Packing and covering the balanced complete bipartite multigraph with cycles and stars

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Packing and covering the balanced complete bipartite multigraph with cycles and stars

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Let C_k denote a cycle of length k and let S_k denote a star with k edges. For multigraphs F, G and H, an (F, G)-decomposition of H is an edge decomposition of H into copies of F and G using at least one of each. For $L \subseteq H$ and $R \subseteq rH$, an (F, G)-packing (resp. (F, G)-covering) of H with leave L (resp. padding R) is an (F, G)-decomposition of $H - E(L)$ (resp. H+E(R)). An (F, G)-packing (resp. (F, G)-covering) of H with the largest (resp. smallest) cardinality is a maximum (F, G)-packing (resp. minimum (F, G)-covering), and its cardinality is referred to as the (F, G)-packing number (resp. (F, G)-covering number) of H. In this paper, we determine the packing number and the covering number of $\lambda K_{n,n}$ with C_k's and S_k's for any λ, n and k, and give the complete solution of the maximum packing and the minimum covering of $\lambda K_{n,n}$ with 4-cycles and 4-stars for any λ and n with all possible leaves and paddings.

Keywords: complete bipartite multigraph, cycle, star, packing, covering

1 Introduction

For positive integers m and n, $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n. If $m = n$, the complete bipartite graph is referred to as *balanced*. A k-cycle, denoted by C_k, is a cycle of length k. A k-star, denoted by S_k, is the complete bipartite graph $K_{1,k}$. A *k-path*, denoted by P_k, is a path with k vertices. For a graph H and a positive integer λ, we use λH to denote the multigraph obtained from H by replacing each edge e by λ edges each having the same endpoints as e. When $\lambda = 1$, $1H$ is simply written as H.

Let F, G, and H be multigraphs. A *decomposition* of H is a set of edge-disjoint subgraphs of H whose union is H. An (F, G)-*decomposition* of H is a decomposition of H into copies of F and G using at least one of each. If H has an (F, G)-decomposition, we say that H is (F, G)-*decomposable* and write $(F, G)|H$. If H does not admit an (F, G)-decomposition, two natural questions arise:

(1) What is the minimum number of edges needed to be removed from the edge set of H so that the resulting graph is (F, G)-decomposable, and what does the collection of removed edges look like?

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(2) What is the minimum number of edges needed to be added to the edge set of H so that the resulting graph is (F, G)-decomposable, and what does the collection of added edges look like?

These questions are respectively called the maximum packing problem and the minimum covering problem of H with F and G.

Let F, G, and H be multigraphs. For $L \subseteq H$ and $R \subseteq rH$, an (F, G)*-packing* of H with *leave* L is an (F, G)-decomposition of $H - E(L)$, and an (F, G)*-covering* with *padding* R is an (F, G)-decomposition of $H + E(R)$. For an (F, G)-packing $\mathscr{P}$ of H with leave L, if $|\mathscr{P}|$ is as large as possible (so that $|L|$ is as small as possible), then $\mathscr{P}$ and L are referred to as a *maximum* (F, G)*-packing* and a *minimum leave*, respectively. Moreover, the cardinality of the maximum (F, G)-packing of H is called the (F, G)*-packing number* of H, denoted by $p(H; F, G)$. For an (F, G)-covering $\mathscr{C}$ of H with padding R, if $|\mathscr{C}|$ is as small as possible (so that $|R|$ is as small as possible), then $\mathscr{C}$ and R are referred to as a *minimum covering* and a *minimum padding*, respectively. Moreover, the cardinality of the minimum (F, G)-covering of H is called the (F, G)*-covering number* of H, denoted by $c(H; F, G)$. Clearly, an (F, G)-decomposition of H is a maximum (F, G)-packing with leave the empty graph, and also a minimum (F, G)-covering with padding the empty graph.

Recently, decomposition into a pair of graphs has attracted a fair share of interest. Abueida and Daven [3] investigated the problem of (K_k, S_k)-decomposition of the complete graph K_n. Abueida and Daven [4] investigated the problem of the (C_4, E_2)-decomposition of several graph products where E_2 denotes two vertex disjoint edges. Abueida and O'Neil [7] settled the existence problem for (C_k, S_{k-1})-decomposition of the complete multigraph λK_n for $k \in \{3, 4, 5\}$. Priyadharsini and Muthusamy [12, 13] gave necessary and sufficient conditions for the existence of (G_n, H_n)-decompositions of λK_n and $\lambda K_{n,n}$ where $G_n, H_n \in \{C_n, P_n, S_{n-1}\}$. A *graph-pair* (G, H) *of order* m is a pair of non-isomorphic graphs G and H on m non-isolated vertices such that $G \cup H$ is isomorphic to K_m. Abueida and Daven [2] and Abueida, Daven and Roblee [5] completely determined the values of n for which λK_n admits a (G, H)-decomposition where (G, H) is a graph-pair of order 4 or 5. Abueida, Clark and Leach [1] and Abueida and Hampson [6] considered the existence of decompositions of $K_n - F$ for the graph-pair of order 4 and 5, respectively, where F is a Hamiltonian cycle, a 1-factor, or almost 1-factor. Furthermore, Shyu [14] investigated the problem of decomposing K_n into paths and stars with k edges, giving a necessary and sufficient condition for $k = 3$. In [15, 16], Shyu considered the existence of a decomposition of K_n into paths and cycles with k edges, giving a necessary and sufficient condition for $k \in \{3, 4\}$. Shyu [17] investigated the problem of decomposing K_n into cycles and stars with k edges, settling the case $k = 4$. In [18], Shyu considered the existence of a decomposition of $K_{m,n}$ into paths and stars with k edges, giving a necessary and sufficient condition for $k = 3$. Recently, Lee [9] and Lee and Lin [10] established necessary and sufficient conditions for the existence of (C_k, S_k)-decompositions of the complete bipartite graph and the complete bipartite graph with a 1-factor removed, respectively. However, much less work has been done on the problem of packing and covering graphs with a pair of graphs. Abueida and Daven [3] obtained the maximum packing and the minimum covering of the complete graph K_n with (K_k, S_k). Abueida and Daven [2] and Abueida, Daven and Roblee [5] gave the maximum packing and the minimum covering of K_n and λK_n with G and H, respectively, where (G, H) is a graph-pair of order 4 or 5. In this paper, we determine the packing number and the covering number of $\lambda K_{n,n}$ with k-cycles and k-stars for any λ, n and k, and give the complete solution of the maximum packing and the minimum covering of $\lambda K_{n,n}$ with 4-cycles and 4-stars for any λ and n with all possible leaves and paddings.

2 Preliminaries

In this section we first collect some needed terminology and notation, and then present a result which is useful for our discussions to follow.

Let G be a multigraph. The *degree* of a vertex x of G, denoted by $\deg_G x$, is the number of edges incident with x. The vertex of degree k in S_k is the *center* of S_k and any vertex of degree 1 is an *endvertex* of S_k. For $W \subseteq V(G)$, we use $G[W]$ to denote the subgraph of G induced by W. Furthermore, $\mu(uv)$ denotes the number of edges of G joining u and v, $(v_1, \ldots, v_k)$ and $v_1 \ldots v_k$ denote the k-cycle and the k-path through vertices $v_1, \ldots, v_k$ in order, respectively, and $(x; y_1, \ldots, y_k)$ denotes the k-star with center x and endvertices $y_1, \ldots, y_k$. When $G_1, G_2, \ldots, G_t$ are multigraphs, not necessarily disjoint, we write $G_1 \cup G_2 \cup \cdots \cup G_t$ or $\bigcup_{i=1}^t G_i$ for the graph with vertex set $\bigcup_{i=1}^t V(G_i)$ and edge set $\bigcup_{i=1}^t E(G_i)$. When the edge sets are disjoint, $G = \bigcup_{i=1}^t G_i$ expresses the decomposition of G into $G_1, G_2, \ldots, G_t$. Given an S_k-decomposition of G, a *central function* c from $V(G)$ to the set of non-negative integers is defined as follows. For each $v \in V(G)$, $c(v)$ is the number of k-stars in the decomposition whose center is v.

The following result is essential to our proof.

Proposition 2.1 (Hoffman [8]) *For a positive integer k, a multigraph H has an S_k-decomposition with central function c if and only if*

(i) $k \sum_{v \in V(H)} c(v) = |E(H)|$,
(ii) *for all* $x, y \in V(H)$, $\mu(xy) \le c(x) + c(y)$,
(iii) *for all* $S \subseteq V(H)$, $k \displaystyle\sum_{v \in S} c(v) \le \varepsilon(S) + \sum_{x \in S, y \in V(H)-S} \min\{c(x), \mu(xy)\}$.

where $\varepsilon(S)$ denotes the number of edges of H with both ends in S.

In the sequel of the paper, (A, B) denotes the bipartition of $\lambda K_{n,n}$, where $A = \{a_0, a_1, \ldots, a_{n-1}\}$ and $B = \{b_0, b_1, \ldots, b_{n-1}\}$.

3 Packing numbers and covering numbers

In this section the packing number and the covering number of the balanced complete bipartite multigraph with k-cycles and k-stars are determined. We begin with a criterion for decomposing the complete bipartite graph into k-cycles.

Proposition 3.1 (Sotteau [19]) *For positive integers m, n, and k, the graph $K_{m,n}$ is C_k-decomposable if and only if m, n, and k are even, $k \ge 4$, $\min\{m, n\} \ge k/2$, and k divides mn.*

Let $K^*_{m,n}$ denote the symmetric complete bipartite digraph with parts of size m and n, and let $\overrightarrow{C_k}$ denote the directed k-cycle.

Proposition 3.2 (Sotteau [19]) *For positive integers m, n, and k, the digraph $K^*_{m,n}$ is $\overrightarrow{C_k}$-decomposable if and only if k is even, $k \ge 4$, $\min\{m, n\} \ge k/2$, and k divides $2mn$.*

Removing the directions from the arcs of directed cycles in a $\overrightarrow{C_k}$-decomposition of $K^*_{m,n}$, we obtain the following result by Proposition 3.2.

Lemma 3.3 *For positive integers m, n, and k, the multigraph $2K_{m,n}$ is C_k-decomposable if k is even, $k \geq 4$, $\min\{m,n\} \geq k/2$, and k divides $2mn$.*

Lemma 3.4 *Let λ, k, m, and n be positive integers with $\lambda m \equiv \lambda n \equiv k \equiv 0 \pmod 2$ and $\min\{m,n\} \geq k/2 \geq 2$. If m or n is divisible by k, then $\lambda K_{m,n}$ is C_k-decomposable.*

Proof: Since $\lambda K_{m,n}$ is isomorphic to $\lambda K_{n,m}$, it suffices to show that the result holds for $k \mid m$. If λ is odd, then m and n are even from the assumption $\lambda m \equiv \lambda n \equiv 0 \pmod 2$. Since k divides mn, Proposition 3.1 implies that $K_{m,n}$ is C_k-decomposable. If λ is even, then $2K_{m,n} \mid \lambda K_{m,n}$. Since k divides $2mn$, $2K_{m,n}$ is C_k-decomposable by Lemma 3.3. Hence $\lambda K_{m,n}$ is C_k-decomposable. □

Lemma 3.5 *If k is a positive even integer with $k \geq 4$, then $\lambda K_{k,k}$ is (C_k, S_k)-decomposable.*

Proof: Note that $\lambda K_{k,k} = \lambda K_{k,k-2} \cup \lambda K_{k,2}$. By Lemma 3.4, $\lambda K_{k,k-2}$ is C_k-decomposable. Trivially, $\lambda K_{k,2}$ is C_k-decomposable. Therefore, $\lambda K_{k,k}$ is (C_k, S_k)-decomposable. □

Lemma 3.6 *Let k be a positive even integer and let n be a positive integer with $4 \leq k < n < 2k$. If $\lambda(n-k)^2 < k$, then $\lambda K_{n,n}$ has a (C_k, S_k)-packing with leave $\lambda K_{n-k,n-k}$ and a (C_k, S_k)-covering with padding $P_{k-\lambda(n-k)^2+1}$.*

Proof: Let $n = k + r$. The assumption $k < n < 2k$ implies $0 < r < k$. We first give the required packing. Note that

$$\lambda K_{n,n} = \lambda K_{k,k} \cup \lambda K_{k,r} \cup \lambda K_{r,k} \cup \lambda K_{r,r}.$$

By Lemma 3.4, $\lambda K_{k,k}$ has a C_k-decomposition $\mathscr{D}_1$. Trivially, $\lambda K_{k,r}$ and $\lambda K_{r,k}$ have S_k-decompositions $\mathscr{D}_2$ and $\mathscr{D}_3$, respectively. Thus $\bigcup_{i=1}^{3} \mathscr{D}_i$ is a (C_k, S_k)-packing of $\lambda K_{n,n}$ with leave $\lambda K_{r,r}$, as desired.

Now we give the required covering. Let $s = \lambda r^2$. Let $A_0 = \{a_0, a_1, \ldots, a_{\lfloor (s-1)/2 \rfloor}\}$, $A_1 = A - A_0$, $B_0 = \{b_0, b_1, \ldots, b_{k-1}\}$ and $B_1 = B - B_0$. Define a k-cycle C and a $(k-s+1)$-path P as follows:

$$C = (b_0, a_0, b_1, a_1, \ldots, b_{k/2-1}, a_{k/2-1})$$

$$P = \begin{cases} b_0 a_{k/2-1} b_{k/2-1} a_{k/2-2} \ldots b_{(s+1)/2} a_{(s-1)/2} & \text{if } s \text{ is odd}, \\ b_0 a_{k/2-1} b_{k/2-1} a_{k/2-2} \ldots a_{s/2} b_{s/2} & \text{if } s \text{ is even}. \end{cases}$$

Let

$$H = \lambda K_{n,n} - E(C) + E(P).$$

Note that $V(H) = V(\lambda K_{n,n})$, $|E(H)| = \lambda n^2 - k + (k-s) = \lambda n^2 - \lambda r^2 = \lambda k(k+2r)$, and $\mu(uv) \leq \lambda$ for all $u, v \in V(H)$. Furthermore, for $H' = H[A \cup B_0]$, we have

$$\deg_{H'} v = \begin{cases} \lambda k - 2 & \text{if } v \in A_0 - \{a_{\lfloor (s-1)/2 \rfloor}\}, \\ \lambda k - \rho & \text{if } v = a_{\lfloor (s-1)/2 \rfloor}, \\ \lambda k & \text{if } v \in A_1, \end{cases}$$

where $\rho = 1$ if s is odd, and $\rho = 2$ if s is even. Define a function $c : V(H) \to \mathbb{N}$ as follows:

$$c(v) = \begin{cases} 0 & \text{if } v \in B_0, \\ \lambda & \text{otherwise}. \end{cases}$$

Now we show that there exists an S_k-decomposition of H with central function c by Proposition 2.1.

First, $k\sum_{v\in V(H)} c(v) = k\lambda(k+2r) = |E(H)|$. This proves (i). Next, if $u, v \in B_0$, then $c(u)+c(v) = 0 = \mu(uv)$; otherwise, $c(u)+c(v) \geq \lambda \geq \mu(uv)$. This proves (ii). Finally, for $S \subseteq V(H)$ and $i \in \{0,1\}$, let $S \cap A_i = X_i$ and $S \cap B_i = Y_i$. Moreover, let $X = X_0 \cup X_1$ and $Y = Y_0 \cup Y_1$. Define a set T of ordered pairs of vertices as follows:

$$T = \{(u,v) | u \in X, v \in B_1 - Y_1 \text{ or } u \in X_1, v \in B_0 - Y_0 \text{ or } u \in Y_1, v \in A - X\}.$$

Note that

$$k\sum_{w\in S} c(w) = k\lambda(|X| + |Y_1|), \tag{1}$$

$$\varepsilon(S) = \lambda(|X||Y_1| + |X_1||Y_0|) + \sum_{u\in X_0, v\in Y_0} \mu(uv), \tag{2}$$

and for $u \in S$ and $v \in V(H) - S$

$$\min\{c(u), \mu(uv)\} = \begin{cases} \lambda & \text{if } (u,v) \in T, \\ \mu(uv) & \text{if } u \in X_0, v \in B_0 - Y_0, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

For $S \subseteq V(H)$, let

$$g(S) = \varepsilon(S) + \sum_{u\in S, v\in V(H)-S} \min\{c(u), \mu(uv)\} - k\sum_{w\in S} c(w).$$

Note that

$$\begin{aligned} & \sum_{u\in X_0, v\in Y_0} \mu(uv) + \sum_{u\in X_0, v\in B_0-Y_0} \mu(uv) \\ = & \sum_{u\in X_0, v\in B_0} \mu(uv) \\ = & \begin{cases} |X_0|(\lambda k - 2) & \text{if } a_{\lfloor (s-1)/2\rfloor} \notin X_0, \\ |X_0|(\lambda k - 2) + 2 - \rho & \text{if } a_{\lfloor (s-1)/2\rfloor} \in X_0. \end{cases} \end{aligned}$$

By (1)–(3) and $|X_0| + |X_1| = |X|$, we have

$$\begin{aligned} g(S) &= \lambda(|X||Y_1| + |X_1||Y_0|) + \sum_{u\in X_0, v\in Y_0} \mu(uv) \\ &\quad +\lambda(|X|(r - |Y_1|) + |X_1|(k - |Y_0|) + |Y_1|(k + r - |X|)) \\ &\quad + \sum_{u\in X_0, v\in B_0-Y_0} \mu(uv) - k\lambda(|X| + |Y_1|) \\ &= \begin{cases} \lambda(r|X| + |Y_1|(r - |X|)) - 2|X_0| & \text{if } a_{\lfloor (s-1)/2\rfloor} \notin X_0, \\ \lambda(r|X| + |Y_1|(r - |X|)) - 2|X_0| + 2 - \rho & \text{if } a_{\lfloor (s-1)/2\rfloor} \in X_0. \end{cases} \end{aligned}$$

If $a_{\lfloor (s-1)/2 \rfloor} \notin X_0$, then $|X_0| \le \lfloor (s-1)/2 \rfloor$, which implies $-2|X_0| \ge -s$. If $a_{\lfloor (s-1)/2 \rfloor} \in X_0$, then $|X_0| \le \lfloor (s-1)/2 \rfloor + 1$, which implies $-2|X_0| + 2 - \rho \ge -2\lfloor (s-1)/2 \rfloor - \rho = -2(s-\rho)/2 - \rho = -s$. Thus for $|X| \ge r$, we have

$$\begin{aligned} g(S) &\ge \lambda(r|X| - |Y_1|(|X| - r)) - s \\ &= \lambda(r|X| - |Y_1|(|X| - r)) - \lambda r^2 \\ &= \lambda(|X| - r)(r - |Y_1|) \\ &\ge 0. \end{aligned}$$

If $\lambda r = 1$ and $|X| < r$, then $|X_0| = |X| = 0$, which implies $-2|X_0| = -\lambda r|X_0|$. If $\lambda r \ge 2$, then $-2|X_0| \ge -\lambda r|X_0|$. Note that $2 - \rho \ge 0$. Hence for $|X| < r$, we have

$$\begin{aligned} g(S) &\ge \lambda(r|X| + |Y_1|(r - |X|)) - \lambda r|X_0| \\ &= \lambda(r|X_1| + |Y_1|(r - |X|)) \\ &\ge 0. \end{aligned}$$

This settles (iii) and completes the proof. □

Before going on, the following results are needed.

Proposition 3.7 (Ma et al. [11]) *For positive integers k and n, the graph obtained by deleting a 1-factor from $K_{n,n}$ is C_k-decomposable if and only if n is odd, k is even, $4 \le k \le 2n$, and $n(n-1)$ is divisible by k.*

Lemma 3.8 *If λ and p are positive integers and k is a positive even integer with $k \ge 4$, then there exist $\lambda pk/2 - p$ edge-disjoint k-cycles in $\lambda K_{k/2,pk}$ $(also\ in\ \lambda K_{pk,k/2})$.*

Proof: It suffices to show that the result holds for $\lambda K_{k/2,pk}$. If λ or $k/2$ is even, then by Lemma 3.4 there exists a C_k-decomposition $\mathscr{D}$ of $\lambda K_{k/2,pk}$ with $|\mathscr{D}| = \lambda pk/2$, in which k-cycles are edge-disjoint. If $k/2$ is odd, then by Proposition 3.7 there exists a C_k-decomposition $\mathscr{D}'$ of $K_{k/2,k/2} - I$ with $|\mathscr{D}'| = (k-2)/4$, where I is a 1-factor of $K_{k/2,k/2}$. Since $K_{k/2,pk}$ can be decomposed into $2p$ copies of $K_{k/2,k/2}$, there exist $2p|\mathscr{D}'| = pk/2 - p$ edge-disjoint k-cycles in $K_{k/2,pk}$. For odd λ with $\lambda \ge 3$, $\lambda K_{k/2,k} = (\lambda-1)K_{k/2,k} \cup K_{k/2,k}$. By Lemma 3.4 there exists a C_k-decomposition $\mathscr{D}''$ of $(\lambda-1)K_{k/2,pk}$ with $|\mathscr{D}''| = (\lambda-1)pk/2$. Hence there exist $(\lambda-1)pk/2 + pk/2 - p = \lambda pk/2 - p$ edge-disjoint k-cycles in $\lambda K_{k/2,pk}$. □

Lemma 3.9 *Let λ and r be positive integers and let k be a positive even integer with $k \ge 4$ and $r < k$. If $t = \lfloor \lambda r^2/k \rfloor$, then there exist $\lceil t/2 \rceil$ edge-disjoint k-cycles in $\lambda K_{k/2,k}$. Moreover, if $\lambda \ge 2$ or $r \le k-2$ and $\lambda r^2 \ge k$, then $\lfloor t/2 \rfloor + 1 \le \lambda r/2$ and there exist $\lfloor t/2 \rfloor + 1$ edge-disjoint k-cycles in $\lambda K_{k/2,k}$.*

Proof: Since $r < k$, we have $t < \lambda r$. Thus $t + 1 \le \lambda r$; in turn, $\lceil t/2 \rceil \le (t+1)/2 \le \lambda r/2 < \lambda k/2$, which implies $\lceil t/2 \rceil \le \lambda k/2 - 1$. By Lemma 3.8, there exist $\lceil t/2 \rceil$ edge-disjoint k-cycles in $\lambda K_{k/2,k}$. When $\lambda r^2 = k$, the result is trivial. When $\lambda r^2 > k$, we have $r > 2/\sqrt{\lambda}$ since $k \ge 4$. For $\lambda \ge 2$,

$$\frac{\lambda r^2}{k} \le \frac{\lambda r^2}{r+1} = \lambda r - \frac{\lambda}{1 + 1/r} < \lambda r - \frac{2\lambda}{2 + \sqrt{\lambda}} < \lambda r - \frac{4}{2 + \sqrt{2}}.$$

For $r \leq k-2$,

$$\frac{\lambda r^2}{k} \leq \frac{\lambda r^2}{r+2} = \lambda r - \frac{2\lambda}{1+2/r} < \lambda r - \frac{2\lambda}{1+\sqrt{\lambda}} < \lambda r - 1.$$

Therefore, $t = \lfloor \lambda r^2/k \rfloor \leq \lambda r - 2$. In turn, $\lfloor t/2 \rfloor + 1 \leq t/2 + 1 \leq \lambda r/2$ for $\lambda \geq 2$ or $r \leq k-2$. It implies $\lfloor t/2 \rfloor + 1 < \lambda k/2$. Hence $\lfloor t/2 \rfloor + 1 \leq \lambda k/2 - 1$ for $\lambda \geq 2$ or $r \leq k-2$. This assures us that there exist $\lfloor t/2 \rfloor + 1$ edge-disjoint k-cycles in $\lambda K_{k/2,k}$ by Lemma 3.8. □

Lemma 3.10 *Let k be a positive even integer and let n be a positive integer with $4 \leq k < n < 2k$. If $\lambda(n-k)^2 \geq k$, then $\lambda K_{n,n}$ has a (C_k, S_k)-packing $\mathscr{P}$ with $|\mathscr{P}| = \lfloor \lambda n^2/k \rfloor$ and a (C_k, S_k)-covering $\mathscr{C}$ with $|\mathscr{C}| = \lceil \lambda n^2/k \rceil$.*

Proof: Let $n = k + r$. From the assumption $k < n < 2k$, we have $0 < r < k$. Let $\lambda r^2 = tk + s$ such that s and t are integers with $0 \leq s < k$. Note that $t = \lfloor \lambda r^2/k \rfloor$. Hence $\lfloor \lambda n^2/k \rfloor = \lfloor \lambda(k+r)^2/k \rfloor = \lambda(k+2r) + t$ and

$$\left\lceil \frac{\lambda n^2}{k} \right\rceil = \left\lceil \frac{\lambda(k+r)^2}{k} \right\rceil = \begin{cases} \lambda(k+2r) + t & \text{if } s = 0 \\ \lambda(k+2r) + t + 1 & \text{if } s > 0. \end{cases}$$

Since $\lambda(n-k)^2 \geq k$, $t \geq 1$. Let $p_0 = \lceil t/2 \rceil$ and $p_1 = \lfloor t/2 \rfloor$. We have $p_0 = 1$ and $p_1 = 0$ for $t = 1$, and $p_0 \geq p_1 \geq 1$ for $t \geq 2$. In the sequel, we will show that $\lambda K_{n,n}$ has a packing $\mathscr{P}$ consisting of t copies of k-cycles and $\lambda(k+2r)$ copies of k-stars with leave P_{s+1} (except in the case $s = 0$, in which the leave is the empty graph), and a covering $\mathscr{C}$ with $|\mathscr{C}| = \lceil \lambda n^2/k \rceil$.

Let $A_0 = \{a_0, a_1, \ldots, a_{k/2-1}\}$, $A_1 = \{a_{k/2}, a_{k/2+1}, \ldots, a_{k-1}\}$, $A_2 = A - (A_0 \cup A_1)$, $B_0 = \{b_0, b_1, \ldots, b_{k-1}\}$ and $B_1 = B - B_0$. In addition, letting $A_1' = \{a_{k/2}, a_{k/2+1}, \ldots, a_{\lceil (k+s)/2 \rceil - 1}\}$ for $s > 0$ and $G_i = \lambda K_{n,n}[A_i \cup B_1]$ for $i = 0, 1$. Clearly, G_0 and G_1 are isomorphic to $\lambda K_{k/2,k}$. By Lemma 3.9, there exist p_i edge-disjoint k-cycles in G_i for $i \in \{0,1\}$, and there exist $p_1 + 1$ edge-disjoint k-cycles in G_1 for $\lambda \geq 2$ or $r \leq k-2$. Let $\delta = 0$ for $p_1 = 0$ and $\delta = 1$ for $p_1 \geq 1$. Suppose that $Q_{i,0}, Q_{i,1}, \ldots, Q_{i,p_i-1}$ are edge-disjoint k-cycles in G_i for $0 \leq i \leq \delta$. Moreover, for $\lambda \geq 2$ or $r \leq k-2$, let Q be a k-cycle in G_1 which is edge-disjoint with $Q_{1,j}$ for $0 \leq j \leq p_1 - 1$. Without loss of generality, we assume that

$$Q = (b_{j_1}, a_{k/2}, b_{j_2}, a_{k/2+1}, \ldots, b_{j_{k/2}}, a_{k-1}).$$

Note, for $\lambda = 1$ and $r = k-1$, that $\lambda r^2 = (k-1)^2 = k(k-2) + 1$, which implies $t = k-2$ and $s = 1$. For $s > 0$, define an $(s+1)$-path P as follows:

$$P = \begin{cases} a_{k/2} b_\ell & \text{if } \lambda = 1, r = k-1, \\ b_{j_1} a_{k/2} b_{j_2} a_{k/2+1} \ldots b_{j_{s/2}} a_{(k+s)/2-1} b_{j_{s/2+1}} & \text{if } \lambda \geq 2 \text{ or } r \leq k-2, s \text{ is even}, \\ b_{j_1} a_{k/2} b_{j_2} a_{k/2+1} \ldots b_{j_{(s+1)/2}} a_{(k+s+1)/2-1} & \text{if } \lambda \geq 2 \text{ or } r \leq k-2, s \text{ is odd}, \end{cases}$$

where $a_{k/2} b_\ell$ is any edge (incident with $a_{k/2}$) not in $Q_{1,0}, Q_{1,1}, \ldots, Q_{1,p_1-1}$. Let

$$H = \lambda K_{n,n} - E\Big(\bigcup_{i=0}^{\delta} \big(\bigcup_{h=0}^{p_i - 1} Q_{i,h}\big) \cup P\Big).$$

Note that $V(H) = V(\lambda K_{n,n})$, $|E(H)| = \lambda n^2 - (tk+s) = \lambda n^2 - \lambda r^2 = \lambda k(k+2r)$, and $\mu(uv) \le \lambda$ for all $u, v \in V(H)$. Moreover, for $H' = H[A \cup B_0]$, we have

$$\deg_{H'} v = \begin{cases} \lambda k - 2\lceil t/2 \rceil & \text{if } v \in A_0, \\ \lambda k - 2(\lfloor t/2 \rfloor + 1) & \text{if } s > 0 \text{ and } v \in A_1' - \{a_{\lceil (k+s)/2 \rceil - 1}\}, \\ \lambda k - 2\lfloor t/2 \rfloor - \rho & \text{if } s > 0 \text{ and } v = a_{\lceil (k+s)/2 \rceil - 1}, \\ \lambda k - 2\lfloor t/2 \rfloor & \text{if } s > 0 \text{ and } v \in A_1 - A_1', \text{ or } s = 0 \text{ and } v \in A_1, \\ \lambda k & \text{if } v \in A_2, \end{cases}$$

where $\rho = 1$ if s is odd, and $\rho = 2$ if s is even. Define a function $c : V(H) \to \mathbb{N}$ as follows:

$$c(v) = \begin{cases} 0 & \text{if } v \in B_0, \\ \lambda & \text{otherwise.} \end{cases}$$

Now we show that there exists an S_k-decomposition $\mathscr{D}$ of H with central function c by Proposition 2.1.

First, $k \sum_{v \in V(H)} c(v) = k\lambda(k+2r) = |E(H)|$. This proves (i). Next, if $u, v \in B_0$, then $c(u)+c(v) = 0 = \mu(uv)$; otherwise, $c(u) + c(v) \ge \lambda \ge \mu(uv)$. This proves (ii). Finally, for $S \subseteq V(H)$, $i \in \{0,1,2\}$, and $j \in \{0,1\}$, let $S \cap A_i = X_i$ and $S \cap B_j = Y_j$. Moreover, letting $S \cap A_1' = X_1'$, $X = X_0 \cup X_1 \cup X_2$, and $Y = Y_0 \cup Y_1$. Define a set T of ordered pairs of vertices as follows:

$$T = \{(u,v) | u \in X, v \in B_1 - Y_1 \text{ or } u \in X_2, v \in B_0 - Y_0 \text{ or } u \in Y_1, v \in A - X\}.$$

Note that

$$k \sum_{w \in S} c(w) = k\lambda(|X| + |Y_1|), \tag{4}$$

$$\varepsilon(S) = \lambda(|X||Y_1| + |X_2||Y_0|) + \sum_{u \in X_0 \cup X_1, v \in Y_0} \mu(uv), \tag{5}$$

and for $u \in S$ and $v \in V(H) - S$

$$\min\{c(u), \mu(uv)\} = \begin{cases} \lambda & \text{if } (u,v) \in T, \\ \mu(uv) & \text{if } u \in X_0 \cup X_1, v \in B_0 - Y_0, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

For $S \subseteq V(H)$, let

$$g(S) = \varepsilon(S) + \sum_{u \in S, v \in V(H) - S} \min\{c(u), \mu(uv)\} - k \sum_{w \in S} c(w).$$

Note that

$$\begin{aligned} & \sum_{u \in X_0 \cup X_1, v \in Y_0} \mu(uv) + \sum_{u \in X_0 \cup X_1, v \in B_0 - Y_0} \mu(uv) \\ = & \sum_{u \in X_0 \cup X_1, v \in B_0} \mu(uv) \\ = & \begin{cases} |X_0|(\lambda k - 2\lceil t/2 \rceil) + |X_1|(\lambda k - 2\lfloor t/2 \rfloor) & \text{if } s = 0, \\ |X_0|(\lambda k - 2\lceil t/2 \rceil) + |X_1|(\lambda k - 2\lfloor t/2 \rfloor) - 2|X_1'| & \text{if } s > 0, a_{\lceil (k+s)/2 \rceil - 1} \notin X_1', \\ |X_0|(\lambda k - 2\lceil t/2 \rceil) + |X_1|(\lambda k - 2\lfloor t/2 \rfloor) - 2|X_1'| + 2 - \rho & \text{if } s > 0, a_{\lceil (k+s)/2 \rceil - 1} \in X_1'. \end{cases} \end{aligned}$$

By (4)−(6) and $|X_0|+|X_1|+|X_2|=|X|$, we have

$$\begin{aligned} g(S) &= \lambda(|X||Y_1|+|X_2||Y_0|)+\sum_{u\in X_0\cup X_1, v\in Y_0}\mu(uv) \\ &\quad +\lambda(|X|(r-|Y_1|)+|X_2|(k-|Y_0|)+|Y_1|(k+r-|X|)) \\ &\quad +\sum_{u\in X_0\cup X_1, v\in B_0-Y_0}\mu(uv)-k\lambda(|X|+|Y_1|) \\ &= \lambda(r|X|+|Y_1|(r-|X|))+m, \end{aligned}$$

where

$$m=\begin{cases} -2(|X_0|\lceil t/2\rceil+|X_1|\lfloor t/2\rfloor) & \text{if } s=0, \\ -2(|X_0|\lceil t/2\rceil+|X_1|\lfloor t/2\rfloor)-2|X_1'| & \text{if } s>0, a_{\lceil (k+s)/2\rceil-1}\notin X_1', \\ -2(|X_0|\lceil t/2\rceil+|X_1|\lfloor t/2\rfloor)-2|X_1'|+2-\rho & \text{if } s>0, a_{\lceil (k+s)/2\rceil-1}\in X_1'. \end{cases}$$

If $a_{\lceil (k+s)/2\rceil-1}\notin X_1'$, then $|X_1'|\le|A_1'|-1=\lceil s/2\rceil-1$. Hence $-2|X_1'|\ge-2(\lceil s/2\rceil-1)\ge-s$. If $a_{\lceil (k+s)/2\rceil-1}\in X_1$, then $|X_1'|\le|A_1'|=\lceil s/2\rceil$. In addition, $\rho=1$ for odd s and $\rho=2$ for even s. Therefore, $-2|X_1'|+2-\rho\ge-2\lceil s/2\rceil+2-\rho=-s$. Together with the fact $\max\{|X_0|,|X_1|\}\le k/2$, we have

$$m\ge-2(k/2\lceil t/2\rceil+k/2\lfloor t/2\rfloor)-s=-(kt+s)=-\lambda r^2.$$

Thus for $|X|\ge r$, we have

$$g(S)\ge\lambda(r|X|-|Y_1|(|X|-r))-\lambda r^2=\lambda(|X|-r)(r-|Y_1|)\ge0.$$

So it remains to consider the case $|X|<r$. Recall that $t=k-2$ and $s=1$ for $(\lambda,r)=(1,k-1)$. Thus $\lceil t/2\rceil=\lfloor t/2\rfloor=(\lambda r-1)/2$. In addition, $|X_1'|=0$ for $a_{\lceil (k+s)/2\rceil-1}\notin X_1'$, and $\rho=1$ as well as $|X_1'|=1$ (which implies $|X_1|\ge1$) for $a_{\lceil (k+s)/2\rceil-1}\in X_1'$. Hence for $a_{\lceil (k+s)/2\rceil-1}\notin X_1'$,

$$m=-2(|X_0|+|X_1|)(\lambda r-1)/2\ge-\lambda r(|X_0|+|X_1|),$$

and for $a_{\lceil (k+s)/2\rceil-1}\in X_1'$,

$$\begin{aligned} m &= -2(|X_0|+|X_1|)(\lambda r-1)/2-1 \\ &= -\lambda r(|X_0|+|X_1|)+|X_0|+|X_1|-1 \\ &\ge -\lambda r(|X_0|+|X_1|). \end{aligned}$$

On the other hand, for $\lambda\ge2$ or $r\le k-2$, we have $\lfloor t/2\rfloor+1\le\lambda r/2$ by Lemma 3.9, this implies

$$\begin{aligned} m &\ge -2(|X_0|\lceil t/2\rceil+|X_1'|(\lfloor t/2\rfloor+1)+(|X_1|-|X_1'|)\lfloor t/2\rfloor) \\ &\ge -2(|X_0|+|X_1|)(\lambda r/2) \\ &= -\lambda r(|X_0|+|X_1|). \end{aligned}$$

Therefore, for $|X|<r$, we have

$$g(S)\ge\lambda(r|X|+|Y_1|(r-|X|))-\lambda r(|X_0|+|X_1|)=\lambda(r|X_2|+|Y_1|(r-|X|))\ge0.$$

This settles (iii).

Let $\mathscr{P} = \mathscr{D} \cup_{i=0}^{\delta} \{Q_{i,0}, Q_{i,1}, \ldots, Q_{i,p_i-1}\}$. Clearly, $\mathscr{P}$ is the required packing. Let

$$\mathscr{C} = \begin{cases} \mathscr{P} & \text{if } s = 0, \\ \mathscr{P} \cup \{Q\} & \text{if } s \geq 1. \end{cases}$$

It is easy to check that $\mathscr{C}$ is the covering as required. □

Now, we are ready for the main result of this section.

Theorem 3.11 *If λ and n are positive integers and k is a positive even integer with $4 \leq k \leq n$, then $p(\lambda K_{n,n}; C_k, S_k) = \lfloor \lambda n^2/k \rfloor$ and $c(\lambda K_{n,n}; C_k, S_k) = \lceil \lambda n^2/k \rceil$.*

Proof: Obviously,

$$p(\lambda K_{n,n}; C_k, S_k) \leq \left\lfloor \frac{\lambda n^2}{k} \right\rfloor \leq \left\lceil \frac{\lambda n^2}{k} \right\rceil \leq c(\lambda K_{n,n}; C_k, S_k),$$

Let $n = qk + r$ where q and r are integers with $0 \leq r < k$. For $q = 1$, the result follows from Lemmas 3.5, 3.6, and 3.10. If $q \geq 2$,then $\lambda K_{n,n} = \lambda K_{k+r,k+r} \cup \lambda K_{k+r,(q-1)k} \cup \lambda K_{(q-1)k,n}$. Note that $\lambda K_{k+r,k+r}$ has a (C_k, S_k)-packing $\mathscr{P}$ with $|\mathscr{P}| = \lfloor \lambda(k+r)^2/k \rfloor$ and a (C_k, S_k)-covering $\mathscr{C}$ with $|\mathscr{C}| = \lceil \lambda(k+r)^2/k \rceil$. Trivially, $\lambda K_{k+r,(q-1)k}$ and $\lambda K_{(q-1)k,n}$ have S_k-decompositions $\mathscr{D}$ and $\mathscr{D}'$ with $|\mathscr{D}| = \lambda(k+r)(q-1)$ and $|\mathscr{D}'| = \lambda(q-1)n$, respectively. Since $\lambda(k+r)^2/k + \lambda(k+r)(q-1) + \lambda(q-1)n = \lambda(qk+r)^2/k = \lambda n^2/k$, $\mathscr{P} \cup \mathscr{D} \cup \mathscr{D}'$ is a (C_k, S_k)-packing of $\lambda K_{n,n}$ with cardinality $\lfloor \lambda n^2/k \rfloor$ and $\mathscr{C} \cup \mathscr{D} \cup \mathscr{D}'$ is a (C_k, S_k)-covering of $\lambda K_{n,n}$ with cardinality $\lceil \lambda n^2/k \rceil$. This completes the proof. □

Clearly, if $\lambda K_{n,n}$ admits a (C_k, S_k)-decomposition, then $4 \leq k \leq n$ and k is even and λn^2 is divisible by k. When k divides λn^2, a (C_k, S_k)-packing $\mathscr{P}$ with $|\mathscr{P}| = \lfloor \lambda n^2/k \rfloor$ is a (C_k, S_k)-decomposition. Therefore, with the aid of Theorem 3.11, we have the following.

Corollary 3.12 *For positive integers λ, k and n, the balanced complete bipartite multigraph $\lambda K_{n,n}$ is (C_k, S_k)-decomposable if and only if $4 \leq k \leq n$, k is even, and λn^2 is divisible by k.*

4 Packing and covering with 4-cycles and 4-stars

In this section a complete solution to the maximum packing and minimum covering problem of $\lambda K_{n,n}$ with C_4 and S_4 is given. Before that, we need more notations. For multigraphs G and H, $G \uplus H$ denotes the disjoint union of G and H, $G \odot H$ denotes the union of G and H with a common vertex. For a set $\mathscr{R}$ and a positive integer t, $t\mathscr{R}$ denotes the multiset in which each element in $\mathscr{R}$ appears t times. In addition, M_t denotes the graph induced by t nonadjacent edges. We begin with the discussion for the possible minimum leaves and paddings of $\lambda K_{n,n}$ with C_4 and S_4.

Note that $|E(\lambda K_{n,n})| = \lambda n^2$. If $\lambda \equiv 0 \pmod 4$ or $n \equiv 0 \pmod 2$, then $|E(\lambda K_{n,n})| \equiv 0 \pmod 4$. By Corollary 3.12, both of the possible minimum leave and the possible minimum padding are the empty graph. If $\lambda \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 2$, then $|E(\lambda K_{n,n})| \equiv 1 \pmod 4$. This implies that the possible minimum leave is only P_2, and the possible minimum paddings are S_3, P_4, $P_3 \uplus P_2$, M_3, $2P_2 \uplus P_2$, $2P_2 \odot P_2$, and $3P_2$. If $\lambda \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 2$, then $|E(\lambda K_{n,n})| \equiv 2 \pmod 4$. This implies that the possible minimum leaves are P_3, M_2, and $2P_2$, so are the possible minimum paddings. If $\lambda \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 2$, then $|E(\lambda K_{n,n})| \equiv 3 \pmod 4$. This implies that the possible minimum leaves are S_3, P_4, $P_3 \uplus P_2$, M_3, $2P_2 \uplus P_2$, $2P_2 \odot P_2$, and $3P_2$, and the possible minimum padding is only P_2.

Lemma 4.1 *$K_{5,5}$ has no (C_4, S_4)-covering with padding $3P_2$.*

Proof: It suffices to show that $K_{5,5} + 3\{a_0b_0\}$ is not (C_4, S_4)-decomposable. Suppose, to the contrary of the conclusion, that there exists a (C_4, S_4)-decomposition $\mathscr{D}$ of $K_{5,5} + 3\{a_0b_0\}$. Since there are at most two star with center a_0 (or b_0) and each edge joining a_0 and b_0 lies in exactly one subgraph in $\mathscr{D}$, there are exactly three possibilities for the edges joining a_0 and b_0 to lie in the decomposition: in four 4-cycles, in three 4-cycles and a 4-star, or in two 4-cycles and two 4-stars. Let G_1 be the graph obtained from $K_{5,5} + 3\{a_0b_0\}$ by deleting the edges of four 4-cycles, and let G_2 be the graph obtained from $K_{5,5} + 3\{a_0b_0\}$ by deleting the edges of three 4-cycles or deleting the edges of two 4-cycles. Note that $\deg_{G_1} x = 3$ for $x \notin \{a_0, b_0\}$, which implies that there is no 4-star in G_1. Since $\deg_{G_2} x \leq 3$ for $x \in \{a_0, b_0\}$, there is no 4-star with center at a_0 or b_0 in G_2. This leads to a contradiction and completes the proof. $\square$

We summarize the results discussed above in Table 1.

Tab. 1: The possible minimum leaves and paddings of $\lambda K_{n,n}$ with C_4 and S_4

$\lambda \pmod 4$ $n \pmod 2$	$\lambda \equiv 0$ or $n \equiv 0$	$\lambda \equiv 1$ and $n \equiv 1$	$\lambda \equiv 2$ and $n \equiv 1$	$\lambda \equiv 3$ and $n \equiv 1$
Leave	$\emptyset$	P_2	$P_3, M_2, 2P_2$	$S_3, P_4, P_3 \uplus P_2,$ $M_3, 2P_2 \uplus P_2,$ $2P_2 \odot P_2, 3P_2$
Padding	$\emptyset$	$S_3, P_4, P_3 \uplus P_2,$ $M_3, 2P_2 \uplus P_2,$ $2P_2 \odot P_2, 3P_2$ ($3P_2$ for $\lambda \neq 1$)	$P_3, M_2, 2P_2$	P_2

Lemma 4.2 *Let $r \in \{1, 2, 3, 5\}$.*
(a) *There exists a (C_4, S_4)-packing of $rK_{5,5}$ with leave L where*

$$\begin{cases} L = P_2 & \text{if } r = 1 \text{ or } r = 5, \\ L \in \{P_3, M_2, 2P_2\} & \text{if } r = 2, \\ L \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2, 3P_2\} & \text{if } r = 3. \end{cases}$$

(b) *There exists a (C_4, S_4)-covering of $rK_{5,5}$ with padding R where*

$$\begin{cases} R \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2\} & \text{if } r = 1, \\ R \in \{P_3, M_2, 2P_2\} & \text{if } r = 2, \\ R = P_2 & \text{if } r = 3, \\ R \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2, 3P_2\} & \text{if } r = 5. \end{cases}$$

Proof: The proof is divided into four parts according to the value of r.
Case 1. $r = 1$.

Let $A_1 = \{a_1, a_2, a_3, a_4\}$ and $B_1 = \{b_1, b_2, b_3, b_4\}$, and let $H = K_{5,5}[A_1 \cup B_1]$. Trivially, H is isomorphic to $K_{4,4}$. By Corollary 3.12, there exists a (C_4, S_4)-decomposition $\mathscr{D}$ of $K_{4,4}$. Let $\mathscr{P}$ =

$\mathscr{D} \cup \{(a_0; b_1, b_2, b_3, b_4), (b_0; a_1, a_2, a_3, a_4)\}$. Clearly, $\mathscr{P}$ is a (C_4, S_4)-packing of $K_{5,5}$ with leave P_2 : $\{a_0b_0\}$.

Now we give the required coverings of $K_{5,5}$. Note that $\mathscr{P} \cup \{(a_0; b_0, b_1, b_2, b_3)\}$ is a (C_4, S_4)-covering of $K_{5,5}$ with padding $S_3 : \{(a_0; b_1, b_2, b_3)\}$, and $\mathscr{P} \cup \{(a_0, b_1, a_1, b_0)\}$ is a (C_4, S_4)-covering of $K_{5,5}$ with padding $P_4 : \{a_0b_1a_1b_0\}$. Without loss of generality, we assume that $\mathscr{P}$ contains a 4-star $(a_4; b_1, b_2, b_3, b_4)$. Thus $\mathscr{P} - \{(a_0; b_1, b_2, b_3, b_4), (a_4; b_1, b_2, b_3, b_4)\} \cup \{(a_0, b_3, a_4, b_4), (a_0; b_0, b_1, b_2, b_3), (a_4; b_0, b_1, b_2, b_4)\}$ is a (C_4, S_4)-covering of $K_{5,5}$ with padding $P_3 \uplus P_2 : \{b_0a_4b_4, a_0b_3\}$. In addition, $\{(a_3, b_3, a_4, b_4), (a_0; b_0, b_2, b_3, b_4), (a_1; b_0, b_1, b_3, b_4), (a_2; b_1, b_2, b_3, b_4), (b_0; a_0, a_2, a_3, a_4), (b_1; a_0, a_1, a_3, a_4), (b_2; a_1, a_2, a_3, a_4)\}$ is a (C_4, S_4)-covering of $K_{5,5}$ with padding $M_3 : \{a_0b_0, a_1b_1, a_2b_2\}$, $\mathscr{P} - \{(a_4; b_1, b_2, b_3, b_4)\} \cup \{(a_0, b_0, a_4, b_4), (a_4; b_0, b_1, b_2, b_3)\}$ is a (C_4, S_4)-covering of $K_{5,5}$ with padding $2P_2 \uplus P_2 : 2\{b_0a_4\} \cup \{a_0b_4\}$, and $\mathscr{P} - \{(a_0; b_1, b_2, b_3, b_4), (a_4; b_1, b_2, b_3, b_4)\} \cup \{(a_0, b_0, a_4, b_4), (a_0; b_0, b_1, b_2, b_3), (a_4; b_0, b_1, b_2, b_3)\}$ is a (C_4, S_4)-covering of $K_{5,5}$ with padding $2P_2 \odot P_2 : 2\{b_0a_4\} \cup \{b_0a_0\}$.

Case 2. $r = 2$.

First, we use $\mathscr{P}$ to construct the required packings of $2K_{5,5}$. Exchanging b_0 with b_1 in $\mathscr{P}$, we obtain a packing $\mathscr{P}'$ of $K_{5,5}$ with leave a_0b_1. Let $\mathscr{P}_1 = \mathscr{P} \cup \mathscr{P}'$. One can see that $\mathscr{P}_1$ is a packing of $2K_{5,5}$ with leave $P_3 : \{b_0a_0b_1\}$. Next, rename the vertices a_0, a_1, b_0, b_1 in $\mathscr{P}$ to a_1, a_0, b_1, b_0, respectively, we obtain a packing $\mathscr{P}''$ of $K_{5,5}$ with leave a_1b_1. Let $\mathscr{P}_2 = \mathscr{P} \cup \mathscr{P}''$. It is easy to see that $\mathscr{P}_2$ is a packing of $2K_{5,5}$ with leave $M_2 : \{a_0b_0, a_1b_1\}$. Finally, $2\mathscr{P}$ is clearly a packing of $2K_{5,5}$ with leave $2P_2 : 2\{a_0b_0\}$.

Now we use packings to construct the required coverings of $2K_{5,5}$. Note that $\mathscr{P}_1 \cup \{(a_0; b_0, b_1, b_2, b_3)\}$ is a (C_4, S_4)-covering of $2K_{5,5}$ with padding $P_3 : \{b_2a_0b_3\}$, and $\mathscr{P}_2 \cup \{(a_0, b_0, a_1, b_1)\}$ is a (C_4, S_4)-covering of $2K_{5,5}$ with padding $M_2 : \{a_0b_1, a_1b_0\}$. Moreover, $2\mathscr{P} - \{(a_0; b_1, b_2, b_3, b_4), (a_4; b_1, b_2, b_3, b_4)\} \cup \{(a_0, b_0, a_4, b_4), (a_0; b_0, b_1, b_2, b_3), (a_4; b_0, b_1, b_2, b_3)\}$ is a (C_4, S_4)-covering of $2K_{5,5}$ with padding $2P_2 : 2\{b_0a_4\}$.

Case 3. $r = 3$.

First, we use packings of $K_{5,5}$ and $2K_{5,5}$ to construct the required packings of $3K_{5,5}$. Exchanging b_0 with b_2 in $\mathscr{P}$, we obtain a packing $\mathscr{R}$ of $K_{5,5}$ with leave a_0b_2. Hence $\mathscr{P}_1 \cup \mathscr{R}$ is a packing of $3K_{5,5}$ with leave $S_3 : \{(a_0; b_0, b_1, b_2)\}$. Next, rename the vertices a_0, a_2, b_0, b_2 in $\mathscr{P}$ to a_2, a_0, b_2, b_0, respectively, we obtain a packing $\mathscr{R}'$ of $K_{5,5}$ with leave a_2b_2. Thus $\mathscr{P}_2 \cup \mathscr{R}'$ is a packing of $3K_{5,5}$ with leave $M_3 : \{a_0b_0, a_1b_1, a_2b_2\}$. Note that $\mathscr{P}_1 \cup \mathscr{P}''$ is a packing of $3K_{5,5}$ with leave $P_4 : \{b_0a_0b_1a_1)\}$. In addition, $\mathscr{P}_1 \cup \mathscr{R}'$ is a packing of $3K_{5,5}$ with leave $P_3 \uplus P_2 : \{b_0a_0b_1\} \cup \{a_2b_2\}$, $2\mathscr{P} \cup \mathscr{R}'$ is a packing of $3K_{5,5}$ with leave $2P_2 \uplus P_2 : 2\{a_0b_0\} \cup \{a_2b_2\}$, $2\mathscr{P} \cup \mathscr{R}$ is a packing of $3K_{5,5}$ with leave $2P_2 \odot P_2 : 2\{a_0b_0\} \cup \{a_0b_2\}$, and $3\mathscr{P}$ is clearly a packing of $3K_{5,5}$ with leave $3P_2 : 3\{a_0b_0\}$.

Finally, since $3(5-4)^2 = 3 < 4$, there exists a (C_4, S_4)-covering of $3K_{5,5}$ with leave P_2 by Lemma 3.6.

Case 4. $r = 5$.

By Corollary 3.12, $(C_4, S_4) \mid 4K_{5,5}$. Since $5K_{5,5} = K_{5,5} \cup 4K_{5,5}$, it suffices to show that there exists a (C_4, S_4)-covering of $5K_{5,5}$ with padding $3P_2$. Note that $5K_{5,5} = 2K_{5,5} \cup 3K_{5,5}$. Since $2K_{5,5}$ has a (C_4, S_4)-covering with padding $2P_2 : 2\{b_0a_4\}$ and $3K_{5,5}$ has a (C_4, S_4)-covering with padding P_2 (say $\{b_0a_4\}$), we have the required covering. □

Lemma 4.3 *Let r be a positive integer and let m be a positive odd integer with $m \geq 5$. If $rK_{m,m}$ has a (C_4, S_4)-packing (resp. (C_4, S_4)-covering) with leave L (resp. padding R), then $rK_{m+2,m+2}$ also has a (C_4, S_4)-packing (resp. (C_4, S_4)-covering) with leave L (resp. padding R).*

Proof: Let $m = 2t + 1$ where t is a positive integer with $t \geq 2$. Let $A_1 = \{a_0, a_1, \ldots, a_{2t}\}$ and

$B_1 = \{b_0, b_1, \ldots, b_{2t}\}$. Letting $G_1 = K_{m+2,m+2}[A_1 \cup B_1]$ and $G_2 = K_{m+2,m+2} - E(G_1)$. Clearly, G_1 is isomorphic to $K_{m,m}$. Note that $\{(a_{2t+1}, b_{2i}, a_{2t+2}, b_{2i+1}), (a_{2i}, b_{2t+1}, a_{2i+1}, b_{2t+2}) : i = 0, 1, \ldots, t-2\} \cup \{(a_{2t+1}; b_{2t-2}, b_{2t-1}, b_{2t}, b_{2t+1}), (a_{2t+2}; b_{2t-2}, b_{2t-1}, b_{2t}, b_{2t+2}), (b_{2t+1}; a_{2t-2}, a_{2t-1}, a_{2t}, a_{2t+2}), (b_{2t+2}; a_{2t-2}, a_{2t-1}, a_{2t}, a_{2t+1})\}$ is a (C_4, S_4)-decomposition of G_2. Since $rK_{m+2,m+2} = rG_1 \cup rG_2$, $rK_{m+2,m+2}$ has the required packings and coverings. □

Now, we are ready for the main result of this section.

Theorem 4.4 *Let λ and n be positive integers with $n \geq 4$.*
(A) *$\lambda K_{n,n}$ has a maximum (C_4, S_4)-packing with leave L if and only if*

$$\begin{cases} L = \emptyset & \text{if } \lambda n^2 \equiv 0 \pmod 4, \\ L = P_2 & \text{if } \lambda n^2 \equiv 1 \pmod 4, \\ L \in \{P_3, M_2, 2P_2\} & \text{if } \lambda n^2 \equiv 2 \pmod 4, \\ L \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2, 3P_2\} & \text{if } \lambda n^2 \equiv 3 \pmod 4. \end{cases}$$

(B) *$\lambda K_{n,n}$ has a minimum (C_4, S_4)-covering with padding R if and only if*

$$\begin{cases} R = \emptyset & \text{if } \lambda n^2 \equiv 0 \pmod 4, \\ R \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2\} & \text{if } \lambda n^2 \equiv 1 \pmod 4 \text{ and } \lambda = 1, \\ R \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2, 3P_2\} & \text{if } \lambda n^2 \equiv 1 \pmod 4 \text{ and } \lambda \geq 5, \\ R \in \{P_3, M_2, 2P_2\} & \text{if } \lambda n^2 \equiv 2 \pmod 4, \\ R = P_2 & \text{if } \lambda n^2 \equiv 3 \pmod 4. \end{cases}$$

Proof: The necessity follows from the arguments above Table 1. It suffices to show that $\lambda K_{n,n}$ has required packings and coverings. The result for $\lambda n^2 \equiv 0 \pmod 4$ follows from Corollary 3.12 immediately. So it remains to consider the case $\lambda n^2 \equiv r \pmod 4$ for $r \in \{1, 2, 3\}$. Note that $\lambda n^2 \equiv r \pmod 4$ if and only if $\lambda \equiv r \pmod 4$ and $n \equiv 1 \pmod 2$. When $\lambda \in \{1, 2, 3, 5\}$, the result for $n = 5$ follows from Lemma 4.2, and the result for $n > 5$ can be obtained by using Lemma 4.3 recursively. Now consider $\lambda \equiv r \pmod 4$ and $\lambda > 5$. Note that $\lambda K_{n,n} = rK_{n,n} \cup (\lambda - r)K_{n,n}$. Since $(\lambda - r)K_{n,n}$ is (C_4, S_4)-decomposable by Corollary 3.12, we have the result. □.

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