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► **To cite this version:**

Kyungyong Lee, Li Li. On the diagonal ideal of $(C^2)^n$ and q, t -Catalan numbers. Billey, Sara and Reiner, Victor. 22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010), 2010, San Francisco, United States. Discrete Mathematics and Theoretical Computer Science, DMTCS Proceedings vol. AN, 22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010), pp.881-888, 2010, DMTCS Proceedings. <hal-01186266>

HAL Id: hal-01186266

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Submitted on 24 Aug 2015

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On the diagonal ideal of $(\mathbb{C}^2)^n$ and q, t -Catalan numbers

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Abstract. Let I_n be the (big) diagonal ideal of $(\mathbb{C}^2)^n$. Haiman proved that the q, t -Catalan number is the Hilbert series of the graded vector space $M_n = \bigoplus_{d_1, d_2} (M_n)_{d_1, d_2}$ spanned by a minimal set of generators for I_n . We give simple upper bounds on $\dim (M_n)_{d_1, d_2}$ in terms of partition numbers, and find all bi-degrees (d_1, d_2) such that $\dim (M_n)_{d_1, d_2}$ achieve the upper bounds. For such bi-degrees, we also find explicit bases for $(M_n)_{d_1, d_2}$.

Résumé. Soit I_n l'idéal de la (grande) diagonale de $(\mathbb{C}^2)^n$. Haiman a démontré que le q, t -nombre de Catalan est la série de Hilbert de l'espace vectoriel gradué $M_n = \bigoplus_{d_1, d_2} (M_n)_{d_1, d_2}$ engendré par un ensemble minimal de générateurs de I_n . Nous obtenons des bornes supérieures simples pour $\dim (M_n)_{d_1, d_2}$ en termes de nombres de partitions, ainsi que tous les bi-degrés (d_1, d_2) pour lesquels ces bornes supérieures sont atteintes. Pour ces bi-degrés, nous trouvons aussi des bases explicites de $(M_n)_{d_1, d_2}$.

Keywords: q, t -Catalan number, diagonal ideal

1 introduction

1.1 Background

The goal of this paper is to study the q, t -Catalan numbers and the (thick) diagonal ideal in $(\mathbb{C}^2)^n$, and discuss some technique that we have developed recently.

Let n be a positive integer. Consider the set of n -tuples $\{(x_i, y_i)\}_{1 \leq i \leq n}$ in the plane \mathbb{C}^2 . They form an affine space $(\mathbb{C}^2)^n$ with coordinate ring $\mathbb{C}[\mathbf{x}, \mathbf{y}] = \mathbb{C}[x_1, y_1, \dots, x_n, y_n]$. There is a natural symmetric group S_n acting on $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ by permuting the coordinates in \mathbf{x}, \mathbf{y} simultaneously. With this group action, a polynomial $f \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is called *alternating* if

$$\sigma(f) = \text{sgn}(\sigma)f \quad \text{for all } \sigma \in S_n.$$

Define $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$ to be the vector space of alternating polynomials in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$.

[†]Research of the first author partially supported by NSF grant DMS 0901367.

There is a more combinatorial way to describe $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$. Denote by \mathbb{N} the set of nonnegative integers. Let \mathfrak{D}_n be the set of n -tuples $D = \{(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\} \subset \mathbb{N} \times \mathbb{N}$. For $D \in \mathfrak{D}_n$, define

$$\Delta(D) := \det \begin{bmatrix} x_1^{\alpha_1} y_1^{\beta_1} & x_1^{\alpha_2} y_1^{\beta_2} & \dots & x_1^{\alpha_n} y_1^{\beta_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\alpha_1} y_n^{\beta_1} & x_n^{\alpha_2} y_n^{\beta_2} & \dots & x_n^{\alpha_n} y_n^{\beta_n} \end{bmatrix}$$

Then $\{\Delta(D)\}_{D \in \mathfrak{D}_n}$ forms a basis for the \mathbb{C} -vector space $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$.

It is easy to see that any alternating polynomial vanishes on the thick diagonal of $(\mathbb{C}^2)^n$. (By thick diagonal we mean the set of n -tuples of points in \mathbb{C}^2 where at least two points coincide.) A theorem of Haiman asserts that the converse is also true: any polynomial that vanishes on the diagonal of $(\mathbb{C}^2)^n$ can be generated by alternating polynomials, i.e.

$$\bigcap_{1 \leq i < j \leq n} (x_i - x_j, y_i - y_j) = \text{ideal generated by } \Delta(D)\text{'s.}$$

We call the above ideal the **diagonal ideal** and denote it by I_n . the number of minimal generators of I_n , which is the same as the dimension of the vector space $M_n = I_n/(\mathbf{x}, \mathbf{y})I_n$, is equal to the n -th Catalan number. The space M_n is doubly graded as $\oplus_{d_1, d_2} (M_n)_{d_1, d_2}$. The q, t -Catalan number, originally introduced by A.M.Garsia and M.Haiman in [4], can be defined as

$$C_n(q, t) = \sum_{d_1, d_2} t^{d_1} q^{d_2} \dim(M_n)_{d_1, d_2}.$$

The q, t -Catalan number $C_n(q, t)$ also has a combinatorial interpretation using Dyck paths. To be more precise, take the $n \times n$ square whose southwest corner is $(0, 0)$ and northeast corner is (n, n) . Let \mathcal{D}_n be the collection of Dyck paths, i.e. lattice paths from $(0, 0)$ to (n, n) that proceed by NORTH or EAST steps and never go below the diagonal. For any Dyck path Π , let $a_i(\Pi)$ be the number of squares in the i -th row that lie in the region bounded by Π and the diagonal. A.M.Garsia and J.Haglund ([2], [3]) among others showed that

$$C_n(q, t) = \sum_{\Pi \in \mathcal{D}_n} q^{\text{area}(\Pi)} t^{\text{dinv}(\Pi)},$$

where

$$\text{dinv}(\Pi) := |\{(i, j) \mid i < j \text{ and } a_i(\Pi) = a_j(\Pi)\}| + |\{(i, j) \mid i < j \text{ and } a_i(\Pi) + 1 = a_j(\Pi)\}|.$$

Haiman posed a question asking for a rule that associate to each Dyck path Π an element $D(\Pi) \in \mathfrak{D}_n$ such that $\deg_x \Delta(D(\Pi)) = \text{area}(\Pi)$, $\deg_y \Delta(D(\Pi)) = \text{dinv}(\Pi)$, and that the set $\{\Delta(D(\Pi))\}$ generates I_n . The last condition is equivalent to requiring the images $\{\overline{\Delta(D(\Pi))}\}$ form a basis of M_n . It is natural to ask the following more general question:

Question 1.1 Given a bi-degree (d_1, d_2) , is there a combinatorially significant construction of the basis for each $(M_n)_{d_1, d_2}$?

1.2 Main results

This paper initiates the approach to the study of M_n by comparing it with $M_{n'}$ for a large integer n' . On the one hand, there is a natural map $M_n \rightarrow M_{n'}$ for any $n' > n$. On the other hand, for n' sufficiently large, the basis of $(M_{n'})_{d'_1, d_2}$ becomes “stable” if we fix d_2 and fix

$$k = \binom{n}{2} - d_1 - d_2 = \binom{n'}{2} - d'_1 - d_2.$$

Therefore we can take the “limit” of such basis for $n' \rightarrow \infty$. This basis is indexed by the partitions of k . As a consequence, $(M_{n'})_{d'_1, d_2}$ can be imbedded as a subspace of the polynomial ring with infinite many variables $\mathbb{C}[\rho_1, \rho_2, \dots]$. The induced map

$$\bar{\varphi} : (M_n)_{d_1, d_2} \rightarrow \mathbb{C}[\rho_1, \rho_2, \dots],$$

which will be defined explicitly in subsection 1.2.3, provides a powerful tool to study M_n .

1.2.1 Asymptotic behavior when $k \ll n$

We shall show that if $k \ll n$, then $(M_n)_{d_1, d_2}$ has a basis $\{\overline{\Delta(D)}\}$ where D are so-called **minimal staircase forms** that will be defined later.

The essential step is to observe the following three linear relations that turn the questions into combinatorial games. First we introduce some notations.

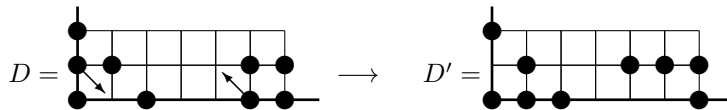
- For $D = \{P_1, \dots, P_n\} \in \mathfrak{D}_n$ where $P_i = (\alpha_i, \beta_i)$, define $|P_i| = \alpha_i + \beta_i$.

Relation 1. Given positive integers $1 \leq i \neq j \leq n$ such that $|P_i| = i - 1, |P_{i+1}| = i, |P_j| = j - 1, |P_{j+1}| = j, \beta_i > 0, \alpha_j > 0$ (we assume $|P_{n+1}| = n$). Let D' be obtained from D by moving P_i to southeast and P_j to northwest, i.e.

$$D' = \{P_1, \dots, P_{i-1}, P_i + (1, -1), P_{i+1}, \dots, P_{j-1}, P_j + (-1, 1), P_{j+1}, \dots, P_n\}.$$

Then $\overline{\Delta(D)} = \overline{\Delta(D')}$.

Example: $n = 9, i = 2, j = 6$.

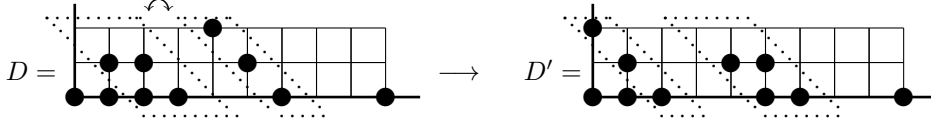


Relation 2. Given positive integers h, ℓ and m such that $2 \leq h < h + \ell + m \leq n + 1, |P_h| = h - 1, |P_{h+\ell}| = h + \ell - 1, |P_{h+\ell+m}| = h + \ell + m - 1$ (by convention, the last equality holds if $h + \ell + m = n + 1$) and $\alpha_{h+\ell}, \dots, \alpha_{h+\ell+m-1} \geq \ell$. Let D' be obtained from D by moving the m points $P_{h+\ell}, \dots, P_{h+\ell+m-1}$ to the left by ℓ units and moving the ℓ points $P_h, \dots, P_{h+\ell-1}$ to the right by m units, i.e.

$$D' = \{P_1, P_2, \dots, P_{h-1}, P_{h+\ell} - (\ell, 0), P_{h+\ell+1} - (\ell, 0), \dots, P_{h+\ell+m-1} - (\ell, 0), P_h + (m, 0), P_{h+1} + (m, 0), \dots, P_{h+\ell-1} + (m, 0), P_{h+\ell+m}, \dots, P_n\}.$$

Then $\overline{\Delta(D)} = \overline{\Delta(D')}$.

Example: $n = 10, h = 3, \ell = 4, m = 3$.



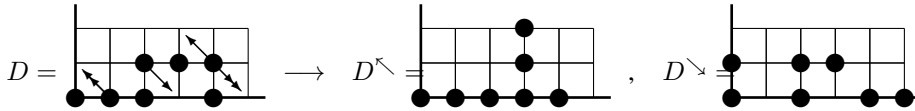
Relation 3. Given positive integers j and s . Suppose P_{s_0} is the last point in D satisfying $|P_i| = i - 1$. Define $j = (s_0 - 1 - |P_{s_0}|) + (s_0 - |P_{s_0+1}|) + \dots + (n - 1 - |P_n|)$. Suppose $|P_i| = i - 1$ for $1 \leq i \leq j + 2$, $P_2 = (1, 0)$, $s_0 \leq s \leq n$, and $\alpha_s, \beta_s \geq 1$. Let

$$D^{\nwarrow} = \{P_1, \dots, P_{j+1}, P_{j+2} + (1, -1), P_{j+3}, \dots, P_{s-1}, P_s + (-1, 1), P_{s+1}, \dots, P_n\},$$

$$D^{\searrow} = \{P_1, (0, 1), P_3, \dots, P_{s-1}, P_s + (1, -1), P_{s+2}, \dots, P_n\}.$$

Then $2\overline{\Delta(D)} = \overline{\Delta(D^{\nwarrow})} + \overline{\Delta(D^{\searrow})}$.

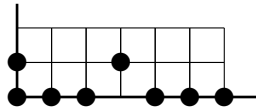
Example: $n = 9, i = 2, j = 6$.



We call $D = \{P_1, \dots, P_n\}$ a *minimal staircase form* if $|P_i| = i - 1$ or $i - 2$ for every $1 \leq i \leq n$. For a minimal staircase form D , let $\{i_1 < i_2 < \dots < i_\ell\}$ be the set of i 's such that $|P_i| = i - 1$, we define the *partition type* of D to be the partition of $\binom{n}{2} - \sum |P_i|$ consisting of all the positive integers in the sequence

$$(i_1 - 1, i_2 - i_1 - 1, i_3 - i_2 - 1, \dots, i_\ell - i_{\ell-1} - 1, n - i_\ell).$$

Example: Let $n = 8$ and $D = \{P_1, \dots, P_8\}$ satisfying $(|P_1|, \dots, |P_8|) = (0, 1, 1, 2, 4, 4, 5, 6)$. Then D is a minimal staircase form. The set $\{i \mid |P_i| = i - 1\}$ equals $\{1, 2, 5\}$. The positive integers in the sequence $(1 - 1, 2 - 1 - 1, 5 - 2 - 1, 8 - 5)$ are $(2, 3)$, so the partition type of D is $(2, 3)$.



Let $p(k)$ denote the number of partitions of an integer k and Π_k denote the set of partitions of k .

Theorem 1.2 Let k be any positive integer. There are positive constants $c_1 = 8k + 5, c_2 = 2k + 1$ such that the following holds:

For integers n, d_1, d_2 satisfying $n \geq c_1, d_1 \geq c_2 n, d_2 \geq c_2 n$ and $d_1 + d_2 = \binom{n}{2} - k$, the vector space $(M_n)_{d_1, d_2}$ has dimension $p(k)$, and the $p(k)$ elements

$$\{ \text{a minimal staircase form of bi-degree } (d_1, d_2) \text{ and of partition type } \mu \}_{\mu \in \Pi_k}$$

form a basis of $(M_n)_{d_1, d_2}$.

Note that N.Bergeron and Z.Chen have found explicit bases for $(M_n)_{d_1, d_2}$ for certain bi-degrees using a different method [1].

1.2.2 For arbitrary k and n

Denote by $p(k)$ the partition number of k and by convention $p(0) = 1$ and $p(k) = 0$ for $k < 0$. Denote by $p(b, k)$ the partition number of k into no more than b parts, and by convention $p(0, k) = 0$ for $k > 0$, $p(b, 0) = 1$ for $b \geq 0$. One of our main results is as follows.

Theorem 1.3 *Let d_1, d_2 be non-negative integers d_1, d_2 with $d_1 + d_2 \leq \binom{n}{2}$. Define $k = \binom{n}{2} - d_1 - d_2$ and $\delta = \min(d_1, d_2)$. Then the coefficient of $q^{d_1}t^{d_2}$ in $C_n(q, t)$ is less than or equal to $p(\delta, k)$, and the equality holds if and only if one the following conditions holds:*

- $k \leq n - 3$, or
- $k = n - 2$ and $\delta = 1$, or
- $\delta = 0$.

This theorem is a consequence of Theorem C. It contains [8, Theorem 6] and a result of N.Bergeron and Z.Chen [1, Corollary 8.3.1] as special cases. In fact it proves [8, Conjecture 8]. We feel that the coefficient of $q^{d_1}t^{d_2}$ for general k can also be expressed in terms of partition numbers, only that the expression might be complicated. For example, we give the following conjecture which is verified for $6 \leq n \leq 10$.

Conjecture. Let n, d_1, d_2, δ, k be as in Theorem 1.3. If $n - 2 \leq k \leq 2n - 8$ and $\delta \geq k$, then the coefficient of $q^{d_1}t^{d_2}$ in $C_n(q, t)$ equals

$$p(k) - 2[p(0) + p(1) + \dots + p(k - n + 1)] - p(k - n + 2).$$

As a corollary of Theorem 1.3 , we can compute some higher degree terms of the specialization at $t = q$.

Corollary 1.4

$$C_n(q, q) = \sum_{k=0}^{n-3} \left(p(k) \left(\binom{n}{2} - 3k + 1 \right) + 2 \sum_{i=1}^{k-1} p(i, k) \right) q^{\binom{n}{2}-k} + (\text{lower degree terms}).$$

The following theorem immediately implies Theorem 1.3.

Theorem 1.5 *Let d_1, d_2 be non-negative integers d_1, d_2 with $d_1 + d_2 \leq \binom{n}{2}$. Define $k = \binom{n}{2} - d_1 - d_2$ and $\delta = \min(d_1, d_2)$. Then $\dim(M_n)_{d_1, d_2} \leq p(\delta, k)$, and the equality holds if and only if one the following conditions holds:*

- $k \leq n - 3$, or
- $k = n - 2$ and $\delta = 1$, or
- $\delta = 0$.

In case the equality holds, there is an explicit construction of a basis of $(M_n)_{d_1, d_2}$.

The idea of the construction of the basis in the above theorem consists of two parts:

(1) Prove that

$$\dim(M_n)_{d_1, d_2} \leq p(\delta, k)$$

using a new characterization of q, t -Catalan numbers. The characterization is as follows, and is discovered independently by A. Woo [10].

Let $\mathfrak{D}_n^{\text{catalan}}$ be the set consisting of $D \subset \mathbb{N} \times \mathbb{N}$, where D contains n points satisfying the following conditions.

- (a) If $(p, 0) \in D$ then $(i, 0) \in D, \forall i \in [0, p]$.
- (b) For any $p \in \mathbb{N}$,

$$\#\{j \mid (p + 1, j) \in D\} + \#\{j \mid (p, j) \in D\} \geq \max\{j \mid (p, j) \in D\} + 1.$$

(If $\{j \mid (p, j) \in D\} = \emptyset$, then we require that no point $(i, j) \in D$ satisfies $i \geq p$.) Denote by $\text{deg}_x D$ (resp. $\text{deg}_y D$) the sum of the first (resp. second) components of the n points in D .

Proposition 1.6 *The coefficient of $q^{d_1} t^{d_2}$ in the q, t -Catalan number $C_n(q, t)$ is equal to*

$$\#\{D \in \mathfrak{D}_n^{\text{catalan}} \mid \text{deg}_x D = d_1, \text{deg}_y D = d_2\}.$$

(2) Construct a set of $p(\delta, k)$ linearly independent elements in $(M_n)_{d_1, d_2}$. It seems difficult (as least to the authors) to test directly whether a given set of elements in $(M_n)_{d_1, d_2}$ are linearly independent. We define a map φ sending an alternating polynomial $f \in \mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$ to a polynomial ring

$$\mathbb{C}[\rho] := \mathbb{C}[\rho_1, \rho_2, \rho_3, \dots].$$

The map has two desirable properties: (i) for many f , $\varphi(f)$ can be easily computed, and (ii) for each bi-degree (d_1, d_2) , φ induces a morphism $\bar{\varphi} : (M_n)_{d_1, d_2} \rightarrow \mathbb{C}[\rho]$ of \mathbb{C} -modules. Then we use the fact the linear dependency is easier to check in $\mathbb{C}[\rho]$ than in $(M_n)_{d_1, d_2}$. The map φ is defined as below.

1.2.3 Maps φ and $\bar{\varphi}$.

(a) Define the map $\varphi : \mathfrak{D}_n \rightarrow \mathbb{Z}[\rho]$ as follows. Let $D = \{(a_1, b_1), \dots, (a_n, b_n)\} \in \mathfrak{D}_n, k = \binom{n}{2} - \sum_{i=1}^n (a_i + b_i)$, and define

$$\varphi(D) := (-1)^k \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n \left(\sum \rho_{w_1} \rho_{w_2} \cdots \rho_{w_{b_i}} \right),$$

where (w_1, \dots, w_{b_i}) in the sum $\sum \rho_{w_1} \rho_{w_2} \cdots \rho_{w_{b_i}}$ runs through the set

$$\{(w_1, \dots, w_{b_i}) \in \mathbb{N}^{b_i} \mid w_1 + \dots + w_{b_i} = \sigma(i) - 1 - a_i - b_i\},$$

with the convention that

$$\sum \rho_{w_1} \cdots \rho_{w_{b_i}} = \begin{cases} 0 & \text{if } \sigma(i) - 1 - a_i - b_i < 0; \\ 0 & \text{if } b_i = 0 \text{ and } \sigma(i) - 1 - a_i - b_i > 0; \\ 1 & \text{if } b_i = 0 \text{ and } \sigma(i) - 1 - a_i - b_i = 0. \end{cases}$$

(b) Here is an equivalent definition of $\varphi(D)$. Define the weight of ρ_i to be i for $i \in \mathbb{N}^+$ and define the weight of $\rho_0 = 1$ to be 0. Naturally the weight of any monomial $c\rho_{i_1}\dots\rho_{i_n}$ ($c \in \mathbb{Z}$) is defined to be $i_1 + \dots + i_n$. For $w \in \mathbb{N}$ and a power series $f \in \mathbb{Z}[[\rho_1, \rho_2, \dots]]$, denote by $\{f\}_w$ the sum of terms of weight- w in f , which is a polynomial. Define

$$h(b, w) := \{(1 + \rho_1 + \rho_2 + \dots)^b\}_w, \quad b \in \mathbb{N}, w \in \mathbb{Z}.$$

Naturally $h(b, w) = 0$ if $w < 0$. Also assume $(1 + \rho_1 + \rho_2 + \dots)^0 = 1$. Then

$$\varphi(D) = (-1)^k \begin{vmatrix} h(b_1, -|P_1|) & h(b_1, 1 - |P_1|) & h(b_1, 2 - |P_1|) & \dots & h(b_1, n - 1 - |P_1|) \\ h(b_2, -|P_2|) & h(b_2, 1 - |P_2|) & h(b_2, 2 - |P_2|) & \dots & h(b_2, n - 1 - |P_2|) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h(b_n, -|P_n|) & h(b_n, 1 - |P_n|) & h(b_n, 2 - |P_n|) & \dots & h(b_n, n - 1 - |P_n|) \end{vmatrix}.$$

(c) Let $D_1, \dots, D_\ell \in D'$ be of the same bi-degree and $\sum_{i=1}^\ell c_i D_i$ be the formal sum for any $c_i \in \mathbb{C}$ ($1 \leq i \leq \ell$). Define

$$\varphi\left(\sum_{i=1}^\ell c_i D_i\right) := \sum_{i=1}^\ell c_i \varphi(D_i).$$

For any bi-homogeneous alternating polynomials $f = \sum_{i=1}^\ell c_i \Delta(D_i) \in \mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$, we define

$$\varphi(f) := \varphi\left(\sum_{i=1}^\ell c_i D_i\right) = \sum_{i=1}^\ell c_i \varphi(D_i)$$

by abuse of notation. □

Proposition 1.7 Fix any pair of nonnegative integers (d_1, d_2) , the map φ induces a well-defined linear map

$$\bar{\varphi} : (M_n)_{d_1, d_2} \longrightarrow \mathbb{C}[\rho].$$

Moreover, this map $\bar{\varphi}$ is conjecturally injective. And our future work is to generalizing it to the case $I_n^m / (\mathbf{x}, \mathbf{y}) I_n^m$ for any positive integer m .

Acknowledgements

We are grateful to François Bergeron, Mahir Can, Jim Haglund, Nick Loehr, Alex Woo and Alex Yong for valuable discussions and correspondence.

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