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# Packing and covering the balanced complete bipartite multigraph with cycles and stars

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Let  $C_k$  denote a cycle of length  $k$  and let  $S_k$  denote a star with  $k$  edges. For multigraphs  $F$ ,  $G$  and  $H$ , an  $(F, G)$ -decomposition of  $H$  is an edge decomposition of  $H$  into copies of  $F$  and  $G$  using at least one of each. For  $L \subseteq H$  and  $R \subseteq rH$ , an  $(F, G)$ -packing (resp.  $(F, G)$ -covering) of  $H$  with leave  $L$  (resp. padding  $R$ ) is an  $(F, G)$ -decomposition of  $H - E(L)$  (resp.  $H + E(R)$ ). An  $(F, G)$ -packing (resp.  $(F, G)$ -covering) of  $H$  with the largest (resp. smallest) cardinality is a maximum  $(F, G)$ -packing (resp. minimum  $(F, G)$ -covering), and its cardinality is referred to as the  $(F, G)$ -packing number (resp.  $(F, G)$ -covering number) of  $H$ . In this paper, we determine the packing number and the covering number of  $\lambda K_{n,n}$  with  $C_k$ 's and  $S_k$ 's for any  $\lambda$ ,  $n$  and  $k$ , and give the complete solution of the maximum packing and the minimum covering of  $\lambda K_{n,n}$  with 4-cycles and 4-stars for any  $\lambda$  and  $n$  with all possible leaves and paddings.

**Keywords:** complete bipartite multigraph, cycle, star, packing, covering

## 1 Introduction

For positive integers  $m$  and  $n$ ,  $K_{m,n}$  denotes the complete bipartite graph with parts of sizes  $m$  and  $n$ . If  $m = n$ , the complete bipartite graph is referred to as *balanced*. A  $k$ -cycle, denoted by  $C_k$ , is a cycle of length  $k$ . A  $k$ -star, denoted by  $S_k$ , is the complete bipartite graph  $K_{1,k}$ . A  $k$ -path, denoted by  $P_k$ , is a path with  $k$  vertices. For a graph  $H$  and a positive integer  $\lambda$ , we use  $\lambda H$  to denote the multigraph obtained from  $H$  by replacing each edge  $e$  by  $\lambda$  edges each having the same endpoints as  $e$ . When  $\lambda = 1$ ,  $1H$  is simply written as  $H$ .

Let  $F$ ,  $G$ , and  $H$  be multigraphs. A *decomposition* of  $H$  is a set of edge-disjoint subgraphs of  $H$  whose union is  $H$ . An  $(F, G)$ -*decomposition* of  $H$  is a decomposition of  $H$  into copies of  $F$  and  $G$  using at least one of each. If  $H$  has an  $(F, G)$ -decomposition, we say that  $H$  is  $(F, G)$ -*decomposable* and write  $(F, G)|H$ . If  $H$  does not admit an  $(F, G)$ -decomposition, two natural questions arise:

- (1) What is the minimum number of edges needed to be removed from the edge set of  $H$  so that the resulting graph is  $(F, G)$ -decomposable, and what does the collection of removed edges look like?

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- (2) What is the minimum number of edges needed to be added to the edge set of  $H$  so that the resulting graph is  $(F, G)$ -decomposable, and what does the collection of added edges look like?

These questions are respectively called the maximum packing problem and the minimum covering problem of  $H$  with  $F$  and  $G$ .

Let  $F, G$ , and  $H$  be multigraphs. For  $L \subseteq H$  and  $R \subseteq rH$ , an  $(F, G)$ -packing of  $H$  with leave  $L$  is an  $(F, G)$ -decomposition of  $H - E(L)$ , and an  $(F, G)$ -covering with padding  $R$  is an  $(F, G)$ -decomposition of  $H + E(R)$ . For an  $(F, G)$ -packing  $\mathcal{P}$  of  $H$  with leave  $L$ , if  $|\mathcal{P}|$  is as large as possible (so that  $|L|$  is as small as possible), then  $\mathcal{P}$  and  $L$  are referred to as a *maximum  $(F, G)$ -packing* and a *minimum leave*, respectively. Moreover, the cardinality of the maximum  $(F, G)$ -packing of  $H$  is called the  *$(F, G)$ -packing number* of  $H$ , denoted by  $p(H; F, G)$ . For an  $(F, G)$ -covering  $\mathcal{C}$  of  $H$  with padding  $R$ , if  $|\mathcal{C}|$  is as small as possible (so that  $|R|$  is as small as possible), then  $\mathcal{C}$  and  $R$  are referred to as a *minimum covering* and a *minimum padding*, respectively. Moreover, the cardinality of the minimum  $(F, G)$ -covering of  $H$  is called the  *$(F, G)$ -covering number* of  $H$ , denoted by  $c(H; F, G)$ . Clearly, an  $(F, G)$ -decomposition of  $H$  is a maximum  $(F, G)$ -packing with leave the empty graph, and also a minimum  $(F, G)$ -covering with padding the empty graph.

Recently, decomposition into a pair of graphs has attracted a fair share of interest. Abueida and Daven [3] investigated the problem of  $(K_k, S_k)$ -decomposition of the complete graph  $K_n$ . Abueida and Daven [4] investigated the problem of the  $(C_4, E_2)$ -decomposition of several graph products where  $E_2$  denotes two vertex disjoint edges. Abueida and O'Neil [7] settled the existence problem for  $(C_k, S_{k-1})$ -decomposition of the complete multigraph  $\lambda K_n$  for  $k \in \{3, 4, 5\}$ . Priyadharsini and Muthusamy [12, 13] gave necessary and sufficient conditions for the existence of  $(G_n, H_n)$ -decompositions of  $\lambda K_n$  and  $\lambda K_{n,n}$  where  $G_n, H_n \in \{C_n, P_n, S_{n-1}\}$ . A *graph-pair*  $(G, H)$  of order  $m$  is a pair of non-isomorphic graphs  $G$  and  $H$  on  $m$  non-isolated vertices such that  $G \cup H$  is isomorphic to  $K_m$ . Abueida and Daven [2] and Abueida, Daven and Roblee [5] completely determined the values of  $n$  for which  $\lambda K_n$  admits a  $(G, H)$ -decomposition where  $(G, H)$  is a graph-pair of order 4 or 5. Abueida, Clark and Leach [1] and Abueida and Hampson [6] considered the existence of decompositions of  $K_n - F$  for the graph-pair of order 4 and 5, respectively, where  $F$  is a Hamiltonian cycle, a 1-factor, or almost 1-factor. Furthermore, Shyu [14] investigated the problem of decomposing  $K_n$  into paths and stars with  $k$  edges, giving a necessary and sufficient condition for  $k = 3$ . In [15, 16], Shyu considered the existence of a decomposition of  $K_n$  into paths and cycles with  $k$  edges, giving a necessary and sufficient condition for  $k \in \{3, 4\}$ . Shyu [17] investigated the problem of decomposing  $K_n$  into cycles and stars with  $k$  edges, settling the case  $k = 4$ . In [18], Shyu considered the existence of a decomposition of  $K_{m,n}$  into paths and stars with  $k$  edges, giving a necessary and sufficient condition for  $k = 3$ . Recently, Lee [9] and Lee and Lin [10] established necessary and sufficient conditions for the existence of  $(C_k, S_k)$ -decompositions of the complete bipartite graph and the complete bipartite graph with a 1-factor removed, respectively. However, much less work has been done on the problem of packing and covering graphs with a pair of graphs. Abueida and Daven [3] obtained the maximum packing and the minimum covering of the complete graph  $K_n$  with  $(K_k, S_k)$ . Abueida and Daven [2] and Abueida, Daven and Roblee [5] gave the maximum packing and the minimum covering of  $K_n$  and  $\lambda K_n$  with  $G$  and  $H$ , respectively, where  $(G, H)$  is a graph-pair of order 4 or 5. In this paper, we determine the packing number and the covering number of  $\lambda K_{n,n}$  with  $k$ -cycles and  $k$ -stars for any  $\lambda, n$  and  $k$ , and give the complete solution of the maximum packing and the minimum covering of  $\lambda K_{n,n}$  with 4-cycles and 4-stars for any  $\lambda$  and  $n$  with all possible leaves and paddings.

## 2 Preliminaries

In this section we first collect some needed terminology and notation, and then present a result which is useful for our discussions to follow.

Let  $G$  be a multigraph. The *degree* of a vertex  $x$  of  $G$ , denoted by  $\deg_G x$ , is the number of edges incident with  $x$ . The vertex of degree  $k$  in  $S_k$  is the *center* of  $S_k$  and any vertex of degree 1 is an *endvertex* of  $S_k$ . For  $W \subseteq V(G)$ , we use  $G[W]$  to denote the subgraph of  $G$  induced by  $W$ . Furthermore,  $\mu(uv)$  denotes the number of edges of  $G$  joining  $u$  and  $v$ ,  $(v_1, \dots, v_k)$  and  $v_1 \dots v_k$  denote the  $k$ -cycle and the  $k$ -path through vertices  $v_1, \dots, v_k$  in order, respectively, and  $(x; y_1, \dots, y_k)$  denotes the  $k$ -star with center  $x$  and endvertices  $y_1, \dots, y_k$ . When  $G_1, G_2, \dots, G_t$  are multigraphs, not necessarily disjoint, we write  $G_1 \cup G_2 \cup \dots \cup G_t$  or  $\bigcup_{i=1}^t G_i$  for the graph with vertex set  $\bigcup_{i=1}^t V(G_i)$  and edge set  $\bigcup_{i=1}^t E(G_i)$ . When the edge sets are disjoint,  $G = \bigcup_{i=1}^t G_i$  expresses the decomposition of  $G$  into  $G_1, G_2, \dots, G_t$ . Given an  $S_k$ -decomposition of  $G$ , a *central function*  $c$  from  $V(G)$  to the set of non-negative integers is defined as follows. For each  $v \in V(G)$ ,  $c(v)$  is the number of  $k$ -stars in the decomposition whose center is  $v$ .

The following result is essential to our proof.

**Proposition 2.1** (Hoffman [8]) *For a positive integer  $k$ , a multigraph  $H$  has an  $S_k$ -decomposition with central function  $c$  if and only if*

- (i)  $k \sum_{v \in V(H)} c(v) = |E(H)|$ ,
- (ii) for all  $x, y \in V(H)$ ,  $\mu(xy) \leq c(x) + c(y)$ ,
- (iii) for all  $S \subseteq V(H)$ ,  $k \sum_{v \in S} c(v) \leq \varepsilon(S) + \sum_{x \in S, y \in V(H) - S} \min\{c(x), \mu(xy)\}$ .

where  $\varepsilon(S)$  denotes the number of edges of  $H$  with both ends in  $S$ .

In the sequel of the paper,  $(A, B)$  denotes the bipartition of  $\lambda K_{n,n}$ , where  $A = \{a_0, a_1, \dots, a_{n-1}\}$  and  $B = \{b_0, b_1, \dots, b_{n-1}\}$ .

## 3 Packing numbers and covering numbers

In this section the packing number and the covering number of the balanced complete bipartite multigraph with  $k$ -cycles and  $k$ -stars are determined. We begin with a criterion for decomposing the complete bipartite graph into  $k$ -cycles.

**Proposition 3.1** (Sotteau [19]) *For positive integers  $m, n$ , and  $k$ , the graph  $K_{m,n}$  is  $C_k$ -decomposable if and only if  $m, n$ , and  $k$  are even,  $k \geq 4$ ,  $\min\{m, n\} \geq k/2$ , and  $k$  divides  $mn$ .*

Let  $K_{m,n}^*$  denote the symmetric complete bipartite digraph with parts of size  $m$  and  $n$ , and let  $\overrightarrow{C}_k$  denote the directed  $k$ -cycle.

**Proposition 3.2** (Sotteau [19]) *For positive integers  $m, n$ , and  $k$ , the digraph  $K_{m,n}^*$  is  $\overrightarrow{C}_k$ -decomposable if and only if  $k$  is even,  $k \geq 4$ ,  $\min\{m, n\} \geq k/2$ , and  $k$  divides  $2mn$ .*

Removing the directions from the arcs of directed cycles in a  $\overrightarrow{C}_k$ -decomposition of  $K_{m,n}^*$ , we obtain the following result by Proposition 3.2.

**Lemma 3.3** For positive integers  $m, n$ , and  $k$ , the multigraph  $2K_{m,n}$  is  $C_k$ -decomposable if  $k$  is even,  $k \geq 4$ ,  $\min\{m, n\} \geq k/2$ , and  $k$  divides  $2mn$ .

**Lemma 3.4** Let  $\lambda, k, m$ , and  $n$  be positive integers with  $\lambda m \equiv \lambda n \equiv k \equiv 0 \pmod{2}$  and  $\min\{m, n\} \geq k/2 \geq 2$ . If  $m$  or  $n$  is divisible by  $k$ , then  $\lambda K_{m,n}$  is  $C_k$ -decomposable.

**Proof:** Since  $\lambda K_{m,n}$  is isomorphic to  $\lambda K_{n,m}$ , it suffices to show that the result holds for  $k \mid m$ . If  $\lambda$  is odd, then  $m$  and  $n$  are even from the assumption  $\lambda m \equiv \lambda n \equiv 0 \pmod{2}$ . Since  $k$  divides  $mn$ , Proposition 3.1 implies that  $K_{m,n}$  is  $C_k$ -decomposable. If  $\lambda$  is even, then  $2K_{m,n} \mid \lambda K_{m,n}$ . Since  $k$  divides  $2mn$ ,  $2K_{m,n}$  is  $C_k$ -decomposable by Lemma 3.3. Hence  $\lambda K_{m,n}$  is  $C_k$ -decomposable.  $\square$

**Lemma 3.5** If  $k$  is a positive even integer with  $k \geq 4$ , then  $\lambda K_{k,k}$  is  $(C_k, S_k)$ -decomposable.

**Proof:** Note that  $\lambda K_{k,k} = \lambda K_{k,k-2} \cup \lambda K_{k,2}$ . By Lemma 3.4,  $\lambda K_{k,k-2}$  is  $C_k$ -decomposable. Trivially,  $\lambda K_{k,2}$  is  $C_k$ -decomposable. Therefore,  $\lambda K_{k,k}$  is  $(C_k, S_k)$ -decomposable.  $\square$

**Lemma 3.6** Let  $k$  be a positive even integer and let  $n$  be a positive integer with  $4 \leq k < n < 2k$ . If  $\lambda(n-k)^2 < k$ , then  $\lambda K_{n,n}$  has a  $(C_k, S_k)$ -packing with leave  $\lambda K_{n-k,n-k}$  and a  $(C_k, S_k)$ -covering with padding  $P_{k-\lambda(n-k)^2+1}$ .

**Proof:** Let  $n = k + r$ . The assumption  $k < n < 2k$  implies  $0 < r < k$ . We first give the required packing. Note that

$$\lambda K_{n,n} = \lambda K_{k,k} \cup \lambda K_{k,r} \cup \lambda K_{r,k} \cup \lambda K_{r,r}.$$

By Lemma 3.4,  $\lambda K_{k,k}$  has a  $C_k$ -decomposition  $\mathcal{D}_1$ . Trivially,  $\lambda K_{k,r}$  and  $\lambda K_{r,k}$  have  $S_k$ -decompositions  $\mathcal{D}_2$  and  $\mathcal{D}_3$ , respectively. Thus  $\bigcup_{i=1}^3 \mathcal{D}_i$  is a  $(C_k, S_k)$ -packing of  $\lambda K_{n,n}$  with leave  $\lambda K_{r,r}$ , as desired.

Now we give the required covering. Let  $s = \lambda r^2$ . Let  $A_0 = \{a_0, a_1, \dots, a_{\lfloor (s-1)/2 \rfloor}\}$ ,  $A_1 = A - A_0$ ,  $B_0 = \{b_0, b_1, \dots, b_{k-1}\}$  and  $B_1 = B - B_0$ . Define a  $k$ -cycle  $C$  and a  $(k-s+1)$ -path  $P$  as follows:

$$C = (b_0, a_0, b_1, a_1, \dots, b_{k/2-1}, a_{k/2-1})$$

$$P = \begin{cases} b_0 a_{k/2-1} b_{k/2-1} a_{k/2-2} \dots b_{(s+1)/2} a_{(s-1)/2} & \text{if } s \text{ is odd,} \\ b_0 a_{k/2-1} b_{k/2-1} a_{k/2-2} \dots a_s/2 b_{s/2} & \text{if } s \text{ is even.} \end{cases}$$

Let

$$H = \lambda K_{n,n} - E(C) + E(P).$$

Note that  $V(H) = V(\lambda K_{n,n})$ ,  $|E(H)| = \lambda n^2 - k + (k-s) = \lambda n^2 - \lambda r^2 = \lambda k(k+2r)$ , and  $\mu(uv) \leq \lambda$  for all  $u, v \in V(H)$ . Furthermore, for  $H' = H[A \cup B_0]$ , we have

$$\deg_{H'} v = \begin{cases} \lambda k - 2 & \text{if } v \in A_0 - \{a_{\lfloor (s-1)/2 \rfloor}\}, \\ \lambda k - \rho & \text{if } v = a_{\lfloor (s-1)/2 \rfloor}, \\ \lambda k & \text{if } v \in A_1, \end{cases}$$

where  $\rho = 1$  if  $s$  is odd, and  $\rho = 2$  if  $s$  is even. Define a function  $c : V(H) \rightarrow \mathbb{N}$  as follows:

$$c(v) = \begin{cases} 0 & \text{if } v \in B_0, \\ \lambda & \text{otherwise.} \end{cases}$$

Now we show that there exists an  $S_k$ -decomposition of  $H$  with central function  $c$  by Proposition 2.1.

First,  $k \sum_{v \in V(H)} c(v) = k\lambda(k+2r) = |E(H)|$ . This proves (i). Next, if  $u, v \in B_0$ , then  $c(u) + c(v) = 0 = \mu(uv)$ ; otherwise,  $c(u) + c(v) \geq \lambda \geq \mu(uv)$ . This proves (ii). Finally, for  $S \subseteq V(H)$  and  $i \in \{0, 1\}$ , let  $S \cap A_i = X_i$  and  $S \cap B_i = Y_i$ . Moreover, let  $X = X_0 \cup X_1$  and  $Y = Y_0 \cup Y_1$ . Define a set  $T$  of ordered pairs of vertices as follows:

$$T = \{(u, v) | u \in X, v \in B_1 - Y_1 \text{ or } u \in X_1, v \in B_0 - Y_0 \text{ or } u \in Y_1, v \in A - X\}.$$

Note that

$$k \sum_{w \in S} c(w) = k\lambda(|X| + |Y_1|), \quad (1)$$

$$\varepsilon(S) = \lambda(|X||Y_1| + |X_1||Y_0|) + \sum_{u \in X_0, v \in Y_0} \mu(uv), \quad (2)$$

and for  $u \in S$  and  $v \in V(H) - S$

$$\min\{c(u), \mu(uv)\} = \begin{cases} \lambda & \text{if } (u, v) \in T, \\ \mu(uv) & \text{if } u \in X_0, v \in B_0 - Y_0, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

For  $S \subseteq V(H)$ , let

$$g(S) = \varepsilon(S) + \sum_{u \in S, v \in V(H) - S} \min\{c(u), \mu(uv)\} - k \sum_{w \in S} c(w).$$

Note that

$$\begin{aligned} & \sum_{u \in X_0, v \in Y_0} \mu(uv) + \sum_{u \in X_0, v \in B_0 - Y_0} \mu(uv) \\ &= \sum_{u \in X_0, v \in B_0} \mu(uv) \\ &= \begin{cases} |X_0|(\lambda k - 2) & \text{if } a_{\lfloor (s-1)/2 \rfloor} \notin X_0, \\ |X_0|(\lambda k - 2) + 2 - \rho & \text{if } a_{\lfloor (s-1)/2 \rfloor} \in X_0. \end{cases} \end{aligned}$$

By (1)–(3) and  $|X_0| + |X_1| = |X|$ , we have

$$\begin{aligned} g(S) &= \lambda(|X||Y_1| + |X_1||Y_0|) + \sum_{u \in X_0, v \in Y_0} \mu(uv) \\ &\quad + \lambda(|X|(r - |Y_1|) + |X_1|(k - |Y_0|) + |Y_1|(k + r - |X|)) \\ &\quad + \sum_{u \in X_0, v \in B_0 - Y_0} \mu(uv) - k\lambda(|X| + |Y_1|) \\ &= \begin{cases} \lambda(r|X| + |Y_1|(r - |X|)) - 2|X_0| & \text{if } a_{\lfloor (s-1)/2 \rfloor} \notin X_0, \\ \lambda(r|X| + |Y_1|(r - |X|)) - 2|X_0| + 2 - \rho & \text{if } a_{\lfloor (s-1)/2 \rfloor} \in X_0. \end{cases} \end{aligned}$$

If  $a_{\lfloor (s-1)/2 \rfloor} \notin X_0$ , then  $|X_0| \leq \lfloor (s-1)/2 \rfloor$ , which implies  $-2|X_0| \geq -s$ . If  $a_{\lfloor (s-1)/2 \rfloor} \in X_0$ , then  $|X_0| \leq \lfloor (s-1)/2 \rfloor + 1$ , which implies  $-2|X_0| + 2 - \rho \geq -2\lfloor (s-1)/2 \rfloor - \rho = -2(s-\rho)/2 - \rho = -s$ . Thus for  $|X| \geq r$ , we have

$$\begin{aligned} g(S) &\geq \lambda(r|X| - |Y_1|(|X| - r)) - s \\ &= \lambda(r|X| - |Y_1|(|X| - r)) - \lambda r^2 \\ &= \lambda(|X| - r)(r - |Y_1|) \\ &\geq 0. \end{aligned}$$

If  $\lambda r = 1$  and  $|X| < r$ , then  $|X_0| = |X| = 0$ , which implies  $-2|X_0| = -\lambda r|X_0|$ . If  $\lambda r \geq 2$ , then  $-2|X_0| \geq -\lambda r|X_0|$ . Note that  $2 - \rho \geq 0$ . Hence for  $|X| < r$ , we have

$$\begin{aligned} g(S) &\geq \lambda(r|X| + |Y_1|(r - |X|)) - \lambda r|X_0| \\ &= \lambda(r|X_1| + |Y_1|(r - |X|)) \\ &\geq 0. \end{aligned}$$

This settles (iii) and completes the proof.  $\square$

Before going on, the following results are needed.

**Proposition 3.7** (Ma et al. [11]) *For positive integers  $k$  and  $n$ , the graph obtained by deleting a 1-factor from  $K_{n,n}$  is  $C_k$ -decomposable if and only if  $n$  is odd,  $k$  is even,  $4 \leq k \leq 2n$ , and  $n(n-1)$  is divisible by  $k$ .*

**Lemma 3.8** *If  $\lambda$  and  $p$  are positive integers and  $k$  is a positive even integer with  $k \geq 4$ , then there exist  $\lambda pk/2 - p$  edge-disjoint  $k$ -cycles in  $\lambda K_{k/2, pk}$  (also in  $\lambda K_{pk, k/2}$ ).*

**Proof:** It suffices to show that the result holds for  $\lambda K_{k/2, pk}$ . If  $\lambda$  or  $k/2$  is even, then by Lemma 3.4 there exists a  $C_k$ -decomposition  $\mathcal{D}$  of  $\lambda K_{k/2, pk}$  with  $|\mathcal{D}| = \lambda pk/2$ , in which  $k$ -cycles are edge-disjoint. If  $k/2$  is odd, then by Proposition 3.7 there exists a  $C_k$ -decomposition  $\mathcal{D}'$  of  $K_{k/2, k/2} - I$  with  $|\mathcal{D}'| = (k-2)/4$ , where  $I$  is a 1-factor of  $K_{k/2, k/2}$ . Since  $K_{k/2, pk}$  can be decomposed into  $2p$  copies of  $K_{k/2, k/2}$ , there exist  $2p|\mathcal{D}'| = pk/2 - p$  edge-disjoint  $k$ -cycles in  $K_{k/2, pk}$ . For odd  $\lambda$  with  $\lambda \geq 3$ ,  $\lambda K_{k/2, k} = (\lambda-1)K_{k/2, k} \cup K_{k/2, k}$ . By Lemma 3.4 there exists a  $C_k$ -decomposition  $\mathcal{D}''$  of  $(\lambda-1)K_{k/2, pk}$  with  $|\mathcal{D}''| = (\lambda-1)pk/2$ . Hence there exist  $(\lambda-1)pk/2 + pk/2 - p = \lambda pk/2 - p$  edge-disjoint  $k$ -cycles in  $\lambda K_{k/2, pk}$ .  $\square$

**Lemma 3.9** *Let  $\lambda$  and  $r$  be positive integers and let  $k$  be a positive even integer with  $k \geq 4$  and  $r < k$ . If  $t = \lfloor \lambda r^2/k \rfloor$ , then there exist  $\lceil t/2 \rceil$  edge-disjoint  $k$ -cycles in  $\lambda K_{k/2, k}$ . Moreover, if  $\lambda \geq 2$  or  $r \leq k-2$  and  $\lambda r^2 \geq k$ , then  $\lfloor t/2 \rfloor + 1 \leq \lambda r/2$  and there exist  $\lfloor t/2 \rfloor + 1$  edge-disjoint  $k$ -cycles in  $\lambda K_{k/2, k}$ .*

**Proof:** Since  $r < k$ , we have  $t < \lambda r$ . Thus  $t+1 \leq \lambda r$ ; in turn,  $\lceil t/2 \rceil \leq (t+1)/2 \leq \lambda r/2 < \lambda k/2$ , which implies  $\lceil t/2 \rceil \leq \lambda k/2 - 1$ . By Lemma 3.8, there exist  $\lceil t/2 \rceil$  edge-disjoint  $k$ -cycles in  $\lambda K_{k/2, k}$ . When  $\lambda r^2 = k$ , the result is trivial. When  $\lambda r^2 > k$ , we have  $r > 2/\sqrt{\lambda}$  since  $k \geq 4$ . For  $\lambda \geq 2$ ,

$$\frac{\lambda r^2}{k} \leq \frac{\lambda r^2}{r+1} = \lambda r - \frac{\lambda}{1+1/r} < \lambda r - \frac{2\lambda}{2+\sqrt{\lambda}} < \lambda r - \frac{4}{2+\sqrt{2}}.$$

For  $r \leq k - 2$ ,

$$\frac{\lambda r^2}{k} \leq \frac{\lambda r^2}{r+2} = \lambda r - \frac{2\lambda}{1+2/r} < \lambda r - \frac{2\lambda}{1+\sqrt{\lambda}} < \lambda r - 1.$$

Therefore,  $t = \lfloor \lambda r^2/k \rfloor \leq \lambda r - 2$ . In turn,  $\lfloor t/2 \rfloor + 1 \leq t/2 + 1 \leq \lambda r/2$  for  $\lambda \geq 2$  or  $r \leq k - 2$ . It implies  $\lfloor t/2 \rfloor + 1 < \lambda k/2$ . Hence  $\lfloor t/2 \rfloor + 1 \leq \lambda k/2 - 1$  for  $\lambda \geq 2$  or  $r \leq k - 2$ . This assures us that there exist  $\lfloor t/2 \rfloor + 1$  edge-disjoint  $k$ -cycles in  $\lambda K_{k/2,k}$  by Lemma 3.8.  $\square$

**Lemma 3.10** *Let  $k$  be a positive even integer and let  $n$  be a positive integer with  $4 \leq k < n < 2k$ . If  $\lambda(n-k)^2 \geq k$ , then  $\lambda K_{n,n}$  has a  $(C_k, S_k)$ -packing  $\mathcal{P}$  with  $|\mathcal{P}| = \lfloor \lambda n^2/k \rfloor$  and a  $(C_k, S_k)$ -covering  $\mathcal{C}$  with  $|\mathcal{C}| = \lceil \lambda n^2/k \rceil$ .*

**Proof:** Let  $n = k + r$ . From the assumption  $k < n < 2k$ , we have  $0 < r < k$ . Let  $\lambda r^2 = tk + s$  such that  $s$  and  $t$  are integers with  $0 \leq s < k$ . Note that  $t = \lfloor \lambda r^2/k \rfloor$ . Hence  $\lfloor \lambda n^2/k \rfloor = \lfloor \lambda(k+r)^2/k \rfloor = \lambda(k+2r) + t$  and

$$\left\lceil \frac{\lambda n^2}{k} \right\rceil = \left\lceil \frac{\lambda(k+r)^2}{k} \right\rceil = \begin{cases} \lambda(k+2r) + t & \text{if } s = 0 \\ \lambda(k+2r) + t + 1 & \text{if } s > 0. \end{cases}$$

Since  $\lambda(n-k)^2 \geq k$ ,  $t \geq 1$ . Let  $p_0 = \lceil t/2 \rceil$  and  $p_1 = \lfloor t/2 \rfloor$ . We have  $p_0 = 1$  and  $p_1 = 0$  for  $t = 1$ , and  $p_0 \geq p_1 \geq 1$  for  $t \geq 2$ . In the sequel, we will show that  $\lambda K_{n,n}$  has a packing  $\mathcal{P}$  consisting of  $t$  copies of  $k$ -cycles and  $\lambda(k+2r)$  copies of  $k$ -stars with leave  $P_{s+1}$  (except in the case  $s = 0$ , in which the leave is the empty graph), and a covering  $\mathcal{C}$  with  $|\mathcal{C}| = \lceil \lambda n^2/k \rceil$ .

Let  $A_0 = \{a_0, a_1, \dots, a_{k/2-1}\}$ ,  $A_1 = \{a_{k/2}, a_{k/2+1}, \dots, a_{k-1}\}$ ,  $A_2 = A - (A_0 \cup A_1)$ ,  $B_0 = \{b_0, b_1, \dots, b_{k-1}\}$  and  $B_1 = B - B_0$ . In addition, letting  $A'_1 = \{a_{k/2}, a_{k/2+1}, \dots, a_{\lceil (k+s)/2 \rceil - 1}\}$  for  $s > 0$  and  $G_i = \lambda K_{n,n}[A_i \cup B_1]$  for  $i = 0, 1$ . Clearly,  $G_0$  and  $G_1$  are isomorphic to  $\lambda K_{k/2,k}$ . By Lemma 3.9, there exist  $p_i$  edge-disjoint  $k$ -cycles in  $G_i$  for  $i \in \{0, 1\}$ , and there exist  $p_1 + 1$  edge-disjoint  $k$ -cycles in  $G_1$  for  $\lambda \geq 2$  or  $r \leq k - 2$ . Let  $\delta = 0$  for  $p_1 = 0$  and  $\delta = 1$  for  $p_1 \geq 1$ . Suppose that  $Q_{i,0}, Q_{i,1}, \dots, Q_{i,p_i-1}$  are edge-disjoint  $k$ -cycles in  $G_i$  for  $0 \leq i \leq \delta$ . Moreover, for  $\lambda \geq 2$  or  $r \leq k - 2$ , let  $Q$  be a  $k$ -cycle in  $G_1$  which is edge-disjoint with  $Q_{1,j}$  for  $0 \leq j \leq p_1 - 1$ . Without loss of generality, we assume that

$$Q = (b_{j_1}, a_{k/2}, b_{j_2}, a_{k/2+1}, \dots, b_{j_{k/2}}, a_{k-1}).$$

Note, for  $\lambda = 1$  and  $r = k - 1$ , that  $\lambda r^2 = (k - 1)^2 = k(k - 2) + 1$ , which implies  $t = k - 2$  and  $s = 1$ . For  $s > 0$ , define an  $(s + 1)$ -path  $P$  as follows:

$$P = \begin{cases} a_{k/2} b_\ell & \text{if } \lambda = 1, r = k - 1, \\ b_{j_1} a_{k/2} b_{j_2} a_{k/2+1} \dots b_{j_{s/2}} a_{(k+s)/2-1} b_{j_{s/2+1}} & \text{if } \lambda \geq 2 \text{ or } r \leq k - 2, s \text{ is even,} \\ b_{j_1} a_{k/2} b_{j_2} a_{k/2+1} \dots b_{j_{(s+1)/2}} a_{(k+s+1)/2-1} & \text{if } \lambda \geq 2 \text{ or } r \leq k - 2, s \text{ is odd,} \end{cases}$$

where  $a_{k/2} b_\ell$  is any edge (incident with  $a_{k/2}$ ) not in  $Q_{1,0}, Q_{1,1}, \dots, Q_{1,p_1-1}$ . Let

$$H = \lambda K_{n,n} - E\left(\bigcup_{i=0}^{\delta} \bigcup_{h=0}^{p_i-1} Q_{i,h}\right) \cup P.$$



Note that  $V(H) = V(\lambda K_{n,n})$ ,  $|E(H)| = \lambda n^2 - (tk + s) = \lambda n^2 - \lambda r^2 = \lambda k(k + 2r)$ , and  $\mu(uv) \leq \lambda$  for all  $u, v \in V(H)$ . Moreover, for  $H' = H[A \cup B_0]$ , we have

$$\deg_{H'} v = \begin{cases} \lambda k - 2\lceil t/2 \rceil & \text{if } v \in A_0, \\ \lambda k - 2(\lfloor t/2 \rfloor + 1) & \text{if } s > 0 \text{ and } v \in A'_1 - \{a_{\lceil (k+s)/2 \rceil - 1}\}, \\ \lambda k - 2\lfloor t/2 \rfloor - \rho & \text{if } s > 0 \text{ and } v = a_{\lceil (k+s)/2 \rceil - 1}, \\ \lambda k - 2\lfloor t/2 \rfloor & \text{if } s > 0 \text{ and } v \in A_1 - A'_1, \text{ or } s = 0 \text{ and } v \in A_1, \\ \lambda k & \text{if } v \in A_2, \end{cases}$$

where  $\rho = 1$  if  $s$  is odd, and  $\rho = 2$  if  $s$  is even. Define a function  $c : V(H) \rightarrow \mathbb{N}$  as follows:

$$c(v) = \begin{cases} 0 & \text{if } v \in B_0, \\ \lambda & \text{otherwise.} \end{cases}$$

Now we show that there exists an  $S_k$ -decomposition  $\mathcal{D}$  of  $H$  with central function  $c$  by Proposition 2.1.

First,  $k \sum_{v \in V(H)} c(v) = k\lambda(k + 2r) = |E(H)|$ . This proves (i). Next, if  $u, v \in B_0$ , then  $c(u) + c(v) = 0 = \mu(uv)$ ; otherwise,  $c(u) + c(v) \geq \lambda \geq \mu(uv)$ . This proves (ii). Finally, for  $S \subseteq V(H)$ ,  $i \in \{0, 1, 2\}$ , and  $j \in \{0, 1\}$ , let  $S \cap A_i = X_i$  and  $S \cap B_j = Y_j$ . Moreover, letting  $S \cap A'_1 = X'_1$ ,  $X = X_0 \cup X_1 \cup X_2$ , and  $Y = Y_0 \cup Y_1$ . Define a set  $T$  of ordered pairs of vertices as follows:

$$T = \{(u, v) \mid u \in X, v \in B_1 - Y_1 \text{ or } u \in X_2, v \in B_0 - Y_0 \text{ or } u \in Y_1, v \in A - X\}.$$

Note that

$$k \sum_{w \in S} c(w) = k\lambda(|X| + |Y_1|), \quad (4)$$

$$\varepsilon(S) = \lambda(|X||Y_1| + |X_2||Y_0|) + \sum_{u \in X_0 \cup X_1, v \in Y_0} \mu(uv), \quad (5)$$

and for  $u \in S$  and  $v \in V(H) - S$

$$\min\{c(u), \mu(uv)\} = \begin{cases} \lambda & \text{if } (u, v) \in T, \\ \mu(uv) & \text{if } u \in X_0 \cup X_1, v \in B_0 - Y_0, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

For  $S \subseteq V(H)$ , let

$$g(S) = \varepsilon(S) + \sum_{u \in S, v \in V(H) - S} \min\{c(u), \mu(uv)\} - k \sum_{w \in S} c(w).$$

Note that

$$\begin{aligned} & \sum_{u \in X_0 \cup X_1, v \in Y_0} \mu(uv) + \sum_{u \in X_0 \cup X_1, v \in B_0 - Y_0} \mu(uv) \\ = & \sum_{u \in X_0 \cup X_1, v \in B_0} \mu(uv) \\ = & \begin{cases} |X_0|(\lambda k - 2\lceil t/2 \rceil) + |X_1|(\lambda k - 2\lfloor t/2 \rfloor) & \text{if } s = 0, \\ |X_0|(\lambda k - 2\lceil t/2 \rceil) + |X_1|(\lambda k - 2\lfloor t/2 \rfloor) - 2|X'_1| & \text{if } s > 0, a_{\lceil (k+s)/2 \rceil - 1} \notin X'_1, \\ |X_0|(\lambda k - 2\lceil t/2 \rceil) + |X_1|(\lambda k - 2\lfloor t/2 \rfloor) - 2|X'_1| + 2 - \rho & \text{if } s > 0, a_{\lceil (k+s)/2 \rceil - 1} \in X'_1. \end{cases} \end{aligned}$$

By (4)–(6) and  $|X_0| + |X_1| + |X_2| = |X|$ , we have

$$\begin{aligned} g(S) &= \lambda(|X||Y_1| + |X_2||Y_0|) + \sum_{u \in X_0 \cup X_1, v \in Y_0} \mu(uv) \\ &\quad + \lambda(|X|(r - |Y_1|) + |X_2|(k - |Y_0|) + |Y_1|(k + r - |X|)) \\ &\quad + \sum_{u \in X_0 \cup X_1, v \in B_0 - Y_0} \mu(uv) - k\lambda(|X| + |Y_1|) \\ &= \lambda(r|X| + |Y_1|(r - |X|)) + m, \end{aligned}$$

where

$$m = \begin{cases} -2(|X_0|\lceil t/2 \rceil + |X_1|\lfloor t/2 \rfloor) & \text{if } s = 0, \\ -2(|X_0|\lceil t/2 \rceil + |X_1|\lfloor t/2 \rfloor) - 2|X'_1| & \text{if } s > 0, a_{\lceil (k+s)/2 \rceil - 1} \notin X'_1, \\ -2(|X_0|\lceil t/2 \rceil + |X_1|\lfloor t/2 \rfloor) - 2|X'_1| + 2 - \rho & \text{if } s > 0, a_{\lceil (k+s)/2 \rceil - 1} \in X'_1. \end{cases}$$

If  $a_{\lceil (k+s)/2 \rceil - 1} \notin X'_1$ , then  $|X'_1| \leq |A'_1| - 1 = \lceil s/2 \rceil - 1$ . Hence  $-2|X'_1| \geq -2(\lceil s/2 \rceil - 1) \geq -s$ . If  $a_{\lceil (k+s)/2 \rceil - 1} \in X_1$ , then  $|X'_1| \leq |A'_1| = \lceil s/2 \rceil$ . In addition,  $\rho = 1$  for odd  $s$  and  $\rho = 2$  for even  $s$ . Therefore,  $-2|X'_1| + 2 - \rho \geq -2\lceil s/2 \rceil + 2 - \rho = -s$ . Together with the fact  $\max\{|X_0|, |X_1|\} \leq k/2$ , we have

$$m \geq -2(k/2\lceil t/2 \rceil + k/2\lfloor t/2 \rfloor) - s = -(kt + s) = -\lambda r^2.$$

Thus for  $|X| \geq r$ , we have

$$g(S) \geq \lambda(r|X| - |Y_1|(|X| - r)) - \lambda r^2 = \lambda(|X| - r)(r - |Y_1|) \geq 0.$$

So it remains to consider the case  $|X| < r$ . Recall that  $t = k - 2$  and  $s = 1$  for  $(\lambda, r) = (1, k - 1)$ . Thus  $\lceil t/2 \rceil = \lfloor t/2 \rfloor = (\lambda r - 1)/2$ . In addition,  $|X'_1| = 0$  for  $a_{\lceil (k+s)/2 \rceil - 1} \notin X'_1$ , and  $\rho = 1$  as well as  $|X'_1| = 1$  (which implies  $|X_1| \geq 1$ ) for  $a_{\lceil (k+s)/2 \rceil - 1} \in X'_1$ . Hence for  $a_{\lceil (k+s)/2 \rceil - 1} \notin X'_1$ ,

$$m = -2(|X_0| + |X_1|)(\lambda r - 1)/2 \geq -\lambda r(|X_0| + |X_1|),$$

and for  $a_{\lceil (k+s)/2 \rceil - 1} \in X'_1$ ,

$$\begin{aligned} m &= -2(|X_0| + |X_1|)(\lambda r - 1)/2 - 1 \\ &= -\lambda r(|X_0| + |X_1|) + |X_0| + |X_1| - 1 \\ &\geq -\lambda r(|X_0| + |X_1|). \end{aligned}$$

On the other hand, for  $\lambda \geq 2$  or  $r \leq k - 2$ , we have  $\lfloor t/2 \rfloor + 1 \leq \lambda r/2$  by Lemma 3.9, this implies

$$\begin{aligned} m &\geq -2(|X_0|\lceil t/2 \rceil + |X'_1|(\lfloor t/2 \rfloor + 1) + (|X_1| - |X'_1|)\lfloor t/2 \rfloor) \\ &\geq -2(|X_0| + |X_1|)(\lambda r/2) \\ &= -\lambda r(|X_0| + |X_1|). \end{aligned}$$

Therefore, for  $|X| < r$ , we have

$$g(S) \geq \lambda(r|X| + |Y_1|(r - |X|)) - \lambda r(|X_0| + |X_1|) = \lambda(r|X_2| + |Y_1|(r - |X|)) \geq 0.$$

This settles (iii).

Let  $\mathcal{P} = \mathcal{D} \cup_{i=0}^{\delta} \{Q_{i,0}, Q_{i,1}, \dots, Q_{i,p_i-1}\}$ . Clearly,  $\mathcal{P}$  is the required packing. Let

$$\mathcal{C} = \begin{cases} \mathcal{P} & \text{if } s = 0, \\ \mathcal{P} \cup \{Q\} & \text{if } s \geq 1. \end{cases}$$

It is easy to check that  $\mathcal{C}$  is the covering as required.  $\square$

Now, we are ready for the main result of this section.

**Theorem 3.11** *If  $\lambda$  and  $n$  are positive integers and  $k$  is a positive even integer with  $4 \leq k \leq n$ , then  $p(\lambda K_{n,n}; C_k, S_k) = \lfloor \lambda n^2/k \rfloor$  and  $c(\lambda K_{n,n}; C_k, S_k) = \lceil \lambda n^2/k \rceil$ .*

**Proof:** Obviously,

$$p(\lambda K_{n,n}; C_k, S_k) \leq \left\lfloor \frac{\lambda n^2}{k} \right\rfloor \leq \left\lceil \frac{\lambda n^2}{k} \right\rceil \leq c(\lambda K_{n,n}; C_k, S_k),$$

Let  $n = qk + r$  where  $q$  and  $r$  are integers with  $0 \leq r < k$ . For  $q = 1$ , the result follows from Lemmas 3.5, 3.6, and 3.10. If  $q \geq 2$ , then  $\lambda K_{n,n} = \lambda K_{k+r, k+r} \cup \lambda K_{k+r, (q-1)k} \cup \lambda K_{(q-1)k, n}$ . Note that  $\lambda K_{k+r, k+r}$  has a  $(C_k, S_k)$ -packing  $\mathcal{P}$  with  $|\mathcal{P}| = \lfloor \lambda(k+r)^2/k \rfloor$  and a  $(C_k, S_k)$ -covering  $\mathcal{C}$  with  $|\mathcal{C}| = \lceil \lambda(k+r)^2/k \rceil$ . Trivially,  $\lambda K_{k+r, (q-1)k}$  and  $\lambda K_{(q-1)k, n}$  have  $S_k$ -decompositions  $\mathcal{D}$  and  $\mathcal{D}'$  with  $|\mathcal{D}| = \lambda(k+r)(q-1)$  and  $|\mathcal{D}'| = \lambda(q-1)n$ , respectively. Since  $\lambda(k+r)^2/k + \lambda(k+r)(q-1) + \lambda(q-1)n = \lambda(qk+r)^2/k = \lambda n^2/k$ ,  $\mathcal{P} \cup \mathcal{D} \cup \mathcal{D}'$  is a  $(C_k, S_k)$ -packing of  $\lambda K_{n,n}$  with cardinality  $\lfloor \lambda n^2/k \rfloor$  and  $\mathcal{C} \cup \mathcal{D} \cup \mathcal{D}'$  is a  $(C_k, S_k)$ -covering of  $\lambda K_{n,n}$  with cardinality  $\lceil \lambda n^2/k \rceil$ . This completes the proof.  $\square$

Clearly, if  $\lambda K_{n,n}$  admits a  $(C_k, S_k)$ -decomposition, then  $4 \leq k \leq n$  and  $k$  is even and  $\lambda n^2$  is divisible by  $k$ . When  $k$  divides  $\lambda n^2$ , a  $(C_k, S_k)$ -packing  $\mathcal{P}$  with  $|\mathcal{P}| = \lfloor \lambda n^2/k \rfloor$  is a  $(C_k, S_k)$ -decomposition. Therefore, with the aid of Theorem 3.11, we have the following.

**Corollary 3.12** *For positive integers  $\lambda$ ,  $k$  and  $n$ , the balanced complete bipartite multigraph  $\lambda K_{n,n}$  is  $(C_k, S_k)$ -decomposable if and only if  $4 \leq k \leq n$ ,  $k$  is even, and  $\lambda n^2$  is divisible by  $k$ .*

## 4 Packing and covering with 4-cycles and 4-stars

In this section a complete solution to the maximum packing and minimum covering problem of  $\lambda K_{n,n}$  with  $C_4$  and  $S_4$  is given. Before that, we need more notations. For multigraphs  $G$  and  $H$ ,  $G \uplus H$  denotes the disjoint union of  $G$  and  $H$ ,  $G \odot H$  denotes the union of  $G$  and  $H$  with a common vertex. For a set  $\mathcal{R}$  and a positive integer  $t$ ,  $t\mathcal{R}$  denotes the multiset in which each element in  $\mathcal{R}$  appears  $t$  times. In addition,  $M_t$  denotes the graph induced by  $t$  nonadjacent edges. We begin with the discussion for the possible minimum leaves and paddings of  $\lambda K_{n,n}$  with  $C_4$  and  $S_4$ .

Note that  $|E(\lambda K_{n,n})| = \lambda n^2$ . If  $\lambda \equiv 0 \pmod{4}$  or  $n \equiv 0 \pmod{2}$ , then  $|E(\lambda K_{n,n})| \equiv 0 \pmod{4}$ . By Corollary 3.12, both of the possible minimum leave and the possible minimum padding are the empty graph. If  $\lambda \equiv 1 \pmod{4}$  and  $n \equiv 1 \pmod{2}$ , then  $|E(\lambda K_{n,n})| \equiv 1 \pmod{4}$ . This implies that the possible minimum leave is only  $P_2$ , and the possible minimum paddings are  $S_3$ ,  $P_4$ ,  $P_3 \uplus P_2$ ,  $M_3$ ,  $2P_2 \uplus P_2$ ,  $2P_2 \odot P_2$ , and  $3P_2$ . If  $\lambda \equiv 2 \pmod{4}$  and  $n \equiv 1 \pmod{2}$ , then  $|E(\lambda K_{n,n})| \equiv 2 \pmod{4}$ . This implies that the possible minimum leaves are  $P_3$ ,  $M_2$ , and  $2P_2$ , so are the possible minimum paddings. If  $\lambda \equiv 3 \pmod{4}$  and  $n \equiv 1 \pmod{2}$ , then  $|E(\lambda K_{n,n})| \equiv 3 \pmod{4}$ . This implies that the possible minimum leaves are  $S_3$ ,  $P_4$ ,  $P_3 \uplus P_2$ ,  $M_3$ ,  $2P_2 \uplus P_2$ ,  $2P_2 \odot P_2$ , and  $3P_2$ , and the possible minimum padding is only  $P_2$ .

**Lemma 4.1**  $K_{5,5}$  has no  $(C_4, S_4)$ -covering with padding  $3P_2$ .

**Proof:** It suffices to show that  $K_{5,5} + 3\{a_0b_0\}$  is not  $(C_4, S_4)$ -decomposable. Suppose, to the contrary of the conclusion, that there exists a  $(C_4, S_4)$ -decomposition  $\mathcal{D}$  of  $K_{5,5} + 3\{a_0b_0\}$ . Since there are at most two star with center  $a_0$  (or  $b_0$ ) and each edge joining  $a_0$  and  $b_0$  lies in exactly one subgraph in  $\mathcal{D}$ , there are exactly three possibilities for the edges joining  $a_0$  and  $b_0$  to lie in the decomposition: in four 4-cycles, in three 4-cycles and a 4-star, or in two 4-cycles and two 4-stars. Let  $G_1$  be the graph obtained from  $K_{5,5} + 3\{a_0b_0\}$  by deleting the edges of four 4-cycles, and let  $G_2$  be the graph obtained from  $K_{5,5} + 3\{a_0b_0\}$  by deleting the edges of three 4-cycles or deleting the edges of two 4-cycles. Note that  $\deg_{G_1} x = 3$  for  $x \notin \{a_0, b_0\}$ , which implies that there is no 4-star in  $G_1$ . Since  $\deg_{G_2} x \leq 3$  for  $x \in \{a_0, b_0\}$ , there is no 4-star with center at  $a_0$  or  $b_0$  in  $G_2$ . This leads to a contradiction and completes the proof.  $\square$

We summarize the results discussed above in Table 1.

**Tab. 1:** The possible minimum leaves and paddings of  $\lambda K_{n,n}$  with  $C_4$  and  $S_4$

$\lambda \pmod{4}$ $n \pmod{2}$	$\lambda \equiv 0$ or $n \equiv 0$	$\lambda \equiv 1$ and $n \equiv 1$	$\lambda \equiv 2$ and $n \equiv 1$	$\lambda \equiv 3$ and $n \equiv 1$
Leave	$\emptyset$	$P_2$	$P_3, M_2, 2P_2$	$S_3, P_4, P_3 \uplus P_2,$ $M_3, 2P_2 \uplus P_2,$ $2P_2 \odot P_2, 3P_2$
Padding	$\emptyset$	$S_3, P_4, P_3 \uplus P_2,$ $M_3, 2P_2 \uplus P_2,$ $2P_2 \odot P_2, 3P_2$ ( $3P_2$ for $\lambda \neq 1$ )	$P_3, M_2, 2P_2$	$P_2$

**Lemma 4.2** Let  $r \in \{1, 2, 3, 5\}$ .

(a) There exists a  $(C_4, S_4)$ -packing of  $rK_{5,5}$  with leave  $L$  where

$$\begin{cases} L = P_2 & \text{if } r = 1 \text{ or } r = 5, \\ L \in \{P_3, M_2, 2P_2\} & \text{if } r = 2, \\ L \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2, 3P_2\} & \text{if } r = 3. \end{cases}$$

(b) There exists a  $(C_4, S_4)$ -covering of  $rK_{5,5}$  with padding  $R$  where

$$\begin{cases} R \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2\} & \text{if } r = 1, \\ R \in \{P_3, M_2, 2P_2\} & \text{if } r = 2, \\ R = P_2 & \text{if } r = 3, \\ R \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2, 3P_2\} & \text{if } r = 5. \end{cases}$$

**Proof:** The proof is divided into four parts according to the value of  $r$ .

**Case 1.**  $r = 1$ .

Let  $A_1 = \{a_1, a_2, a_3, a_4\}$  and  $B_1 = \{b_1, b_2, b_3, b_4\}$ , and let  $H = K_{5,5}[A_1 \cup B_1]$ . Trivially,  $H$  is isomorphic to  $K_{4,4}$ . By Corollary 3.12, there exists a  $(C_4, S_4)$ -decomposition  $\mathcal{D}$  of  $K_{4,4}$ . Let  $\mathcal{P} =$

$\mathcal{D} \cup \{(a_0; b_1, b_2, b_3, b_4), (b_0; a_1, a_2, a_3, a_4)\}$ . Clearly,  $\mathcal{P}$  is a  $(C_4, S_4)$ -packing of  $K_{5,5}$  with leave  $P_2 : \{a_0b_0\}$ .

Now we give the required coverings of  $K_{5,5}$ . Note that  $\mathcal{P} \cup \{(a_0; b_0, b_1, b_2, b_3)\}$  is a  $(C_4, S_4)$ -covering of  $K_{5,5}$  with padding  $S_3 : \{(a_0; b_1, b_2, b_3)\}$ , and  $\mathcal{P} \cup \{(a_0, b_1, a_1, b_0)\}$  is a  $(C_4, S_4)$ -covering of  $K_{5,5}$  with padding  $P_4 : \{a_0b_1a_1b_0\}$ . Without loss of generality, we assume that  $\mathcal{P}$  contains a 4-star  $(a_4; b_1, b_2, b_3, b_4)$ . Thus  $\mathcal{P} - \{(a_0; b_1, b_2, b_3, b_4), (a_4; b_1, b_2, b_3, b_4)\} \cup \{(a_0, b_3, a_4, b_4), (a_0; b_0, b_1, b_2, b_3), (a_4; b_0, b_1, b_2, b_4)\}$  is a  $(C_4, S_4)$ -covering of  $K_{5,5}$  with padding  $P_3 \uplus P_2 : \{b_0a_4b_4, a_0b_3\}$ . In addition,  $\{(a_3, b_3, a_4, b_4), (a_0; b_0, b_2, b_3, b_4), (a_1; b_0, b_1, b_3, b_4), (a_2; b_1, b_2, b_3, b_4), (b_0; a_0, a_2, a_3, a_4), (b_1; a_0, a_1, a_3, a_4), (b_2; a_1, a_2, a_3, a_4)\}$  is a  $(C_4, S_4)$ -covering of  $K_{5,5}$  with padding  $M_3 : \{a_0b_0, a_1b_1, a_2b_2\}$ ,  $\mathcal{P} - \{(a_4; b_1, b_2, b_3, b_4)\} \cup \{(a_0, b_0, a_4, b_4), (a_4; b_0, b_1, b_2, b_3)\}$  is a  $(C_4, S_4)$ -covering of  $K_{5,5}$  with padding  $2P_2 \uplus P_2 : 2\{b_0a_4\} \cup \{a_0b_4\}$ , and  $\mathcal{P} - \{(a_0; b_1, b_2, b_3, b_4), (a_4; b_1, b_2, b_3, b_4)\} \cup \{(a_0, b_0, a_4, b_4), (a_0; b_0, b_1, b_2, b_3), (a_4; b_0, b_1, b_2, b_3)\}$  is a  $(C_4, S_4)$ -covering of  $K_{5,5}$  with padding  $2P_2 \odot P_2 : 2\{b_0a_4\} \cup \{b_0a_0\}$ .

**Case 2.**  $r = 2$ .

First, we use  $\mathcal{P}$  to construct the required packings of  $2K_{5,5}$ . Exchanging  $b_0$  with  $b_1$  in  $\mathcal{P}$ , we obtain a packing  $\mathcal{P}'$  of  $K_{5,5}$  with leave  $a_0b_1$ . Let  $\mathcal{P}_1 = \mathcal{P} \cup \mathcal{P}'$ . One can see that  $\mathcal{P}_1$  is a packing of  $2K_{5,5}$  with leave  $P_3 : \{b_0a_0b_1\}$ . Next, rename the vertices  $a_0, a_1, b_0, b_1$  in  $\mathcal{P}$  to  $a_1, a_0, b_1, b_0$ , respectively, we obtain a packing  $\mathcal{P}''$  of  $K_{5,5}$  with leave  $a_1b_1$ . Let  $\mathcal{P}_2 = \mathcal{P} \cup \mathcal{P}''$ . It is easy to see that  $\mathcal{P}_2$  is a packing of  $2K_{5,5}$  with leave  $M_2 : \{a_0b_0, a_1b_1\}$ . Finally,  $2\mathcal{P}$  is clearly a packing of  $2K_{5,5}$  with leave  $2P_2 : 2\{a_0b_0\}$ .

Now we use packings to construct the required coverings of  $2K_{5,5}$ . Note that  $\mathcal{P}_1 \cup \{(a_0; b_0, b_1, b_2, b_3)\}$  is a  $(C_4, S_4)$ -covering of  $2K_{5,5}$  with padding  $P_3 : \{b_2a_0b_3\}$ , and  $\mathcal{P}_2 \cup \{(a_0, b_0, a_1, b_1)\}$  is a  $(C_4, S_4)$ -covering of  $2K_{5,5}$  with padding  $M_2 : \{a_0b_1, a_1b_0\}$ . Moreover,  $2\mathcal{P} - \{(a_0; b_1, b_2, b_3, b_4), (a_4; b_1, b_2, b_3, b_4)\} \cup \{(a_0, b_0, a_4, b_4), (a_0; b_0, b_1, b_2, b_3), (a_4; b_0, b_1, b_2, b_3)\}$  is a  $(C_4, S_4)$ -covering of  $2K_{5,5}$  with padding  $2P_2 : 2\{b_0a_4\}$ .

**Case 3.**  $r = 3$ .

First, we use packings of  $K_{5,5}$  and  $2K_{5,5}$  to construct the required packings of  $3K_{5,5}$ . Exchanging  $b_0$  with  $b_2$  in  $\mathcal{P}$ , we obtain a packing  $\mathcal{R}$  of  $K_{5,5}$  with leave  $a_0b_2$ . Hence  $\mathcal{P}_1 \cup \mathcal{R}$  is a packing of  $3K_{5,5}$  with leave  $S_3 : \{(a_0; b_0, b_1, b_2)\}$ . Next, rename the vertices  $a_0, a_2, b_0, b_2$  in  $\mathcal{P}$  to  $a_2, a_0, b_2, b_0$ , respectively, we obtain a packing  $\mathcal{R}'$  of  $K_{5,5}$  with leave  $a_2b_2$ . Thus  $\mathcal{P}_2 \cup \mathcal{R}'$  is a packing of  $3K_{5,5}$  with leave  $M_3 : \{a_0b_0, a_1b_1, a_2b_2\}$ . Note that  $\mathcal{P}_1 \cup \mathcal{P}''$  is a packing of  $3K_{5,5}$  with leave  $P_4 : \{b_0a_0b_1a_1\}$ . In addition,  $\mathcal{P}_1 \cup \mathcal{R}'$  is a packing of  $3K_{5,5}$  with leave  $P_3 \uplus P_2 : \{b_0a_0b_1\} \cup \{a_2b_2\}$ ,  $2\mathcal{P} \cup \mathcal{R}'$  is a packing of  $3K_{5,5}$  with leave  $2P_2 \uplus P_2 : 2\{a_0b_0\} \cup \{a_2b_2\}$ ,  $2\mathcal{P} \cup \mathcal{R}$  is a packing of  $3K_{5,5}$  with leave  $2P_2 \odot P_2 : 2\{a_0b_0\} \cup \{a_0b_2\}$ , and  $3\mathcal{P}$  is clearly a packing of  $3K_{5,5}$  with leave  $3P_2 : 3\{a_0b_0\}$ .

Finally, since  $3(5-4)^2 = 3 < 4$ , there exists a  $(C_4, S_4)$ -covering of  $3K_{5,5}$  with leave  $P_2$  by Lemma 3.6.

**Case 4.**  $r = 5$ .

By Corollary 3.12,  $(C_4, S_4) \mid 4K_{5,5}$ . Since  $5K_{5,5} = K_{5,5} \cup 4K_{5,5}$ , it suffices to show that there exists a  $(C_4, S_4)$ -covering of  $5K_{5,5}$  with padding  $3P_2$ . Note that  $5K_{5,5} = 2K_{5,5} \cup 3K_{5,5}$ . Since  $2K_{5,5}$  has a  $(C_4, S_4)$ -covering with padding  $2P_2 : 2\{b_0a_4\}$  and  $3K_{5,5}$  has a  $(C_4, S_4)$ -covering with padding  $P_2$  (say  $\{b_0a_4\}$ ), we have the required covering.  $\square$

**Lemma 4.3** *Let  $r$  be a positive integer and let  $m$  be a positive odd integer with  $m \geq 5$ . If  $rK_{m,m}$  has a  $(C_4, S_4)$ -packing (resp.  $(C_4, S_4)$ -covering) with leave  $L$  (resp. padding  $R$ ), then  $rK_{m+2, m+2}$  also has a  $(C_4, S_4)$ -packing (resp.  $(C_4, S_4)$ -covering) with leave  $L$  (resp. padding  $R$ ).*

**Proof:** Let  $m = 2t + 1$  where  $t$  is a positive integer with  $t \geq 2$ . Let  $A_1 = \{a_0, a_1, \dots, a_{2t}\}$  and

$B_1 = \{b_0, b_1, \dots, b_{2t}\}$ . Letting  $G_1 = K_{m+2, m+2}[A_1 \cup B_1]$  and  $G_2 = K_{m+2, m+2} - E(G_1)$ . Clearly,  $G_1$  is isomorphic to  $K_{m, m}$ . Note that  $\{(a_{2t+1}, b_{2i}, a_{2t+2}, b_{2i+1}), (a_{2i}, b_{2t+1}, a_{2i+1}, b_{2t+2}) : i = 0, 1, \dots, t-2\} \cup \{(a_{2t+1}; b_{2t-2}, b_{2t-1}, b_{2t}, b_{2t+1}), (a_{2t+2}; b_{2t-2}, b_{2t-1}, b_{2t}, b_{2t+2}), (b_{2t+1}; a_{2t-2}, a_{2t-1}, a_{2t}, a_{2t+2}), (b_{2t+2}; a_{2t-2}, a_{2t-1}, a_{2t}, a_{2t+1})\}$  is a  $(C_4, S_4)$ -decomposition of  $G_2$ . Since  $rK_{m+2, m+2} = rG_1 \cup rG_2$ ,  $rK_{m+2, m+2}$  has the required packings and coverings.  $\square$

Now, we are ready for the main result of this section.

**Theorem 4.4** *Let  $\lambda$  and  $n$  be positive integers with  $n \geq 4$ .*

(A)  $\lambda K_{n, n}$  has a maximum  $(C_4, S_4)$ -packing with leave  $L$  if and only if

$$\left\{ \begin{array}{ll} L = \emptyset & \text{if } \lambda n^2 \equiv 0 \pmod{4}, \\ L = P_2 & \text{if } \lambda n^2 \equiv 1 \pmod{4}, \\ L \in \{P_3, M_2, 2P_2\} & \text{if } \lambda n^2 \equiv 2 \pmod{4}, \\ L \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2, 3P_2\} & \text{if } \lambda n^2 \equiv 3 \pmod{4}. \end{array} \right.$$

(B)  $\lambda K_{n, n}$  has a minimum  $(C_4, S_4)$ -covering with padding  $R$  if and only if

$$\left\{ \begin{array}{ll} R = \emptyset & \text{if } \lambda n^2 \equiv 0 \pmod{4}, \\ R \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2\} & \text{if } \lambda n^2 \equiv 1 \pmod{4} \text{ and } \lambda = 1, \\ R \in \{S_3, P_4, P_3 \uplus P_2, M_3, 2P_2 \uplus P_2, 2P_2 \odot P_2, 3P_2\} & \text{if } \lambda n^2 \equiv 1 \pmod{4} \text{ and } \lambda \geq 5, \\ R \in \{P_3, M_2, 2P_2\} & \text{if } \lambda n^2 \equiv 2 \pmod{4}, \\ R = P_2 & \text{if } \lambda n^2 \equiv 3 \pmod{4}. \end{array} \right.$$

**Proof:** The necessity follows from the arguments above Table 1. It suffices to show that  $\lambda K_{n, n}$  has required packings and coverings. The result for  $\lambda n^2 \equiv 0 \pmod{4}$  follows from Corollary 3.12 immediately. So it remains to consider the case  $\lambda n^2 \equiv r \pmod{4}$  for  $r \in \{1, 2, 3\}$ . Note that  $\lambda n^2 \equiv r \pmod{4}$  if and only if  $\lambda \equiv r \pmod{4}$  and  $n \equiv 1 \pmod{2}$ . When  $\lambda \in \{1, 2, 3, 5\}$ , the result for  $n = 5$  follows from Lemma 4.2, and the result for  $n > 5$  can be obtained by using Lemma 4.3 recursively. Now consider  $\lambda \equiv r \pmod{4}$  and  $\lambda > 5$ . Note that  $\lambda K_{n, n} = rK_{n, n} \cup (\lambda - r)K_{n, n}$ . Since  $(\lambda - r)K_{n, n}$  is  $(C_4, S_4)$ -decomposable by Corollary 3.12, we have the result.  $\square$

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