adjacent to a block, then $G' = G - w$;
- if a vertex v is adjacent to two terminal blocks Q_1 and Q_2 , then $G' = G - Q_2$;
- if G contains the tangent subgraph $R_6 = R_w^1$, then G' is obtained from G by deleting $V(R_6) \setminus \{u\}$.

Then G is uniquely monopolar-partitionable if and only if G' is. \square

4 Reductions of block graphs

Let G be a block graph. In view of Propositions 3.4, 3.5 and 3.6, we may assume that G satisfies the following two conditions:

- Each bowtie of G is a terminal bowtie and G has no two adjacent terminal bowties.
- G has no induced subgraph R_i for $i = 1, 2, 3, 4, 5$, where u and v in R_3 do not belong to the same big block of G .

Let T be a tree obtained from G by contracting each terminal bowtie, end star, and big block, respectively. Let $v_0 v_1 \cdots v_d$ be a longest path of T . Let $V_i = \{u \in V(T) \mid d(u, v_0) = i\}$ for $i = 0, 1, 2, \dots, d$. Note that (V_0, V_1, \dots, V_d) is a vertex partition of T . From the vertex partition of T , we obtain a vertex partition $(V_G^0, V_G^1, \dots, V_G^d)$ of G as follows: $u \in V_i$ if and only if $u \in V_G^i$ or all the vertices of the corresponding terminal bowtie, end star or big block belong to V_G^i .

For each big block Q of G , if the block belongs to V_G^i of G , then Q is called an i^{th} level big block. Suppose that Q is the i^{th} level big block. If $v \in V(Q)$ is adjacent to a vertex in V_G^{i-1} , then v is called the *upper vertex* of Q , the other vertices are called *down vertices* of Q . For any vertex $v \in V_G^i$, if there exists a vertex $u \in V_G^{i-1}$ such that $vu \in E(G)$, then u is called the *parent* of v . Both Q and v are called *children* of u .

In the section, a family of some special graphs $\{H_i, F_j, Y_k \mid 1 \leq i \leq 4, 1 \leq j \leq 5, k = 2, 5\}$ is given in Fig. 2, Fig. 3 and Fig. 4. Say $d \geq 4$. The basic idea in the section is as follows: By employing a tree trimming techniques, we delete all the vertices of V_G^d . Firstly, we consider each component C in the induced subgraph $G[V_G^{d-1} \cup V_G^d]$ having nonempty intersection with V_G^d . By local structure of C , if C is not isomorphic to H_i for $i \in \{1, 2, 3, 4\}$, either we can determine that G is not uniquely monopolar-partitionable or some blocks of C are deleted. Secondly, we consider each component C' in the induced subgraph $G[V_G^{d-2} \cup V_G^{d-1} \cup V_G^d]$ containing H_i as a tangent subgraph, where $i \in \{1, 2, 3, 4\}$. By local structure of C' , if C' is not isomorphic to F_i for $i \in \{1, 2, 3, 4, 5\}$, either we can determine that G is not uniquely monopolar-partitionable or some blocks of C' are deleted. Thirdly, we consider each component C'' in the induced subgraph $G[V_G^{d-3} \cup V_G^{d-2} \cup V_G^{d-1} \cup V_G^d]$ containing F_j as a tangent subgraph, where $j \in \{1, 2, 3, 4, 5\}$. By local structure of C'' , if C'' is not isomorphic to Y_k for $k \in \{2, 5\}$, either we can determine that G is not uniquely monopolar-partitionable or some blocks of C'' are deleted. Finally, we consider each component C''' in the induced subgraph $G[V_G^{d-4} \cup V_G^{d-3} \cup V_G^{d-2} \cup V_G^{d-1} \cup V_G^d]$ containing Y_j as a tangent subgraph, where $j \in \{2, 5\}$. By local structure of C''' , either we can determine that G is not uniquely monopolar-partitionable or all blocks of C''' belonging to V_G^d are deleted. Then we obtain a new block graph whose associated tree has diameter less than d .

Proposition 4.1 *Let Q be a big block of G , and let G_1, G_2, \dots, G_t be the components of $G - Q$. Assume that the upper vertex v of Q is adjacent to G_1, \dots, G_s . Suppose that G_j is a terminal bowtie, a terminal block, an end star or an isolated vertex for $j = s + 1, \dots, t$.*

- (1) Suppose that there exists a down vertex w of Q such that w is not adjacent to a terminal bowtie. Let $G' = G[V(Q) \cup V(\bigcup_{1 \leq i \leq s} G_i)]$. Then G is uniquely monopolar-partitionable if and only if G' is.
- (2) Suppose that each down vertex of Q is adjacent to a terminal bowtie. Let G' be obtained from G by deleting all the G_k except exactly one terminal bowtie for each down vertex, where $k \in \{s+1, \dots, t\}$. Then G is uniquely monopolar-partitionable if and only if G' is.

Proof: (1) Suppose that G is uniquely monopolar-partitionable. Since G' is an induced subgraph of G , G' is a monopolar graph. For any monopolar partition (A', B') of G' , $v \in A'$. Otherwise, G' has at least two monopolar partitions. One has $V(Q) \subseteq B'$, the other has $V(Q) \setminus \{w\} \subseteq B'$ and $w \in A'$. Both of them can be extended to a monopolar partition of G , which is a contradiction. Since $v \in A'$, any monopolar partition (A', B') of $G[V(Q) \cup V(\bigcup_{1 \leq i \leq s} G_i)]$ can be extended to exactly one monopolar partition of G . So G' is uniquely monopolar-partitionable.

Suppose that G' is uniquely monopolar-partitionable. Let (A', B') be its monopolar partition. By Proposition 3.7, $v \in A'$. Since $v \in A'$, (A', B') can be extended to exactly one monopolar partition of G . By Proposition 3.9, G is uniquely monopolar-partitionable.

- (2) By Proposition 3.9, G is uniquely monopolar-partitionable if and only if G' is. \square

Proposition 4.2 *Let v be a vertex of G not belonging to any big block, and let t denote the parent of v . Suppose v is not adjacent to a terminal bowtie and every child of v is a leaf, a terminal block or end star.*

- (1) *If G is uniquely monopolar-partitionable, then v is not adjacent to an end star.*
- (2) *If G is uniquely monopolar-partitionable, then v is not adjacent to both a leaf and a terminal block.*

Proof: (1) Suppose v is adjacent to an end star. Let u and w be the center and a leaf of the end star, respectively. Let (A, B) be the unique monopolar partition of G . By Proposition 3.7, $u \in A$ and $v \in B$. If $t \in A$, then let $u \in B$ and $N(u) \cup N(v) \setminus \{u, v\} \subseteq A$. So there exists a monopolar partition such that $u \in B$, which is a contradiction. If $t \in B$, let $N(v) \subseteq B$ and $N(u) \subseteq A$, then there exists a monopolar partition such that $u \in B$, which is a contradiction.

(2) Suppose v is adjacent to both a leaf s and a terminal block. Let u and w be the upper vertex and a down vertex of the terminal block, respectively. Let (A, B) be the unique monopolar partition of G . By Proposition 3.7, $u \in A$ and $v \in B$. If $t \in A$, then $s \in A$ or $s \in B$. Then G has two different monopolar partitions, which is a contradiction. If $t \in B$, let $v \in A$ and $N[u] \setminus \{v\} \subseteq B$, then there exists a monopolar partition such that $u \in B$, which is a contradiction. \square

By Proposition 4.1, Proposition 4.2, Corollary 3.8, Corollary 3.10 and the fact that G has no induced subgraph R_1 and R_5 , if $d \geq 2$, then we can assume that each component of $G[V_G^{d-1} \cup V_G^d]$, having nonempty intersection with V_G^d , is isomorphic to H_i for $i \in \{1, 2, 3, 4\}$.

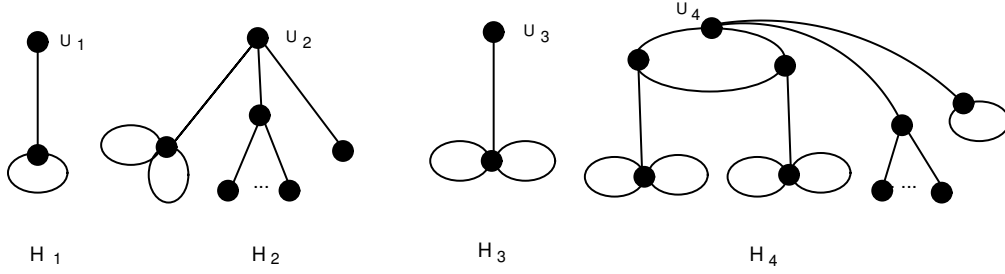


Fig. 2. Each ellipse is a big block, u_2 is adjacent to at least one leaf or end star,

u_4 is adjacent to at least one end star or terminal block.

For each H_i , let w_i denote the parent of u_i . For $i = 1, \dots, 4$, let $F_i = G[V(H_i) \cup \{w_i\}]$. If $w_i = w_j$, then let $F_{ij} = G[V(F_i) \cup V(F_j)]$. That is $F_{ij} = G[V(H_i) \cup V(H_j) \cup \{w_i\}]$. For $k = 1, 2, 3, 4$, let F_j^k denote the graph obtained from F_j by joining w_j to a terminal bowtie, an end star, a terminal block and a leaf, respectively.

By a proof similar to that of Proposition 3.7, we have the following.

Proposition 4.3 *Suppose that (A, B) is the unique monopolar partition of block graph G . If G contains the tangent subgraph $F_i = R_{w_i}^1$ for $i = 1, 4$, then $w_i \in A$. If G contains the tangent subgraph $F_2 = R_{w_2}^1$, then $w_2 \in B$. \square*

By Proposition 4.3 and Proposition 3.7, we have the following corollary.

Corollary 4.4 *Let G be uniquely monopolar-partitionable. Then G does not contain tangent subgraph $F_{12}, F_{24}, F_1^1, F_1^2, F_1^3, F_4^1, F_4^2$ and F_4^3 . \square*

Proposition 4.5 *If G contains the tangent subgraph F_{ij} for $i, j \in \{1, 4\}$, then G is uniquely monopolar-partitionable if and only if $G - H_j$ is.*

Proof: It is obvious that $G - H_j$ is an induced subgraph of G . For any monopolar partition of $G - H_j$, it can be extended to at least a monopolar partition of G . Suppose that $G - H_j$ is uniquely monopolar-partitionable. Let (A', B') be the unique monopolar partition of $G - H_j$. By Proposition 4.3, $w_i \in A'$. Then (A', B') can be extended to exactly one monopolar partition of G . By Proposition 3.9, G is uniquely monopolar-partitionable if and only if $G - H_j$ is. \square

By a proof similar to that of Proposition 4.5, we have the following.

Proposition 4.6 (1) *For any $i \in \{1, 2, 3, 4\}$, if G contains the tangent subgraph F_i^4 , then G is uniquely monopolar-partitionable if and only if $G - t$ is, where $t \in V(F_i^4)$ is the leaf and is adjacent to w_i .*

(2) *Suppose that G contains the tangent subgraph F_2^i for $i = 2, 3$. Let G' be the graph obtained from G by deleting the end star and the terminal block that are adjacent to w_2 . Then G is uniquely monopolar-partitionable if and only if G' is. \square*

Proposition 4.7 *Suppose that G contains the tangent subgraph F_{3j} for $j \in \{1, 4\}$. Let G' be obtained from G by deleting $F_{3j} \setminus \{w_3\}$ and adhering a big block to w_3 . Then G is uniquely monopolar-partitionable if and only if G' is.*

Proof: Let $G'' = G - F_{3j} \setminus \{w_3\}$. Suppose that G is uniquely monopolar-partitionable. Let (A, B) be the unique monopolar partition of G . By Proposition 4.3, $w_3 \in A$. Then $(A \cap V(G''), B \cap V(G''))$ can be extended to a monopolar partition of G' . Hence, G' is a monopolar graph. For any monopolar partition (A', B') of G' , $w_3 \in A'$. Otherwise, $(A' \cap V(G''), B' \cap V(G''))$ can be extended to two different monopolar partitions of G , which is a contradiction. Since $w_3 \in A'$, $(A' \cap V(G''), B' \cap V(G''))$ can be extended to exactly one monopolar partition of G . Hence, G is uniquely monopolar-partitionable.

Suppose that G' is uniquely monopolar-partitionable. Let (A', B') be the unique monopolar partition of G' . By Proposition 3.7, $w_3 \in A'$. Then $(A' \cap V(G''), B' \cap V(G''))$ can be extended to a monopolar partition of G . Hence, G is a monopolar graph. For any monopolar partition (A, B) of G , $w_3 \in A$. Otherwise, G has two different monopolar partitions, which is a contradiction. If G has two different monopolar partitions, then G'' has two different monopolar partitions. So G' has two different monopolar partitions, which is a contradiction. Hence, G is uniquely monopolar-partitionable. \square

By Corollary 4.4, Proposition 4.5, Proposition 4.6, Proposition 4.7 and the fact that G has no induced subgraph R_3 and R_4 , we can assume that the subgraph induced by w_i and its descendant, having nonempty intersection with V_G^d , is isomorphic to F_i in Fig 3 for $i \in \{1, \dots, 5\}$.

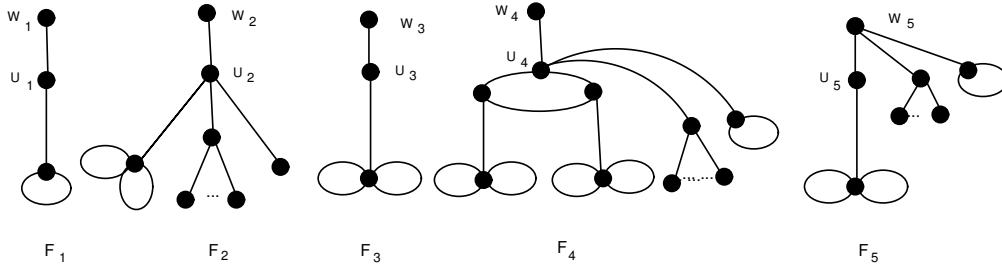


Fig. 3. w_5 is adjacent to at least one terminal block or an end star

Since G has no induced subgraph R_1 , w_2, w_3, w_5 do not belong to a big block. Let $R_7^{H_i}$ and $R_8^{H_i}$ be defined in Fig. 1 for $i \in \{1, 4\}$. For each F_i , $i = 1, 4$, if w_i belongs to a big block of G , then we have the following:

Proposition 4.8 *Let G be a block graph.*

(1) *Suppose that G contains the tangent subgraph $F_i = R_{w_i}^1$, where $i \in \{1, 4\}$. If G is uniquely monopolar-partitionable, then w_i is not a down vertex of a big block.*

(2) *Suppose that G contains the tangent subgraph $R_7^{H_i} = R_w^1$, where $G[V(H_i) \cup \{v\}] = F_i$ and $i \in \{1, 4\}$. Then G is uniquely monopolar-partitionable if and only if $G - H_i$ is.*

(3) *Suppose that G contains the tangent subgraph $R_8^{H_i} = R_w^1$, where $G[V(H_i) \cup \{v\}] = F_i$ and $i \in \{1, 4\}$. Let G' be obtained from G by deleting H_i and all the children of the big block Q . Then G is uniquely monopolar-partitionable if and only if G' is.*

Proof: (1) Suppose that w_i is a down vertex of a big block. Let (A, B) be the unique monopolar partition of G . By Proposition 4.3, $w_i \in A$. Let $G' = G - H_i$. Then $(A \cap V(G'), B \cap V(G'))$ is a monopolar partition of G' . So $(A \cap (V(G') - w_i), (B \cap V(G')) \cup \{w_i\})$ is also a monopolar partition of G' and it can be extended to a monopolar partition of G . Hence, G has two different monopolar partitions, which is a contradiction.

(2) It is obvious that $G - H_i$ is an induced subgraph of G . For any monopolar partition (A', B') of $G - H_i$, it can be extended to at least a monopolar partitions of G . Suppose that $G - H_i$ is uniquely monopolar-partitionable. Let (A', B') be the unique monopolar partition. By Proposition 3.7, $v \in A'$. Then (A', B') can extend to exactly one monopolar partition of G . By Proposition 3.9, G is uniquely monopolar-partitionable if and only if $G - H_i$ is.

(3) A proof similar to that of Case 1 in Proposition 4.1 shows that G is uniquely monopolar-partitionable if and only if G' is. \square

Let m_i denote the parent of w_i and $Y_i = G[V(F_i) \cup \{m_i\}]$ for $i = 1, 2, \dots, 5$. If $m_i = w_j$, let $Y_{ij} = G[V(Y_i) \cup V(F_j)]$ for $i = 2, 5$ and $j = 1, 2, \dots, 5$. If $m_i = m_j$, let $Y_{ij} = G[V(Y_i) \cup V(Y_j)]$ for $i, j = 2, 5$. Let Y_i^j be the graph obtained from Y_i by joining m_i to a terminal bowtie, an end star, a terminal block or an isolated vertex, respectively, for $i = 2, 5$ and $j = 1, 2, 3, 4$. By Proposition 4.8, we can assume that $Y_i = R_{m_i}^1$ is a tangent subgraph of G for $i \in \{1, \dots, 5\}$.

Proposition 4.9 Suppose that G contains the tangent subgraph Y_i , where $i \in \{1, 4\}$. Let H_i be the tangent subgraph of Y_i . Then G is uniquely monopolar-partitionable if and only if $G - H_i \setminus \{u_i\}$ is.

Proof: It is obvious that $G - H_i \setminus \{u_i\}$ is an induced subgraph of G . For any monopolar partition (A', B') of $G - H_i \setminus \{u_i\}$, it can be extended to at least a monopolar partition of G . Suppose that $G - H_i \setminus \{u_i\}$ is uniquely monopolar-partitionable. Let (A', B') be the unique monopolar partition. By Proposition 3.7, $w_i \in A'$ and $u_i \in B'$. Then (A', B') can extend to exactly one monopolar partition of G . By Proposition 3.9, G is uniquely monopolar-partitionable if and only if $G - H_i \setminus \{u_i\}$ is. \square

Proposition 4.10 Suppose that G contains the tangent subgraph Y_3 . Let F_3 be the tangent subgraph of Y_3 . Then G is uniquely monopolar-partitionable if and only if $G - F_3$ is.

Proof: By Proposition 3.9, G is uniquely monopolar-partitionable if and only if $G - F_3$ is. \square

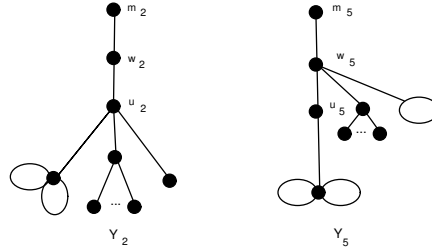


Fig. 4. Subgraphs Y_2 and Y_5

By Proposition 3.7 and Proposition 4.3, we have the following corollary.

Corollary 4.11 *Suppose that block graph G has unique monopolar partition (A, B) . If G contains the tangent subgraph Y_i for $i = 2, 5$, then $m_i \in A$. \square*

By Propositions 4.3 and Corollary 4.11, we have the following corollary.

Corollary 4.12 *Let G be uniquely monopolar-partitionable block graph. Then G does not contain tangent subgraph $YF_{22}, YF_{25}, YF_{52}, YF_{55}, Y_2^j$ and Y_5^j for $j = 1, 2, 3$. \square*

By a proof similar to that of Proposition 4.5, we have the following.

Proposition 4.13 (1) *For $i = 2, 5$ and $j = 1, 4$, suppose that G contains the tangent subgraph YF_{ij} . Let H_j be the tangent subgraph of F_j . Then G is uniquely monopolar-partitionable if and only if $G - H_j$ is.*

(2) *For $i, j = 2, 5$, suppose that G contains the tangent subgraph Y_{ij} . Let F_j be the tangent subgraph of Y_i . Then G is uniquely monopolar-partitionable if and only if $G - F_j$ is.*

(3) *For $i = 2, 5$, if G contains the tangent subgraph Y_i^A , then G is uniquely monopolar-partitionable if and only if $G - t$ is, where $t \in V(Y_i^A)$ is a leaf and is adjacent to m_i .*

By a proof similar to that of Proposition 4.7, we have the following.

Proposition 4.14 *Suppose that G contains the tangent subgraph YF_{i3} for $i \in \{2, 5\}$. Let G' be obtained from G by deleting $YF_{i3} \setminus \{w_3\}$ and adhering a big block to w_3 . Then G is uniquely monopolar-partitionable if and only if G' is.*

Let $R_7^{F_i}$ and $R_8^{F_i}$ be defined in Fig. 1 for $i \in \{2, 5\}$. By a proof similar to that of Proposition 4.8, we have the following.

Proposition 4.15 *Let G be a block graph.*

(1) *Suppose that G contains the tangent subgraph Y_i . If G is uniquely monopolar-partitionable block graph, then m_i is not a down vertex of a big block, where $i \in \{2, 5\}$.*

(2) *Suppose that G contains the tangent subgraph $R_7^{F_i} = R_w^1$, where $G[V(F_i) \cup \{v\}] = Y_i$ and $i \in \{2, 5\}$. Then G is uniquely monopolar-partitionable if and only if $G - F_i$ is.*

(3) *Suppose that G contains the tangent subgraph $R_8^{F_i} = R_w^1$, where $G[V(F_i) \cup \{v\}] = Y_i$ and $i \in \{2, 5\}$. Let G' be obtained from G by deleting F_i and all the children of the big block Q . Then G is uniquely monopolar-partitionable if and only if G' is.*

Let t_i denote the parent of m_i and $X_i = G[V(Y_i) \cup \{t_i\}]$ for $i = 2, 5$. By Corollary 4.12, Proposition 4.13, Proposition 4.14 and Proposition 4.15, we can assume that $X_i = R_{t_i}^1$ is a tangent subgraph of G for $i \in \{2, 5\}$.

By a proof similar to that of Proposition 4.1, we have the following.

Proposition 4.16 *For $i = 2, 5$, suppose that G contains the tangent subgraph X_i . Let F_i be the tangent subgraph of X_i . Then G is uniquely monopolar-partitionable if and only if $G - F_i \setminus \{w_i\}$ is.*

Remark. By above Propositions, we have deleted all of vertices of V_G^d . So we obtain a new block graph whose associated tree has diameter less than d .

5 Uniquely monopolar-partitionable block graphs

In order to determine whether a given block graph has a unique monopolar partition, we first define a family of block graphs. Let Φ be the family of block graphs G satisfying the following conditions:

- (1) Either G is a bowtie or each bowtie of G is a terminal bowtie.
- (2) G has no induced subgraph R_i for $i = 1, 2, 3, 4, 5$, where u and v in R_3 do not belong to the same big block of G .
- (3) G has no two adjacent terminal bowties.

Let G be a block graph. By Proposition 3.4, if G has two bowties $G[V(Q_1 \cup Q_2)]$ and $G[V(Q_2 \cup Q_3)]$ with different centers, then G is not uniquely monopolar-partitionable. Obviously, if G has two adjacent terminal bowties, then G is not uniquely monopolar-partitionable. Without loss of generality, we can assume that G has neither two bowties $G[V(Q_1 \cup Q_2)]$ and $G[V(Q_2 \cup Q_3)]$ with different centers nor two adjacent terminal bowties. By repeatedly applying Proposition 3.5 and Proposition 3.6, we obtain block graph G_1, \dots, G_t such that $G_i \in \Phi$ or $G_i \in \{ \text{a flower}, R_3, R_5 \}$ for $1 \leq i \leq t$. We have the following.

Theorem 5.1 *Let G be a block graph. Suppose that G has neither two bowties $G[V(Q_1 \cup Q_2)]$ and $G[V(Q_2 \cup Q_3)]$ with different centers nor two adjacent terminal bowties. Then G is uniquely monopolar-partitionable if and only if G_i is uniquely monopolar-partitionable for $1 \leq i \leq t$, where G_i is defined as above. \square*

If G_i is a flower or $G_i \in \{R_3, R_5\}$, then G_i is uniquely monopolar-partitionable. Now we determine whether or not a given block graph in Φ is uniquely monopolar-partitionable. Suppose that G is a block graph and $G \in \Phi$. Let T denote the tree structure of G . Now we define some operations on G as follows:

Operation τ_1 : If a block is adjacent to a leaf, delete the leaf; If a vertex v is adjacent to two terminal blocks, delete one terminal block; If G contains the tangent subgraph $R_6 = R_u^1$, delete $V(R_6) \setminus \{u\}$.

Operation τ_2 : Suppose that each down vertex of a big block is only adjacent to a leaf, a terminal bowtie, a terminal block or an end star. If there exists a down vertex such that it is not adjacent to a terminal bowtie, then delete all the children of each down vertex of the block; otherwise, delete all the children of each down vertex except one terminal bowtie.

Operation τ_3 : Suppose that $i = 1, 4$. If G contains the tangent subgraph F_{ij} for $j = 1, 4$, delete H_j ; If G contains tangent subgraph F_j^4 for $j = 1, 2, 3, 4$, delete the leaf; If G contains tangent subgraph F_{3i} , delete $H_i \cup H_3$ and adhere a big block to w_3 ; If G contains tangent subgraph F_2^j for $j = 2, 3$, delete the end star and the terminal block; If G contains the tangent subgraph Y_i , delete $V(H_i) \setminus \{u_i\}$; If G contains the tangent subgraph Y_3 , delete F_3 .

Operation τ_4 : Suppose that $i = 2, 5$. If G contains tangent the subgraph YF_{ij} for $j = 1, 4$, delete H_j ; If G contains tangent subgraph YF_{3i} , delete $F_i \cup H_3$ and adhere a big block to m_3 ; If G contains the tangent subgraph Y_{ij} for $j = 2, 5$, delete F_j ; If G contains the tangent subgraph Y_i^4 , delete the leaf; If G contains the tangent subgraph X_i , delete $V(F_i) \setminus \{w_i\}$.

Operation τ_5 : For $i \in \{1, 4\}$, if G contains the tangent subgraph $R_7^{H_i} = R_u^1$, delete H_i ; if G contains the tangent subgraph $R_8^{H_i} = R_u^1$, delete all the children of the big block of $R_8^{H_i}$. For $i \in \{2, 5\}$, if G contains the tangent subgraph $R_7^{F_i} = R_u^1$, delete F_i ; if G contains the tangent subgraph $R_8^{F_i} = R_u^1$, delete all the children of the big block of $R_8^{F_i}$.

By Propositions in Section 3 and Section 4, we have the following.

Theorem 5.2 *Let G' be the graph obtained from $G \in \Phi$ by some operation τ_i for $i \in \{1, \dots, 5\}$. Then G is uniquely monopolar-partitionable if and only if G' is. \square*

Let G^* be the graph obtained from $G \in \Phi$ by a series of operations τ_i , where $i \in \{1, \dots, 5\}$. It is obvious that $G^* \in \Phi$.

Theorem 5.3 *Let $G \in \Phi$ with $\text{diam}(T) \leq 1$, where T denotes the tree structure of G . Then G is uniquely monopolar-partitionable if and only if G is isomorphic to a bowtie or H_3 . \square*

To present our algorithm for recognizing uniquely monopolar-partitionable block graphs, we introduce the following five properties P_i and two sets \mathcal{G}_i of graphs:

P_1 : there exists a vertex v and a component C of $G - v$ such that $G[V(C) \cup \{v\}]$ is a tree and is not a star;

P_2 : some suspending block is adjacent to a terminal block, an end star or a terminal bowtie;

P_3 : there exists a vertex v such that v neither belongs to a big block nor adjacent to a terminal bowtie, but v is adjacent to either an end star or adjacent to both a leaf and a terminal block;

P_4 : for $i \in \{1, 4\}$, F_i is a tangent subgraph of G and w_i is a down vertex of a big block;

P_5 : for $j \in \{2, 5\}$, Y_j is a tangent subgraph of G and m_j is a down vertex of a big block.

$\mathcal{G}_1 = \{F_{12}, F_{24}, YF_{22}, YF_{25}, YF_{52}, YF_{55}, F_i^j, Y_k^j \mid i = 1, 4, k = 2, 5, j = 1, 2, 3\}$

$\mathcal{G}_2 = \{F_1, F_2, F_3, H_4, Y_2, Y_5\}$

Algorithm

Input: A connected block graph $G \in \Phi$. Let T denote the tree structure of G , and let $v_0v_1 \dots v_d$ be a longest path of T and $(V_G^0, V_G^1, \dots, V_G^d)$ be a vertex partition of G according to T .

Output: Determine whether or not G is uniquely monopolar-partitionable.

Repeatedly apply operation τ_i for $i = 1, \dots, 5$, until one of the following occurs

- G has property P_i for $i \in \{1, \dots, 5\}$ (G is not uniquely monopolar-partitionable);
- G contains a graph in \mathcal{G}_1 as a tangent subgraph (G is not uniquely monopolar-partitionable);
- the reduced graph is in \mathcal{G}_2 (G is not uniquely monopolar-partitionable);
- $\text{diam}(T) \leq 1$ (G is uniquely monopolar-partitionable if $\text{diam}(T) \leq 1$ and G is isomorphic to a bowtie or to H_3 , then G is uniquely monopolar-partitionable; otherwise G is not uniquely monopolar-partitionable).

We now discuss the correctness of the algorithm. In applying operations τ_i , if any property P_i occurs, then G is not uniquely monopolar-partitionable according to Proposition 3.3, Corollary 3.8, and Propositions 4.2, 4.8 and 4.15; if some graph in \mathcal{G}_1 is a tangent subgraph of G , then by Corollaries 4.4 and 4.12, G is not uniquely monopolar-partitionable. Suppose that none of properties P_i occurs and G does not contain any graph of \mathcal{G}_1 as a tangent subgraph. Then the operations applied to G yield either a graph in \mathcal{G}_2 or a graph whose associated T has diameter at most one. If the reduced graph is in \mathcal{G}_2 , then it is obvious that G is not uniquely monopolar-partitionable. When $\text{diam}(T) \leq 1$, by Theorem 5.3, G is uniquely monopolar-partitionable if it is isomorphic to a bowtie or to H_3 ; otherwise G is not uniquely monopolar-partitionable. Moreover all these steps can be implemented in polynomial time. Therefore we have the following:

Theorem 5.4 *There is a polynomial time algorithm to decide if an input block graph is uniquely monopolar-partitionable. \square*

References

- [1] H-J Bandelt and H.M. Mulder, Distance-hereditary graphs, *J. Combinatorial Theory B* 41 (1986) 182 - 208.
- [2] Z.A. Chernyak and A.A. Chernyak, About recognizing (α, β) -classes of polar graphs, *Discrete Math.* 62 (1986) 133 - 138.
- [3] R. Churchley and J. Huang, Line-polar graphs: characterization and recognition, *SIAM Journal on Discrete Mathematics* 25 (2011) 1269 - 1284.
- [4] R. Churchley and J. Huang, List monopolar partitions of claw-free graphs, *Discrete Mathematics* (2011) 2545 - 2549.
- [5] R. Churchley and J. Huang, Solving partition problems with colour-bipartitions, *Graphs and Combinatorics* (2012) DOI 10.1007/s00373-012-1270-z.
- [6] R. Churchley and J. Huang, On the polarity and monopolarity of graphs, *Journal of Graph Theory* (2013) DOI 10.1002/jgt.21755.
- [7] T. Ekim, P. Heggernes, and D. Meister, Polar permutation graphs, J. Fiala, J. Kratochvil, and M. Miller (eds.): *IWOCA 2009, LNCS 5874*, pp 218 - 229, 2009.
- [8] T. Ekim, P. Hell, J. Stacho, and, D. de Werra, Polarity of chordal graphs, *Discrete Applied Math.* 156 (2008) 2469 - 2479.
- [9] T. Ekim and J. Huang, Recognizing line-polar bipartite graphs in time $O(n)$, *Discrete Applied Math.* 158 (2010) 1593 - 1598.
- [10] T. Ekim, N.V.R. Mahadev and D. de Werra, Polar cographs, *Discrete Applied Math.* 156 (2008) 1652 - 1660.
- [11] A. Farrugia, Vertex-partitioning into fixed additive induced-hereditary properties is NP-hard, *Electronic J. Combinatorics* 11 (2004).
- [12] S. Foldes and P.L. Hammer, Split graphs, *Proc. 8th South-Eastern Conf. on Combinatorics, Graph Theory and Computing* (1977) 311 - 315.
- [13] M.R. Garey and D.S. Johnson, **Computers and Intractability**, W.H. Freeman and Company, San Francisco (1979)., *Graph Theory and Computing* (1977) 311 - 315.
- [14] M.C. Golumbic, **Algorithmic Graph Theory and Perfect Graphs**, Academic Press, New York, 1980.
- [15] F. Harary, A characterization of block graphs, *Canadian Mathematical Bulletin* 6 (1963) 1 - 6.
- [16] J. Huang and B. Xu, A forbidden subgraph characterization of line-polar bipartite graphs, *Discrete Applied Math.* 158 (2010) 666 - 680.
- [17] R.I. Tyshkevich and A.A. Chernyak, Decompositions of graphs, *Cybernetics and System Analysis* 21 (1985) 231 - 242.
- [18] D. West, **Introduction to Graph Theory**, Second Edition, Prentice Hall, 2001.