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► **To cite this version:**

Martin Rubey, Christian Stump. Crossings and nestings in set partitions of classical types. 22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010), 2010, San Francisco, United States. pp.1007-1016, 10.46298/dmtcs.2826 . hal-01186254

**HAL Id: hal-01186254**

**<https://inria.hal.science/hal-01186254>**

Submitted on 24 Aug 2015

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# Crossings and nestings in set partitions of classical types

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**Abstract.** In this extended abstract, we investigate bijections on various classes of set partitions of classical types that preserve openers and closers. On the one hand we present bijections for types  $B$  and  $C$  that interchange crossings and nestings, which generalize a construction by Kasraoui and Zeng for type  $A$ . On the other hand we generalize a bijection to type  $B$  and  $C$  that interchanges the cardinality of a maximal crossing with the cardinality of a maximal nesting, as given by Chen, Deng, Du, Stanley and Yan for type  $A$ .

For type  $D$ , we were only able to construct a bijection between non-crossing and non-nesting set partitions. For all classical types we show that the set of openers and the set of closers determine a non-crossing or non-nesting set partition essentially uniquely.

**Résumé.** Dans ce résumé, nous étudions des bijections entre diverses classes de partitions d'ensemble de types classiques qui préservent les "openers" et les "closers". D'un part, nous présentons des bijections pour les types  $B$  et  $C$  qui échangent croisées et emboîtées, qui généralisent une construction de Kasraoui et Zeng pour le type  $A$ . D'autre part, nous généralisons une bijection pour le type  $B$  et  $C$  qui échange la cardinalité d'un croisement maximal avec la cardinalité d'un emboîtement maximal comme il a été fait par Chen, Deng, Du, Stanley et Yan pour le type  $A$ .

Pour le type  $D$ , nous avons seulement construit une bijection entre les partitions non croisées et non emboîtées. Pour tout les types classiques, nous montrons que l'ensemble des "openers" et l'ensemble des "closers" déterminent une partition non croisées ou non emboîtées essentiellement de façon unique.

**Keywords:** non-crossing partitions, non-nesting partitions,  $k$ -crossing partitions,  $k$ -nesting partitions, bijective combinatorics

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## Introduction

The lattice of non-crossing set partitions was first considered by Germain Kreweras in [18]. It was later reinterpreted by Paul Edelman, Rodica Simion and Daniel Ullman as a well-behaved sub-lattice of the intersection lattice for the Coxeter arrangement of type  $A$ , see e.g. [6, 7, 22]. Natural combinatorial interpretations of non-crossing partitions for the classical reflection groups were then given by Christos A. Athanasiadis and Vic Reiner in [3, 20].

On the other hand, non-nesting partitions were simultaneously introduced for all crystallographic reflection groups by Alex Postnikov as anti-chains in the associated root poset, see [20, Remark 2]; for further

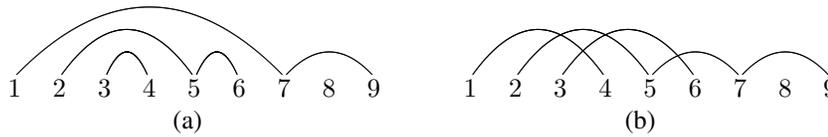


Fig. 1: A non-crossing (a) and a non-nesting (b) set partition of [9].

information on reflection groups as well as for a definition of Coxeter arrangements and root posets see e.g. [14]

Within the last years, several bijections between non-crossing and non-nesting partitions have been constructed. In particular, type (block-size) preserving bijections were given by Christos A. Athanasiadis [2] for type  $A$  and by Alex Fink and Benjamin I. Giraldo [9] for the other classical reflection groups. Recently, Ricardo Mamede and Alessandro Conflitti [5, 19] constructed bijections for types  $A$ ,  $B$  and  $D$  which turn out to be subsumed by the bijections we present here.

In the case of set partitions of type  $A$ , also the *number* of crossings and nestings was considered: Anisse Kasraoui and Jiang Zeng constructed a bijection which interchanges crossings and nestings in [16]. Finally, in a rather different direction, William Y.C. Chen, Eva Y.P. Deng, Rosena R.X. Du, Richard P. Stanley [4] have shown that the number of set partitions where a *maximal crossing* has cardinality  $k$  and a *maximal nesting* has cardinality  $\ell$  is the same as the number of set partitions where a maximal crossing has cardinality  $\ell$  and a maximal nesting has cardinality  $k$ .

In this extended abstract, we present bijections on various classes of set partitions of classical types that preserve openers and closers. In particular, the bijection by Anisse Kasraoui and Jiang Zeng as well as the bijection by William Y.C. Chen, Eva Y.P. Deng, Rosena R.X. Du, Richard P. Stanley enjoy this property. We give generalizations of these bijections for the other classical reflection groups, whenever possible. Furthermore we show that the bijection is in fact (essentially) unique for the class of non-crossing and non-nesting set partitions.

### 1 Set partitions for classical types

A *set partition* of  $[n] := \{1, 2, 3, \dots, n\}$  is a collection  $\mathcal{B}$  of pairwise disjoint, non-empty subsets of  $[n]$ , called *blocks*, whose union is  $[n]$ . We visualize  $\mathcal{B}$  by placing the numbers  $1, 2, \dots, n$  in this order on a line and then joining *consecutive* elements of each block by an arc, see Figure 1 for examples.

The *openers*  $\text{op}(\mathcal{B})$  are the non-maximal elements of the blocks in  $\mathcal{B}$ , whereas the *closers*  $\text{cl}(\mathcal{B})$  are its non-minimal elements. For example, the set partitions in Figure 1 both have  $\text{op}(\mathcal{B}) = \{1, 2, 3, 5, 7\}$  and  $\text{cl}(\mathcal{B}) = \{4, 5, 6, 7, 9\}$ .

A pair  $(\mathcal{O}, \mathcal{C}) \subseteq [n] \times [n]$  is an *opener-closer configuration*, if  $|\mathcal{O}| = |\mathcal{C}|$  and

$$|\mathcal{O} \cap [k]| \geq |\mathcal{C} \cap [k + 1]| \quad \text{for } k \in \{0, 1, \dots, n - 1\},$$

or, equivalently,  $(\mathcal{O}, \mathcal{C}) = (\text{op}(\mathcal{B}), \text{cl}(\mathcal{B}))$  for some set partition  $\mathcal{B}$  of  $n$ .

A set partition of type  $B_n$  or  $C_n$  is a set partition  $\mathcal{B}$  of  $[\pm n] := \{1, 2, \dots, n, -1, -2, \dots, -n\}$ , such that

$$B \in \mathcal{B} \Leftrightarrow -B \in \mathcal{B} \tag{1}$$

and such that there exists at most one block  $B_0 \in \mathcal{B}$  (called the *zero block*) for which  $B_0 = -B_0$ .

A set partition  $\mathcal{B}$  of type  $D_n$  is a set partition of type  $B_n$  (or  $C_n$ ) where the zero block, if present, must not consist of a single pair  $\{i, -i\}$ .

## 2 Crossings and nestings in set partitions of type A

In this extended abstract, we refine the following well known correspondences between non-crossing and non-nesting set partitions. For ordinary set partitions, a *crossing* consists of a pair of arcs  $(i, j)$  and  $(i', j')$  such that  $i < i' < j < j'$ ,



On the other hand, a *nesting* consists of a pair of arcs  $(i, j)$  and  $(i', j')$  such that  $i < i' < j' < j$ ,



A set partition of  $[n]$  is called *non-crossing* (resp. *non-nesting*) if the number of crossings (resp. the number of nestings) equals 0.

It has been known for a long time that the numbers of non-crossing and non-nesting set-partitions of  $[n]$  coincide. More recently, Anisse Kasraoui and Jiang Zeng have shown in [16] that much more is true:

**Theorem 2.1** *There is an explicit bijection on set partitions of  $[n]$ , preserving the set of openers and the set of closers, and interchanging the number of crossings and the number of nestings.*

The construction in [16] also proves the following corollary:

**Corollary 2.2** *For any opener-closer configuration  $(\mathcal{O}, \mathcal{C}) \subseteq [n] \times [n]$ , there exists a unique non-crossing set partition  $\mathcal{B}$  of  $[n]$  and a unique non-nesting set partition  $\mathcal{B}'$  of  $[n]$  such that*

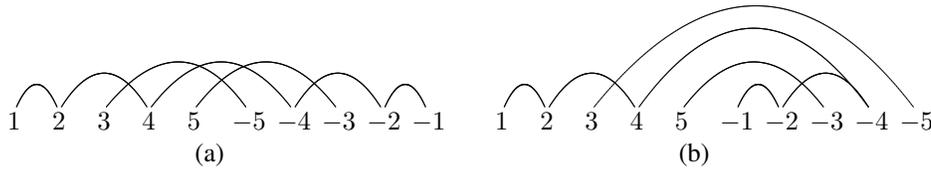
$$\text{op}(\mathcal{B}) = \text{op}(\mathcal{B}') = \mathcal{O} \quad \text{and} \quad \text{cl}(\mathcal{B}) = \text{cl}(\mathcal{B}') = \mathcal{C}.$$

Apart from the number of crossings or nestings, another natural statistic to consider is the cardinality of a ‘maximal crossing’ and of a ‘maximal nesting’: a *maximal crossing* of a set partition is a set of largest cardinality of mutually crossing arcs, whereas a *maximal nesting* is a set of largest cardinality of mutually nesting arcs. For example, in Figure 1(a), the arcs  $\{(1, 7), (2, 5), (3, 4)\}$  form a maximal nesting of cardinality 3. In Figure 1(b) the arcs  $\{(1, 4), (2, 5), (3, 6)\}$  form a maximal crossing.

The following symmetry property was shown by William Y.C. Chen, Eva Y.P. Deng, Rosena R.X. Du, Richard P. Stanley and Catherine H. Yan [4]:

**Theorem 2.3** *There is an explicit bijection on set partitions, preserving the set of openers and the set of closers, and interchanging the cardinalities of a maximal crossing and a maximal nesting.*

Since a ‘maximal crossing’ of a non-crossing partition and a ‘maximal nesting’ of a non-nesting partition both have cardinality 1, Corollary 2.2 implies that this bijection coincides with the bijection by Anisse Kasraoui and Jiang Zeng for non-crossing and non-nesting partitions. In particular, we obtain the curious fact that in this case, the bijection maps non-crossing partitions with  $k$  nestings and maximal nesting having cardinality  $\ell$  to non-nesting partitions with  $k$  crossings and maximal crossing having cardinality  $\ell$ .



**Fig. 2:** The nesting (a) and the crossing (b) diagram of a set partition of type  $C_5$ .

We have to stress however, that in general it is not possible to interchange the number of crossings and the cardinality of a maximal crossing with the number of nestings and the cardinality of a maximal nesting simultaneously.

**Example 2.4** For  $n = 8$ , there is a set partition with one crossing, six nestings and the cardinalities of a maximal crossing and a maximal nesting equal both one, namely  $\{\{1, 7\}, \{2, 8\}, \{3, 4, 5, 6\}\}$ . However, there is no set partition with six crossings, one nesting and cardinalities of a maximal crossing and a maximal nesting equal to one. To check, the four set partitions with six crossings and one nesting are

$$\begin{aligned} & \{\{1, 4, 6\}, \{2, 5, 8\}, \{3, 7\}\}, \quad \{\{1, 4, 7\}, \{3, 5, 8\}, \{2, 6\}\}, \\ & \{\{1, 4, 8\}, \{2, 5, 7\}, \{3, 6\}\}, \quad \{\{1, 5, 8\}, \{2, 4, 7\}, \{3, 6\}\}. \end{aligned}$$

### 3 Crossings and nestings in set partitions of type C

Type independent definitions for *non-crossing* and *non-nesting* set partitions have been available for a while now, see for example [1, 2, 3, 20]. However, it turns out that the notions of crossing and nesting is more tricky, and we do not have a type independent definition. In this section we generalize the results of the previous section to type  $C$ .

We want to associate two pictures to each set partition, namely the ‘crossing’ and the ‘nesting diagram’. To this end, we define two orderings on the set  $[\pm n]$ : the *nesting order* for type  $C_n$  is

$$1 < 2 < \dots < n < -n < \dots < -2 < -1,$$

whereas the *crossing order* is

$$1 < 2 < \dots < n < -1 < -2 < \dots < -n.$$

The *nesting diagram* of a set partition  $\mathcal{B}$  of type  $C_n$  is obtained by placing the numbers in  $[\pm n]$  in *nesting order* on a line and then joining consecutive elements of each block of  $\mathcal{B}$  by an arc, see Figure 2(a) for an example.

The *crossing diagram* of a set partition  $\mathcal{B}$  of type  $C_n$  is obtained from the *nesting diagram* by reversing the order of the negative numbers. More precisely, we place the numbers in  $[\pm n]$  in *crossing order* on a line and then join *consecutive elements in the nesting order* of each block of  $\mathcal{B}$  by an arc, see Figure 2(b) for an example. We stress that the same elements are joined by arcs in both diagrams. Observe furthermore that the symmetry property (1) implies that if  $(i, j)$  is an arc, then its negative  $(-j, -i)$  is also an arc.

A *crossing* is a pair of arcs that crosses in the crossing diagram, and a *nesting* is a pair of arcs that nests in the nesting diagram.

The *openers*  $\text{op}(\mathcal{B})$  are the *positive* non-maximal elements of the blocks in  $\mathcal{B}$ , the *closers*  $\text{cl}(\mathcal{B})$  the *positive* non-minimal elements. Thus, openers and closers are the start and end points of the arcs in the positive part of the nesting (or crossing) diagram. For example, the set partition displayed in Figure 2 has openers  $\{1, 2, 3, 4, 5\}$  and closers  $\{2, 4\}$ . For convenience, we call the negatives of the elements in  $\text{op}(\mathcal{B})$  *negative closers* and the negatives of the elements in  $\text{cl}(\mathcal{B})$  *negative openers*.

In type  $C_n$ ,  $(\mathcal{O}, \mathcal{C}) \subseteq [n] \times [n]$  is an *opener-closer configuration*, if

$$|\mathcal{O} \cap [k]| \geq |\mathcal{C} \cap [k+1]| \quad \text{for } k \in \{0, 1, \dots, n-1\}.$$

Note that we do not require that  $|\mathcal{O}| = |\mathcal{C}|$ .

**Theorem 3.1** *There is an explicit bijection on set partitions of type  $C_n$ , preserving the set of openers and the set of closers, and interchanging the number of crossings and the number of nestings.*

Furthermore, we will also see the following analog of Corollary 2.2:

**Corollary 3.2** *For any opener-closer configuration  $(\mathcal{O}, \mathcal{C}) \subseteq [n] \times [n]$ , there exists a unique non-crossing set partition  $\mathcal{B}$  and a unique non-nesting set partition  $\mathcal{B}'$ , both of type  $C_n$ , such that*

$$\text{op}(\mathcal{B}) = \text{op}(\mathcal{B}') = \mathcal{O} \quad \text{and} \quad \text{cl}(\mathcal{B}) = \text{cl}(\mathcal{B}') = \mathcal{C}.$$

**Theorem 3.3** *There is an explicit bijection on set partitions of type  $C_n$ , preserving the set of openers and the set of closers, and interchanging the cardinalities of a maximal crossing and a maximal nesting.*

The bijection in Theorem 3.1 is constructed in an analogous way as the bijection in Theorem 2.1 whereas the rough idea of our bijection in the Theorem 3.3 is as follows: we render a type  $C_n$  set partition in the language of 0-1-fillings of a certain polyomino. We will do this in such a way that maximal nestings correspond to north-east chains of ones of maximal length.

Interpreting this filling as a growth diagram in the sense of Sergey Fomin and Tom Roby [10, 11, 12, 21] enables us to define a transformation on the filling that maps – technicalities aside – the length of the longest north-east chain to the length of the longest south-east chain. This filling can then again be interpreted as a  $C_n$  set partition, where south-east chains of maximal length correspond to maximal crossings. Many variants of the transformation involved are described in Christian Krattenthaler’s article [17], we employ yet another (slight) variation.

We remark that the bijection in Theorem 3.3 is not an involution and that it does not, as discussed above, exchange the number of crossings and the number of nestings.

**Remark 1** *Using the same methods as in the proof the above theorem, one can also deduce a conjecture due to Daniel Soll and Volkmar Welker [23, Conjecture 13]. Namely, we consider generalized triangulations of the  $2n$ -gon that are invariant under rotation of  $180^\circ$ , and such that at most  $k$  diagonals are allowed to cross mutually. Daniel Soll and Volkmar Welker then conjectured that the number of such triangulations that are maximal, i.e., where one cannot add any diagonal without introducing a  $k+1$  crossing, coincides with the number of fans of  $k$  Dyck paths that are symmetric with respect to a vertical axis.*

*The corresponding theorem for type  $A$  was discovered and proved by Jakob Jonsson [15]. A (nearly) bijective proof very similar to ours was given by Christian Krattenthaler in [17]. A simple bijection for the case of 2-triangulations was recently given by Sergi Elizalde in [8].*

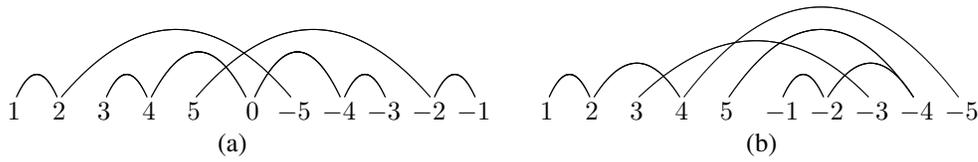


Fig. 3: The nesting (a) and the crossing (b) diagram of a set partition of type  $B_5$ .

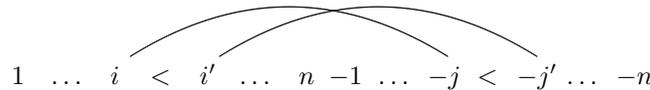
### 4 Crossings and nestings in set partitions of type B

The definition of non-crossing set partitions of type  $B_n$  coincides with the definition in type  $C_n$ , and the *crossing diagram* is also the same. However, the combinatorial model for non-nesting set partitions changes slightly: we define the *nesting order* for type  $B_n$  as

$$1 < 2 < \dots < n < 0 < -n < \dots < -2 < -1.$$

The *nesting diagram* of a set partition  $\mathcal{B}$  is then obtained by placing the numbers in  $[\pm n] \cup \{0\}$  in *nesting order* on a line and joining consecutive elements of each block of  $\mathcal{B}$  by an arc, where the zero block is augmented by the number 0, if present. See Figure 3(a) for an example. The definition of *openers*  $op(\mathcal{B})$  and *closers*  $cl(\mathcal{B})$  is the same as in type  $C$ , the number 0 is neither an opener nor a closer.

These changes are actually dictated by the general, type independent definitions for non-crossing and non-nesting set partitions. Moreover, it turns out that we need to ignore certain crossings and nestings that appear in the diagrams: a *crossing* is a pair of arcs that crosses in the crossing diagram, except if both arcs connect a positive and a negative element and at least one of them connects a positive element with an element smaller in absolute value. Pictorially,



is not a crossing, if  $j < i$  or  $j' < i'$ .

Similarly, a *nesting* is a pair of arcs that nests in the nesting diagram, except if both arcs connect a positive element or 0 with a negative element or 0, and at least one of them connects a positive element with an element smaller in absolute value.

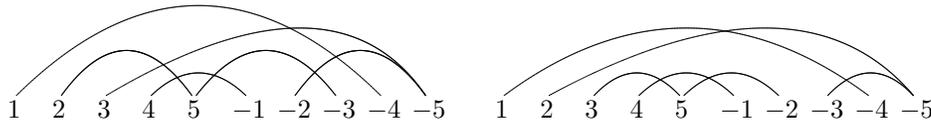
**Example 4.1** *The set partition in Figure 3(b) has three crossings:  $(3, -3)$  crosses  $(2, 4)$ ,  $(4, -5)$ , and  $(-4, -2)$ . It does not cross  $(5, -4)$  by definition.*

*The set partition in Figure 3(a) has three nestings:  $(2, -5)$  nests  $(3, 4)$  and  $(4, 0)$ , and  $(5, -2)$  nests  $(-4, -3)$ . However,  $(5, -2)$  does not nest  $(0, -4)$  by definition.*

With this definition, we have a theorem that is only slightly weaker than in type  $C$ :

**Theorem 4.2** *There is an explicit bijection on set partitions of type  $B_n$ , preserving the set of openers and the set of closers, and mapping the number of nestings to the number of crossings.*

Again, we obtain an analog of Corollary 2.2:



**Fig. 4:** Two non-crossing set partition of type  $D_5$ . Both are obtained from each other by interchanging 5 and  $-5$ .

**Corollary 4.3** For any opener-closer configuration  $(\mathcal{O}, \mathcal{C}) \subseteq [n] \times [n]$ , there exists a unique non-crossing set partition  $\mathcal{B}$  and a unique non-nesting set partition  $\mathcal{B}'$ , both of type  $B_n$ , such that

$$\text{op}(\mathcal{B}) = \text{op}(\mathcal{B}') = \mathcal{O} \quad \text{and} \quad \text{cl}(\mathcal{B}) = \text{cl}(\mathcal{B}') = \mathcal{C}.$$

Using Theorem 3.3 and Theorem 4.2, one can deduce the following theorem:

**Theorem 4.4** There is an explicit bijection on set partitions of type  $B_n$ , preserving the set of openers and the set of closers, and interchanging the cardinalities of a maximal crossing and a maximal nesting.

### 5 Non-crossing and non-nesting set partitions in type D

In type  $D$  we do not have any good notion of crossing or nesting, we can only speak properly about *non-crossing* and *non-nesting* set partitions.

A combinatorial model for non-crossing set partition of type  $D_n$  was given by Christos A. Athanasiadis and Vic Reiner in [3]. For our purposes it is better to use a different description of the same model: let  $\mathcal{B}$  be a set partition of type  $D_n$  and let  $\{(i_1, -j_1), \dots, (i_k, -j_k)\}$  for positive  $i_\ell, j_\ell < n$  be the ordered set of arcs in  $\mathcal{B}$  starting in  $\{1, \dots, n - 1\}$  and ending in its negative.  $\mathcal{B}$  is called *non-crossing* if

- (i)  $(i, -i)$  is an arc in  $\mathcal{B}$  implies  $i = n$ ,

and if it is non-crossing in the sense of type  $C_n$  with the following exceptions:

- (ii) arcs in  $\mathcal{B}$  containing  $n$  must cross all arcs  $(i_\ell, -j_\ell)$  for  $\ell > k/2$ ,
- (iii) arcs in  $\mathcal{B}$  containing  $-n$  must cross all arcs  $(i_\ell, -j_\ell)$  for  $\ell \leq k/2$ ,
- (iv) two arcs in  $\mathcal{B}$  containing  $n$  and  $-n$  may cross.

Here, (i) is equivalent to say that if  $\mathcal{B}$  contains a zero block  $B_0$  then  $n \in B_0$  and observe that (i) together with the non-crossing property of  $\{(i_1, -j_1), \dots, (i_k, -j_k)\}$  imply that  $k/2 \in \mathbb{N}$ , see Figure 4 for an example.

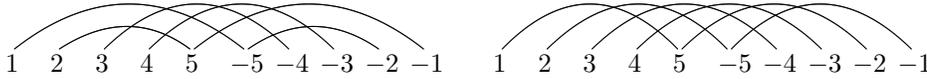
Note that all conditions hold for a set partition  $\mathcal{B}$  if and only if they hold for the set partition obtained from  $\mathcal{B}$  by interchanging  $n$  and  $-n$ .

A set partition of type  $D_n$  is called *non-nesting* if it is non-nesting in the sense of [2]. This translates to our notation as follows: let  $\mathcal{B}$  be a set partition of type  $D_n$ . Then  $\mathcal{B}$  is called non-nesting if

- (i)  $(i, -i)$  is an arc in  $\mathcal{B}$  implies  $i = n$ ,

and if it is non-nesting in the sense of type  $C_n$  with the following exceptions:

- (ii) arcs  $(i, -n)$  and  $(j, n)$  for positive  $i < j < n$  in  $\mathcal{B}$  are allowed to nest, as do



**Fig. 5:** Two non-nesting set partition of type  $D_5$ . Both are obtained from each other by interchanging 5 and  $-5$ .

- (iii) arcs  $(i, -j)$  and  $(n, -n)$  for positive  $k < i, j < n$  in  $\mathcal{B}$  where  $(k, n)$  is another arc in  $\mathcal{B}$  (which exists by the definition of set partitions in type  $D_n$ ).

Again, (i) means that if  $B_0 \in \mathcal{B}$  is a zero block then  $n \in B_0$ . (ii) and (iii) come from the fact that the positive roots  $e_i + e_n$  and  $e_j - e_n$  for  $i \leq j$  are comparable in the root poset of type  $C_n$  but are not comparable in the root poset of type  $D_n$ . As for non-crossing set partitions in type  $D_n$ , all conditions hold if and only if they hold for the set partition obtained by interchanging  $n$  and  $-n$ . See Figure 5 for an example. The definition of *openers*  $\text{op}(\mathcal{B})$ , *closers*  $\text{cl}(\mathcal{B})$  and *opener-closer configuration* is as in type  $C$ .

**Proposition 5.1** *Let  $(\mathcal{O}, \mathcal{C}) \subseteq [n]$  be an opener-closer configuration. Then there exists a non-crossing set partition  $\mathcal{B}$  of type  $D_n$  with  $\text{op}(\mathcal{B}) = \mathcal{O}$  and  $\text{cl}(\mathcal{B}) = \mathcal{C}$  if and only if*

$$|\mathcal{O}| - |\mathcal{C}| \text{ is even or } n \in \mathcal{O}, \mathcal{C}. \quad (2)$$

*Moreover, there exist exactly two non-crossing set partitions of type  $D_n$  having this opener-closer configuration if both conditions hold, otherwise, it is unique.*

As in types  $A, B$  and  $C$ , the analogue proposition holds also for non-nesting set partitions of type  $D_n$ :

**Proposition 5.2** *Let  $(\mathcal{O}, \mathcal{C}) \subseteq [n]$  be an opener-closer configuration. Then there exists a non-nesting set partition  $\mathcal{B}$  of type  $D_n$  with  $\text{op}(\mathcal{B}) = \mathcal{O}$  and  $\text{cl}(\mathcal{B}) = \mathcal{C}$  if and only if*

$$|\mathcal{O}| - |\mathcal{C}| \text{ is even or } n \in \mathcal{O}, \mathcal{C}. \quad (3)$$

*Furthermore, there exist exactly two non-nesting set partitions of type  $D_n$  having this opener-closer configuration if both conditions hold, otherwise, it is unique.*

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