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# Oriented diameter and rainbow connection number of a graph\*

Xiaolong Huang   Hengzhe Li<sup>†</sup>   Xueliang Li   Yuefang Sun

Center for Combinatorics and LPMC-TJKLC, Nankai University, Tianjin, China

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The oriented diameter of a bridgeless graph  $G$  is  $\min\{\text{diam}(H) \mid H \text{ is a strong orientation of } G\}$ . A path in an edge-colored graph  $G$ , where adjacent edges may have the same color, is called rainbow if no two edges of the path are colored the same. The rainbow connection number  $rc(G)$  of  $G$  is the smallest integer number  $k$  for which there exists a  $k$ -edge-coloring of  $G$  such that every two distinct vertices of  $G$  are connected by a rainbow path. In this paper, we obtain upper bounds for the oriented diameter and the rainbow connection number of a graph in terms of  $rad(G)$  and  $\eta(G)$ , where  $rad(G)$  is the radius of  $G$  and  $\eta(G)$  is the smallest integer number such that every edge of  $G$  is contained in a cycle of length at most  $\eta(G)$ . We also obtain constant bounds of the oriented diameter and the rainbow connection number for a (bipartite) graph  $G$  in terms of the minimum degree of  $G$ .

**Keywords:** Diameter, Radius, Oriented diameter, Rainbow connection number, Cycle length, Bipartite graph.

## 1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to the book [2] for notation and terminology not described here. A path  $u = u_1, u_2, \dots, u_k = v$  is called a  $P_{u,v}$  path. Denote by  $u_i P u_j$  the subpath  $u_i, u_{i+1}, \dots, u_j$  for  $i \leq j$ . The length  $\ell(P)$  of a path  $P$  is the number of edges in  $P$ . The distance between two vertices  $x$  and  $y$  in  $G$ , denoted by  $d_G(x, y)$ , is the length of a shortest path between them. The eccentricity of a vertex  $x$  in  $G$  is  $\text{ecc}_G(x) = \max_{y \in V(G)} d(x, y)$ . The radius and diameter of  $G$  are  $rad(G) = \min_{x \in V(G)} \text{ecc}(x)$  and  $\text{diam}(G) = \max_{x \in V(G)} \text{ecc}(x)$ , respectively. A vertex  $u$  is a center of a graph  $G$  if  $\text{ecc}(u) = rad(G)$ . The oriented diameter of a bridgeless graph  $G$  is  $\min\{\text{diam}(H) \mid H \text{ is an orientation of } G\}$ , and the oriented radius of a bridgeless graph  $G$  is  $\min\{rad(H) \mid H \text{ is an orientation of } G\}$ . For any graph  $G$  with edge-connectivity  $\lambda(G) = 0, 1$ ,  $G$  has oriented radius (resp. diameter)  $\infty$ .

In 1939, Robbins solved the One-Way Street Problem and proved that a graph  $G$  admits a strongly connected orientation if and only if  $G$  is bridgeless, that is,  $G$  does not have any cut-edge. Naturally, one hopes that the oriented diameter of a bridgeless graph is as small as possible. Bondy and Murty suggested

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<sup>†</sup>Email: lhz2010@mail.nankai.edu.cn, lhz@htu.cn

to study the quantitative variations on Robbins' theorem. In particular, they conjectured that there exists a function  $f$  such that every bridgeless graph with diameter  $d$  admits an orientation of diameter at most  $f(d)$ .

In 1978, Chvátal and Thomassen [5] obtained some general bounds.

**Theorem 1 (Chvátal and Thomassen 1978 [5])** *For every bridgeless graph  $G$ , there exists an orientation  $H$  of  $G$  such that*

$$\begin{aligned} \text{rad}(H) &\leq \text{rad}(G)^2 + \text{rad}(G), \\ \text{diam}(H) &\leq 2\text{rad}(G)^2 + 2\text{rad}(G). \end{aligned}$$

Moreover, the above bounds are optimal.

There exists a minor error when they constructed the graph  $G_d$  which arrives at the upper bound when  $d$  is odd. Kwok, Liu and West gave a slight correction in [11].

They also showed that determining whether an arbitrary graph can be oriented so that its diameter is at most 2 is NP-complete. Bounds for the oriented diameter of graphs have also been studied in terms of other parameters, for example, radius, dominating number [5, 6, 11, 18], etc. Some classes of graphs have also been studied in [6, 7, 8, 9, 14].

Let  $\eta(G)$  be the smallest integer such that every edge of  $G$  belongs to a cycle of length at most  $\eta(G)$ . In this paper, we show the following result.

**Theorem 2** *For every bridgeless graph  $G$ , there exists an orientation  $H$  of  $G$  such that*

$$\begin{aligned} \text{rad}(H) &\leq \sum_{i=1}^{\text{rad}(G)} \min\{2i, \eta(G) - 1\} \leq \text{rad}(G)(\eta(G) - 1), \\ \text{diam}(H) &\leq 2 \sum_{i=1}^{\text{rad}(G)} \min\{2i, \eta(G) - 1\} \leq 2\text{rad}(G)(\eta(G) - 1). \end{aligned}$$

Note that  $\sum_{i=1}^{\text{rad}(G)} \min\{2i, \eta(G) - 1\} \leq \text{rad}(G)^2 + \text{rad}(G)$  and  $\text{diam}(H) \leq 2\text{rad}(H)$ . So our result implies Chvátal and Thomassen's Theorem 1.

A path in an edge-colored graph  $G$ , where adjacent edges may have the same color, is called *rainbow* if no two edges of the path are colored the same. An edge-coloring of a graph  $G$  is a *rainbow edge-coloring* if every two distinct vertices of the graph  $G$  are connected by a rainbow path. The *rainbow connection number*  $rc(G)$  of  $G$  is the minimum integer  $k$  for which there exists a rainbow  $k$ -edge-coloring of  $G$ . It is easy to see that  $\text{diam}(G) \leq rc(G)$  for any connected graph  $G$ . The rainbow connection number was introduced by Chartrand et al. in [4]. It is of great use in transferring information of high security in multicomputer networks. We refer the readers to [3] for details.

Chakraborty et al. [3] investigated the hardness and algorithms for the rainbow connection number, and showed that given a graph  $G$ , deciding if  $rc(G) = 2$  is NP-complete. Bounds for the rainbow connection number of a graph have also been studied in terms of other graph parameters, for example, radius, dominating number, minimum degree, connectivity, etc. [1, 4, 10]. Cayley graphs and line graphs were studied in [12] and [13], respectively.

A subgraph  $H$  of a graph  $G$  is called *isometric* if the distance between any two distinct vertices in  $H$  is the same as their distance in  $G$ . The size of a largest isometric cycle in  $G$  is denoted by  $\zeta(G)$ .

Clearly, every isometric cycle is an induced cycle and thus  $\zeta(G)$  is not larger than the chordality, where *chordality* is the length of a largest induced cycle in  $G$ . In [1], Basavaraju, Chandran, Rajendraprasad and Ramaswamy got the the following sharp upper bound for the rainbow connection number of a bridgeless graph  $G$  in terms of  $rad(G)$  and  $\zeta(G)$ .

**Theorem 3 (Basavaraju et al. [1])** *For every bridgeless graph  $G$ ,*

$$rc(G) \leq \sum_{i=1}^{rad(G)} \min\{2i + 1, \zeta(G)\} \leq rad(G)\zeta(G).$$

In this paper, we show the following result.

**Theorem 4** *For every bridgeless graph  $G$ ,*

$$rc(G) \leq \sum_{i=1}^{rad(G)} \min\{2i + 1, \eta(G)\} \leq rad(G)\eta(G).$$

From Lemma 2 of Section 2, we will see that  $\eta(G) \leq \zeta(G)$ . Thus our result implies Theorem 3.

This paper is organized as follows: in Section 2, we introduce some new definitions and show several lemmas. In Section 3, we prove Theorem 2 and study upper bounds for the oriented radius (resp. diameter) of plane graphs, edge-transitive graphs and general (bipartite) graphs. In Section 4, we prove Theorem 4 and study upper for the rainbow connection number of plane graphs, edge-transitive graphs and general (bipartite) graphs.

## 2 Preliminaries

In this section, we introduce some definitions and show several lemmas.

**Definition 1** For any  $x \in V(G)$  and  $k \geq 0$ , the *k-step open neighborhood* is  $\{y \mid d(x, y) = k\}$  and denoted by  $N_k(x)$ , the *k-step closed neighborhood* is  $\{y \mid d(x, y) \leq k\}$  and denoted by  $N_k[x]$ . If  $k = 1$ , we simply write  $N(x)$  and  $N[x]$  for  $N_1(x)$  and  $N_1[x]$ , respectively.

**Definition 2** Let  $G$  be a graph and  $H$  be a subset of  $V(G)$  (or a subgraph of  $G$ ). The edges between  $H$  and  $G \setminus H$  are called *legs* of  $H$ . An *H-ear* is a path  $P = (u_0, u_1, \dots, u_k)$  in  $G$  such that  $V(H) \cap V(P) = \{u_0, u_k\}$ . The vertices  $u_0, u_k$  are called the *feet* of  $P$  in  $H$  and  $u_0u_1, u_{k-1}u_k$  are called the *legs* of  $P$ . The *length* of an *H-ear* is the length of the corresponding path. If  $u_0 = u_k$ , then  $P$  is called a *closed H-ear*. For any leg  $e$  of  $H$ , denote by  $\ell(e)$  the smallest number such that there exists an *H-ear* of length  $\ell(e)$  containing  $e$ , and such an *H-ear* is called an *optimal (H, e)-ear*.

Note that for any optimal  $(H, e)$ -ear  $P$  and every pair  $(x, y) \neq (u_0, u_k)$  of distinct vertices of  $P$ ,  $x$  and  $y$  are adjacent on  $P$  if and only if  $x$  and  $y$  are adjacent in  $G$ .

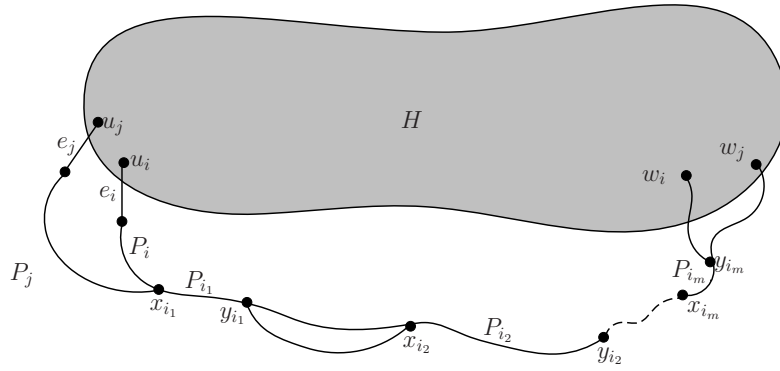
**Definition 3** For any two paths  $P$  and  $Q$ , the joint of  $P$  and  $Q$  are the common vertex and edge of  $P$  and  $Q$ . Paths  $P$  and  $Q$  have *k continuous common segments* if the common vertex and edge are  $k$  disjoint paths.

A common segment is trivial if it has only one vertex.

**Definition 4** Let  $P$  and  $Q$  be two paths in  $G$ . Call  $P$  and  $Q$  *independent* if they has no common internal vertex.

**Lemma 1** Let  $n \geq 1$  be an integer, and let  $G$  be a graph,  $H$  be a subgraph of  $G$  and  $e_i = u_i v_i$  be a leg of  $H$  and  $P_i = P_{u_i w_i}$  be an optimal  $(G, e_i)$ -ear, where  $1 \leq i \leq n$  and  $u_i, w_i$  are the foot of  $P_i$ . Then for any leg  $e_j = u_j v_j$  such that  $e_j \neq e_i$  and  $e_j \notin E(P_i)$ , where  $i \in \{1, 2, \dots, n\}$ , there exists an optimal  $(H, e_j)$ -ear  $P_j = P_{u_j w_j}$  such that either  $P_i$  and  $P_j$  are independent for any  $P_i$ ,  $1 \leq i \leq n$ , or  $P_i$  and  $P_j$  have only one continuous common segment containing  $w_j$  for some  $P_i$ .

**Proof:** Let  $P_j$  be an optimal  $(H, e_j)$ -ear. If  $P_i$  and  $P_j$  are independent for any  $i$ , then we are done. Suppose that  $P_i$  and  $P_j$  have  $m$  continuous common segments for some  $i$ , where  $m \geq 1$ . When  $m \geq 2$ , we first



**Fig. 1:** Two  $H$ -ears  $P_i$  and  $P_j$

construct an optimal  $(H, e_j)$ -ear  $P_j^*$  such that  $P_i$  and  $P_j^*$  has only one continuous common segment. Let  $P_{i_1}, P_{i_2}, \dots, P_{i_m}$  be the  $m$  continuous common segments of  $P_i$  and  $P_j$  and they appear in  $P_i$  in that order. See Figure 1 for details. Furthermore, suppose that  $x_{i_k}$  and  $y_{i_k}$  are the two ends of the path  $P_{i_k}$  and they appear in  $P_i$  successively. We say that the following claim holds.

*Claim 1:*  $\ell(y_k P_i x_{k+1}) = \ell(y_k P_j x_{k+1})$  for any  $1 \leq k \leq m - 1$ .

If not, that is, there exists an integer  $k$  such that  $\ell(y_k P_i x_{k+1}) \neq \ell(y_k P_j x_{k+1})$ . Without loss of generality, we assume  $\ell(y_k P_i x_{k+1}) < \ell(y_k P_j x_{k+1})$ . Then we shall get a more shorter path  $H$ -ear containing  $e_j$  by replacing  $y_k P_j x_{k+1}$  with  $y_k P_i x_{k+1}$ , a contradiction. Thus  $\ell(y_k P_i x_{k+1}) = \ell(y_k P_j x_{k+1})$  for any  $k$ .

Let  $P_j^*$  be the path obtained from  $P_j$  by replacing  $y_k P_j x_{k+1}$  with  $y_k P_i x_{k+1}$ , and let  $P_j = P_j^*$ . If the continuous common segment of  $P_i$  and  $P_j$  does not contain  $w_j$ . Suppose  $x$  and  $y$  are the two ends of the common segment such that  $x$  and  $y$  appeared on  $P$  starting from  $u_i$  to  $w_i$  successively. Similar to Claim 1,  $\ell(y P_i w_i) = \ell(y P_j w_j)$ . Let  $P_j^*$  be the path obtained from  $P_j$  by replacing  $y P_j w_j$  with  $y P_i w_i$ . Clearly,  $P_j^*$  is our desired optimal  $(H, u_j v_j)$ -ear.  $\square$

**Lemma 2** For every bridgeless graph  $G$ ,  $\eta(G) \leq \zeta(G)$ .

**Proof:** Suppose that there exists an edge  $e$  such that the length  $\ell(C)$  of the smallest cycle  $C$  containing  $e$  is larger than  $\zeta(G)$ . Then,  $C$  is not an isometric cycle since the length of a largest isometric cycle is

$\zeta(G)$ . Thus there exist two vertices  $u$  and  $v$  on  $C$  such that  $d_G(u, v) < d_C(u, v)$ . Let  $P$  be a shortest path between  $u$  and  $v$  in  $G$ . Then a closed trail  $C'$  containing  $e$  is obtained from the segment of  $C$  containing  $e$  between  $u$  and  $v$  by adding  $P$ . Clearly, the length  $\ell(C')$  is less than  $\ell(C)$ . We can get a cycle  $C''$  containing  $e$  from  $C'$ . Thus there exists a cycle  $C''$  containing  $e$  with length less than  $\ell(C)$ , a contradiction. Therefore  $\eta(G) \leq \zeta(G)$ .  $\square$

**Lemma 3** *Let  $G$  be a bridgeless graph and  $u$  be a center of  $G$ . For any  $i \leq \text{rad}(G) - 1$  and every leg  $e$  of  $N_i(u)$ , there exists an optimal  $(N_i[u], e)$ -ear with length at most  $\min\{2(\text{rad}(G) - i) + 1, \eta(G)\}$ .*

**Proof:** Let  $P$  be an optimal  $(N_i[u], e)$ -ear. Since  $e$  belongs to a cycle with length at most  $\eta(G)$ ,  $\ell(P) \leq \eta(G)$ . On the other hand, if  $\ell(P) \geq 2(\text{rad}(G) - i) + 1$ , then the middle vertex of  $P$  has distance at least  $\text{rad}(G) - i + 1$  from  $N_i[u]$ , a contradiction.  $\square$

### 3 Oriented diameter

At first, we have the following observation.

**Observation 1** *Let  $G$  be a bridgeless graph and  $H$  be a bridgeless spanning subgraph of  $G$ . Then the oriented radius (resp. diameter) of  $G$  is not larger than the oriented radius (resp. diameter) of  $H$ .*

**Proof of Theorem 2:** We only need to show that  $G$  has an orientation  $H$  such that

$$\text{rad}(H) \leq \sum_{i=1}^{\text{rad}(G)} \min\{2i, \eta(G) - 1\} \leq \text{rad}(G)(\eta(G) - 1).$$

Let  $u$  be a center of  $G$  and let  $H_0$  be the trivial graph with vertex set  $\{u\}$ . We assert that *there exists a subgraph  $G_i$  of  $G$  such that  $N_i[u] \subseteq V(G_i)$  and  $G_i$  has an orientation  $H_i$  satisfying that  $\text{rad}(H_i) \leq \text{ecc}_{H_i}(u) \leq \sum_{j=1}^i \min\{2(\text{rad}(G) - j), \eta(G) - 1\}$ .*

*Basic step:* When  $i = 1$ , we omit the proof since the proof of this step is similar to that of the following induction step.

*Induction step:* Assume that the above assertion holds for  $i - 1$ . Next we show that the above assertion also holds for  $i$ . For any  $v \in N_i(u)$ , either  $v \in V(H_{i-1})$  or  $v \in N_G(V(H_{i-1}))$  since  $N_{i-1}[u] \subseteq V(H_{i-1})$ . If  $N_i(u) \subseteq V(H_{i-1})$ , then let  $H_i = H_{i-1}$  and we are done. Thus, we suppose  $N_i(u) \not\subseteq V(H_{i-1})$  in the following.

Let  $X = N_i(u) \setminus V(H_{i-1})$ . Pick  $x_1 \in X$ , let  $y_1$  be a neighbor of  $x_1$  in  $H_{i-1}$  and let  $P_1 = P_{y_1 z_1}$  be an optimal  $(H_{i-1}, x_1 y_1)$ -ear. We orient  $P$  such that  $P_1$  is a directed path. Pick  $x_2 \in X$  satisfying that all incident edges of  $x_2$  are not oriented. Let  $y_2$  be a neighbor of  $x_2$  in  $H_{i-1}$ . If there exists an optimal  $(H_{i-1}, x_2 y_2)$ -ear  $P_2$  such that  $P_1$  and  $P_2$  are independent, then we can orient  $P_2$  such that  $P_2$  is a directed path. Otherwise, by Lemma 1 there exists an optimal  $(H_{i-1}, x_2 y_2)$ -ear  $P_2 = P_{y_2 z_2}$  such that  $P_1$  and  $P_2$  has only one continuous common segment containing  $z_2$ . Clearly, we can orient the edges in  $E(P_2) \setminus E(P_1)$  such that  $P_2$  is a directed path. We can pick the vertices of  $X$  and oriented optimal  $H$ -ears similar to the above method until that for any  $x \in X$ , at least two incident edges of  $x$  are oriented. Let  $H_i$  be the graph obtained from  $H_{i-1}$  by adding vertices in  $V(G) \setminus V(H_{i-1})$ , which has at least two new oriented incident edges, and adding the new oriented edges. Clearly,  $N_i[u] \subseteq V(H_i) = V(G_i)$ .

Now we show that  $rad(H_i) \leq \sum_{j=1}^i \min\{2(rad(G) - i), \eta(G) - 1\}$ . It suffices to show that for every vertex  $x$  of  $H_i$ ,  $d_{H_i}(H_{i-1}, x) \leq \min\{2(rad(G) - i), \eta(G) - 1\}$  and  $d_{H_i}(x, H_{i-1}) \leq \min\{2(rad(G) - i), \eta(G) - 1\}$ . If  $x \in V(H_{i-1})$ , then the assertion holds by inductive hypothesis. If  $x \notin V(H_{i-1})$ . Let  $P$  be a directed optimal  $(H_i, e)$ -ear containing  $x$ , where  $e$  is some leg of  $H_{i-1}$  (such a leg and such an ear exists by the definition of  $H_i$ ). By Lemma 3,  $\ell(P) \leq \min\{2(rad(G) - i) + 1, \eta(G)\}$ . Thus,  $d_{H_i}(x, H_{i-1}) \leq \min\{2(rad(G) - i), \eta(G) - 1\}$  and  $d_{H_i}(H_{i-1}, x) \leq \min\{2(rad(G) - i), \eta(G) - 1\}$ . Therefore,  $rad(H_i) \leq \sum_{j=1}^i \min\{2(rad(G) - j), \eta(G) - 1\}$ .  $\square$

**Remark 1** The above theorem is optimal since it implies Chvátal and Thomassen's optimal Theorem 1. Readers can see [5, 11] for optimal examples.

The following example shows that our result is better than that of Theorem 1.

**Example 1** Let  $F_3$  be a triangle with one of its vertices designated as a root. In order to construct  $F_r$ , take two copies of  $F_{r-1}$ . Let  $H_r$  be the graph obtained from the triangle  $u_0, u_1, u_2$  by identifying the root of first (resp. second) copy of  $F_{r-1}$  with  $u_1$  (resp.  $u_2$ ), and  $u_0$  be the root of  $F_r$ . Let  $G_r$  be the graph obtained by taking two copies of  $F_r$  and identifying their roots. See Figure 2 for details. It is easy to check that  $G_r$  has radius  $r$  and every edge belongs to a cycle of length  $\eta(G) = 3$ . By Theorem 1,  $G_r$  has an orientation  $H_r$  such that  $rad(H_r) \leq r^2 + r$  and  $diam(H_r) \leq 2r^2 + 2r$ . But, by Theorem 2,  $G_r$  has an orientation  $H_r$  such that  $rad(G) \leq 2r$  and  $diam(G) \leq 4r$ . On the other hand, it is easy to check that all the strong orientations of  $G_r$  has radius  $2r$  and diameter  $4r$ .

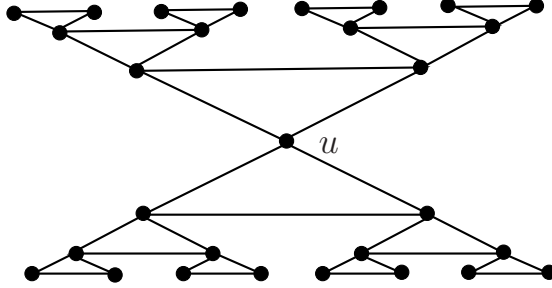


Fig. 2: The graph  $G_3$  which has oriented radius 6 and oriented diameter 12

We have the following result for plane graphs.

**Theorem 5** Let  $G$  be a plane graph. If the length of the boundary of every face is at most  $k$ , then  $G$  has an oriented  $H$  such that  $rad(H) \leq rad(G)(k - 1)$  and  $diam(H) \leq 2rad(G)(k - 1)$ .

Since every edge of a maximal plane (resp. outerplane) graph belongs to a cycle with length 3, the following corollary holds.

**Corollary 1** Let  $G$  be a maximal plane (resp. outerplane) graph. Then there exists an orientation  $H$  of  $G$  such that  $rad(H) \leq 2rad(G)$  and  $diam(H) \leq 4rad(G)$ .

A graph  $G$  is *edge-transitive* if for any  $e_1, e_2 \in E(G)$ , there exists an automorphism  $g$  such that  $g(e_1) = e_2$ . We have the following result for edge-transitive graphs.

**Theorem 6** *Let  $G$  be a bridgeless edge-transitive graph. Then  $G$  has an orientation  $H$  such that  $\text{rad}(H) \leq \text{rad}(G)(g(G) - 1)$  and  $\text{diam}(H) \leq 2\text{rad}(G)(g(G) - 1)$ , where  $g(G)$  is the girth of  $G$ , that is, the length of a smallest induced cycle.*

For general bipartite graphs, the following theorem holds.

**Theorem 7** *Let  $G = (V_1 \cup V_2, E)$  be a bipartite graph with  $|V_1| = n$  and  $|V_2| = m$ . If  $d(x) \geq k > \lceil m/2 \rceil$  for any  $x \in V_1$ ,  $d(y) \geq r > \lceil n/2 \rceil$  for any  $y \in V_2$ , then there exists an orientation  $H$  of  $G$  such that  $\text{rad}(H) \leq 9$ .*

**Proof:** It suffices to show that  $\text{rad}(G) \leq 3$  and  $\eta(G) \leq 4$  by Theorem 2.

First, we show that  $\text{rad}(G) \leq 3$ . Fix a vertex  $x$  in  $G$ , and let  $y$  be any vertex different from  $x$  in  $G$ . If  $x$  and  $y$  belong to the same part, without loss of generality, say  $x, y \in V_1$ . Let  $X$  and  $Y$  be neighborhoods of  $x$  and  $y$  in  $V_2$ , respectively. If  $X \cap Y = \emptyset$ , then  $|V_2| \geq |X| + |Y| \geq 2k > m$ , a contradiction. Thus  $X \cap Y \neq \emptyset$ , that is, there exists a path between  $x$  and  $y$  of length two. If  $x$  and  $y$  belong to different parts, without loss of generality, say  $x \in V_1, y \in V_2$ . Suppose  $x$  and  $y$  are nonadjacent, otherwise there is nothing to prove. Let  $X$  and  $Y$  be the neighborhoods of  $x$  and  $y$  in  $G$ , and let  $X'$  be the set of neighbors of  $X$  except for  $x$  in  $G$ . If  $X' \cap Y = \emptyset$ , then  $|V_1| \geq 1 + |Y| + |X'| \geq 1 + r + (r - 1) = 2r > n$ , a contradiction (Note that  $|X'| \geq r - 1$ ). Thus  $X' \cap Y \neq \emptyset$ , that is, there exists a path between  $x$  and  $y$  of length three in  $G$ .

Next we show that  $\eta(G) \leq 4$ . Let  $xy$  be any edge in  $G$ . Let  $X$  be the set of neighbors of  $x$  except for  $y$  in  $G$ , let  $Y$  be the set of neighbors of  $y$  except for  $x$  in  $G$ , let  $X'$  be the set of neighbors of  $X$  except for  $x$  in  $G$ . If  $X' \cap Y = \emptyset$ , then  $|V_1| \geq 1 + |Y| + |X'| \geq 1 + (r - 1) + (r - 1) = 2r - 1 > n$ , a contradiction (Note that  $|X'| \geq r - 1$ ). Thus  $X' \cap Y \neq \emptyset$ , that is, there exists a cycle containing  $xy$  of length four in  $G$ .  $\square$

**Remark 2** *The degree condition is optimal. Let  $m, n$  be two even numbers with  $n, m \geq 2$ . Since  $K_{n/2, m/2} \cup K_{n/2, m/2}$  is disconnected, the oriented radius (resp. diameter) of  $K_{n/2, m/2} \cup K_{n/2, m/2}$  is  $\infty$ .*

For equal bipartition  $k$ -regular graph, the following corollary holds.

**Corollary 2** *Let  $G = (V_1 \cup V_2, E)$  be a  $k$ -regular bipartite graph with  $|V_1| = |V_2| = n$ . If  $k > n/2$ , then there exists an orientation  $H$  of  $G$  such that  $\text{rad}(H) \leq 9$ .*

The following theorem holds for general graphs.

**Theorem 8** *Let  $G$  be a graph of order  $n$ .*

(i) *If there exists an integer  $k \geq 2$  such that  $|N_k(u)| > n/2 - 1$  for every vertex  $u$  in  $G$ , then  $G$  has an orientation  $H$  such that  $\text{rad}(H) \leq 4k^2$  and  $\text{diam}(H) \leq 8k^2$ .*

(ii) *If  $\delta(G) > n/2$ , then  $G$  has an orientation  $H$  such that  $\text{rad}(H) \leq 4$  and  $\text{diam}(H) \leq 8$ .*

**Proof:** Since methods of the proofs of (i) and (ii) are similar, we only prove (i). For (i), it suffices to show that  $\text{rad}(G) \leq 2k$  and  $\eta(G) \leq 2k + 1$  by Theorem 2.

We first show  $\text{rad}(G) \leq 2k$ . Fix  $u$  in  $G$ , for every  $v \in V(G)$ , if  $v \in N_k[u]$ , then  $d(u, v) \leq k$ . Suppose  $v \notin N_k[u]$ , we have  $N_k(u) \cap N_k(v) \neq \emptyset$ . If not, that is,  $N_k(u) \cap N_k(v) = \emptyset$ , then  $|N_k(u)| + |N_k(v)| + 2 > n$  (a contradiction). Thus  $d(u, v) \leq 2k$ .



Next we show  $\eta(G) \leq 2k + 1$ . Let  $e = uv$  be any edge in  $G$ . If  $N_k(u) \cap N_k(v) = \emptyset$ , then  $|V(G)| \geq |N_k(u)| + |N_k(v)| + 2 > n$ , a contradiction. Thus  $N_k(u) \cap N_k(v) \neq \emptyset$ . Pick  $w \in N_k(u) \cap N_k(v)$ , and let  $P$  (resp.  $Q$ ) be a path between  $u$  and  $w$  (resp. between  $v$  and  $w$ ). Then  $e$  belongs a close trial  $uPwQvu$  of length  $2k + 1$ . Therefore,  $e$  belongs a cycle of length at most  $2k + 1$ .  $\square$

**Remark 3** *The above condition is almost optimal since  $K_{n/2} \cup K_{n/2}$  is disconnected for even  $n$ .*

**Corollary 3** *Let  $G$  be a graph with minimum degree  $\delta(G)$  and girth  $g(G)$ . If there exists an integer  $k$  such that  $k \leq g(G)/2$  and  $\delta(G)(\delta(G) - 1)^{k-1} > n/2 - 1$ , then  $G$  has an orientation  $H$  such that  $\text{rad}(H) \leq 4k^2$ .*

**Proof:** Let  $k$  be an integer such that  $k \leq g(G)/2$  and  $\delta(G)(\delta(G) - 1)^{k-1} > n/2 - 1$ . For any vertex  $u$  of  $G$ , let  $1 \leq i < k$  be any integer and  $x, y \in N_i(u)$ . If  $x$  and  $y$  have a common neighbor  $z$  in  $N_{i+1}(u)$ , then  $G$  has a cycle of length at most  $2i < 2k \leq g(G)/2$ , a contradiction. Thus  $x$  and  $y$  has no common neighbor in  $N_{i+1}(u)$ . Therefore,  $|N_k(u)| \geq \delta(G)(\delta(G) - 1)^{k-1} > n/2 - 1$ . By Theorem 2,  $G$  has an orientation  $H$  such that  $\text{rad}(H) \leq 4k^2$ .  $\square$

## 4 Upper bound for rainbow connection number

At first, we have the following observation.

**Observation 2** *Let  $G$  be a graph and  $H$  be a spanning subgraph of  $G$ . Then  $rc(G) \leq rc(H)$ .*

**Proof of Theorem 4:** Let  $u$  be a center of  $G$  and let  $H_0$  be the trivial graph with vertex set  $\{u\}$ . We assert that *there exists a subgraph  $H_i$  of  $G$  such that  $N_i[u] \subseteq V(H_i)$  and  $rc(H_i) \leq \sum_{j=1}^i \min\{2(\text{rad}(G) - j) + 1, \eta(G)\}$ .*

*Basic step:* When  $i = 1$ , we omit the proof since the proof of this step is similar to that of the following induction step.

*Induction step:* Assume that the above assertion holds for  $i - 1$  and  $c$  is a  $rc(H_{i-1})$ -rainbow coloring of  $H_{i-1}$ . Next we show that the above assertion holds for  $i$ . For any  $v \in N_i(u)$ , either  $v \in V(H_{i-1})$  or  $v \in N_G(V(H_{i-1}))$  since  $N_{i-1}[u] \subseteq V(H_{i-1})$ . If  $N_i(u) \subseteq V(H_{i-1})$ , then let  $H_i = H_{i-1}$  and we are done. Thus, we suppose  $N_i(u) \not\subseteq V(H_{i-1})$  in the following.

Let  $C_1 = \{\alpha_1, \alpha_2, \dots\}$  and  $C_2 = \{\beta_1, \beta_2, \dots\}$  be two pools of colors, none of which are used to color  $H_{i-1}$ , and there exists no common color in  $C_1$  and  $C_2$ . An edge-coloring of an  $H$ -ear  $P = (u_0, u_1, \dots, u_k)$  is a *symmetrical coloring* if its edges are colored by  $\alpha_1, \alpha_2, \dots, \alpha_{\lceil k/2 \rceil}, \beta_{\lfloor k/2 \rfloor}, \dots, \beta_2, \beta_1$  in that order or  $\beta_1, \beta_2, \dots, \beta_{\lfloor k/2 \rfloor}, \alpha_{\lceil k/2 \rceil}, \dots, \alpha_2, \alpha_1$  in that order.

Let  $X = N_i(u) \setminus V(H_{i-1})$  and  $m = \min\{2(\text{rad}(G) - i) + 1, \eta(G)\}$ . Pick  $x_1 \in X$ . Let  $y_1$  be a neighbor of  $x_1$  in  $H_{i-1}$  and  $P_1$  be an optimal  $(H_{i-1}, x_1y_1)$ -ear. We can color  $P$  symmetrically with colors  $\alpha_1, \alpha_2, \dots, \alpha_{\lceil \ell(P)/2 \rceil}, \beta_{\lfloor \ell(P)/2 \rfloor}, \dots, \beta_2, \beta_1$ . Pick  $x_2 \in X$  satisfying that all the incident edges of  $x_2$  are not colored. Let  $y_2$  be a neighbor of  $x_2$  in  $H_{i-1}$ . If there exists an optimal  $(H_{i-1}, x_2y_2)$ -ear  $P_2$  such that  $P_1$  and  $P_2$  are independent, then we can color  $P_2$  symmetrically with colors  $\alpha_1, \alpha_2, \dots, \alpha_{\lceil \ell(P_2)/2 \rceil}, \beta_{\lfloor \ell(P_2)/2 \rfloor}, \dots, \beta_2, \beta_1$ . Otherwise, by Lemma 1, there exists an optimal  $(H_{i-1}, x_2y_2)$ -ear  $P_2 = P_{y_2z_2}$  such that  $P_1$  and  $P_2$  have only one continuous common segment containing  $z_2$ , where  $z_2$  is the other foot of  $P_2$ . Thus we can color  $P_2$  symmetrically with colors

$\alpha_1, \alpha_2, \dots, \alpha_{\lceil \ell(P_2)/2 \rceil}, \beta_{\lfloor \ell(P_2)/2 \rfloor}, \dots, \beta_2, \beta_1$  by preserving the coloring of  $P_1$ . We can pick the vertices of  $X$  and color optimal  $H_i$ -ears until that for any  $x \in X$ , at least two incident edges of  $x$  are colored. Since for any leg  $e$  of  $H_{i-1}$ ,  $\ell(e) \leq m$  by Lemma 3, we use at most  $m$  coloring in the above coloring process.

Let  $H_i$  be the graph obtained from  $H_{i-1}$  by adding all the vertices in  $V(G) \setminus V(H_{i-1})$ , which have at least two new colored incident edges, and adding the new colored edges. Clearly,  $N_i[u] \subseteq V(H_i)$ . It suffices to show that  $H_i$  is rainbow connected. Let  $x$  and  $y$  be two distinct vertices in  $H_i$ . If  $x, y \in V(H_{i-1})$ , then there exists a rainbow path between  $x$  and  $y$  by inductive hypothesis. If exactly one of  $x$  and  $y$  belongs to  $V(H_{i-1})$ , say  $x$ . Let  $P$  be a symmetrical colored  $H_{i-1}$ -ear containing  $y$ , and  $y'$  be a foot of  $P$ . There exists a rainbow path  $Q$  between  $x$  and  $y'$  in  $H_{i-1}$  by inductive hypothesis. Thus,  $xQy'Py$  is a rainbow path between  $x$  and  $y$  in  $H_i$ .

Suppose none of  $x$  and  $y$  belongs to  $H_{i-1}$ . Let  $P$  and  $Q$  be symmetrical colored  $H_{i-1}$ -ear containing  $x$  and  $y$ , respectively. Furthermore, let  $x', x''$  be the feet of  $P$  and  $y', y''$  be the feet of  $Q$ . Without loss of generality, assume that  $P$  is colored from  $x'$  to  $x''$  by  $\alpha_1, \alpha_2, \dots, \alpha_{\lceil \ell(P)/2 \rceil}, \beta_{\lfloor \ell(P)/2 \rfloor}, \dots, \beta_2, \beta_1$  in that order, and  $Q$  is colored from  $y'$  to  $y''$  by  $\alpha_1, \alpha_2, \dots, \alpha_{\lceil \ell(Q)/2 \rceil}, \beta_{\lfloor \ell(Q)/2 \rfloor}, \dots, \beta_2, \beta_1$  in that order. If  $\ell(x'Px) \leq \ell(y'Qy)$ , let  $R$  be a rainbow path between  $x'$  and  $y''$  in  $H_{i-1}$ . Then  $xPx'Ry''Qy$  is a rainbow path between  $x$  and  $y$  in  $H_i$ . Otherwise,  $\ell(x'Px) > \ell(y'Qy)$ . Let  $R$  be a rainbow path between  $y'$  and  $x''$  in  $H_{i-1}$ . Then  $yQy'Rx''Px$  is a rainbow path between  $x$  and  $y$  in  $H_i$ . Thus, there exists a rainbow path between any two distinct vertices in  $H_i$ , that is,  $H_i$  is  $(\sum_{j=1}^i \min\{2(\text{rad}(G) - j) + 1, \eta(G)\})$ -rainbow connected.  $\square$  Readers can see [1] for an optimal example. The following example shows that our result is better than that of Theorem 3.

**Example 2** Let  $r \geq 3, k \geq 2r$  be two integers, and  $W_k = C_k \vee K_1$  be a wheel, where  $V(C_k) = \{u_1, u_2, \dots, u_k\}$  and  $V(K_1) = \{u\}$ . Let  $H$  be the graph obtained from  $W_k$  by inserting  $r - 1$  vertices between every edge  $uu_i$ ,  $1 \leq i \leq k$ . For every edge  $e = xy$  of  $H$ , add a new vertex  $v_e$  and new edges  $v_ex, v_ey$ . Denote by  $G$  the resulting graph. It is easy to check that  $\text{rad}(G) = r$ ,  $\text{diam}(G) = 2r$ ,  $\eta(G) = 3$  and  $\zeta(G) = 2r - 1$ . By Theorem 3, we have  $rc(G) \leq \sum_{i=1}^r \min\{2i + 1, \zeta(G)\} \leq r^2 + 2r - 2$ . But, by Theorem 4 we have  $rc(G) \leq 3r$ . On the other hand,  $rc(G) \geq 2r$  since  $\text{diam}(G) = 2r$ .

The remaining results are similar to those in Section 3.

**Theorem 9** Let  $G$  be a plane graph. If the length of the boundary of every face is at most  $k$ , then  $rc(G) \leq k \text{rad}(G)$ .

**Corollary 4** Let  $G$  be a maximal plane (resp. outerplane) graph. Then  $rc(G) \leq 3 \text{rad}(G)$ .

**Theorem 10** Let  $G$  be a bridgeless edge-transitive graph. Then  $rc(G) \leq \text{rad}(G)g(G)$ , where  $g(G)$  is the girth of  $G$ .

**Theorem 11** Let  $G = (V_1 \cup V_2, E)$  be a bipartite graph with  $|V_1| = n$  and  $|V_2| = m$ . If  $d(x) \geq k > \lceil m/2 \rceil$  for any  $x \in V_1$ ,  $d(y) \geq r > \lceil n/2 \rceil$  for any  $y \in V_2$ , then  $rc(G) \leq 12$ .

**Remark 4** The degree condition is optimal. Let  $m, n$  be two even numbers with  $n, m \geq 2$ . Since  $K_{n/2, m/2} \cup K_{n/2, m/2}$  is disconnected,  $rc(K_{n/2, m/2} \cup K_{n/2, m/2}) = \infty$ .

**Corollary 5** Let  $G = (V_1 \cup V_2, E)$  be a  $k$ -regular bipartite graph with  $|V_1| = |V_2| = n$ . If  $k > \lceil n/2 \rceil$ , then  $rc(G) \leq 12$ .

The following theorem holds for general graphs.

**Theorem 12** Let  $G$  be a graph.

(i) If there exists an integer  $k \geq 2$  such that  $|N_k(u)| > n/2 - 1$  for every vertex  $u$  in  $G$ , then  $rc(G) \leq 4k^2 + 2k$ .

(ii) If  $\delta(G) > n/2$ , then  $rc(G) \leq 6$ .

**Remark 5** The above condition is almost optimal since  $K_{n/2} \cup K_{n/2}$  is disconnected for even  $n$ .

**Corollary 6** Let  $G$  be a graph with minimum degree  $\delta(G)$  and girth  $g(G)$ . If there exists an integer  $k$  such that  $k < g(G)/2$  and  $\delta(G)(\delta(G) - 1)^{k-1} > n/2 - 1$ , then  $rc(G) \leq 4k^2 + 2k$ .

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