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Oriented diameter and rainbow connection number of a graph*

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The oriented diameter of a bridgeless graph G is $\min\{\text{diam}(H) \mid H \text{ is a strong orientation of } G\}$. A path in an edge-colored graph G , where adjacent edges may have the same color, is called rainbow if no two edges of the path are colored the same. The rainbow connection number $rc(G)$ of G is the smallest integer number k for which there exists a k -edge-coloring of G such that every two distinct vertices of G are connected by a rainbow path. In this paper, we obtain upper bounds for the oriented diameter and the rainbow connection number of a graph in terms of $rad(G)$ and $\eta(G)$, where $rad(G)$ is the radius of G and $\eta(G)$ is the smallest integer number such that every edge of G is contained in a cycle of length at most $\eta(G)$. We also obtain constant bounds of the oriented diameter and the rainbow connection number for a (bipartite) graph G in terms of the minimum degree of G .

Keywords: Diameter, Radius, Oriented diameter, Rainbow connection number, Cycle length, Bipartite graph.

1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to the book [2] for notation and terminology not described here. A path $u = u_1, u_2, \dots, u_k = v$ is called a $P_{u,v}$ path. Denote by $u_i P u_j$ the subpath u_i, u_{i+1}, \dots, u_j for $i \leq j$. The length $\ell(P)$ of a path P is the number of edges in P . The distance between two vertices x and y in G , denoted by $d_G(x, y)$, is the length of a shortest path between them. The eccentricity of a vertex x in G is $\text{ecc}_G(x) = \max_{y \in V(G)} d(x, y)$. The radius and diameter of G are $rad(G) = \min_{x \in V(G)} \text{ecc}(x)$ and $\text{diam}(G) = \max_{x \in V(G)} \text{ecc}(x)$, respectively. A vertex u is a center of a graph G if $\text{ecc}(u) = rad(G)$. The oriented diameter of a bridgeless graph G is $\min\{\text{diam}(H) \mid H \text{ is an orientation of } G\}$, and the oriented radius of a bridgeless graph G is $\min\{rad(H) \mid H \text{ is an orientation of } G\}$. For any graph G with edge-connectivity $\lambda(G) = 0, 1$, G has oriented radius (resp. diameter) ∞ .

In 1939, Robbins solved the One-Way Street Problem and proved that a graph G admits a strongly connected orientation if and only if G is bridgeless, that is, G does not have any cut-edge. Naturally, one hopes that the oriented diameter of a bridgeless graph is as small as possible. Bondy and Murty suggested

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to study the quantitative variations on Robbins' theorem. In particular, they conjectured that there exists a function f such that every bridgeless graph with diameter d admits an orientation of diameter at most $f(d)$.

In 1978, Chvátal and Thomassen [5] obtained some general bounds.

Theorem 1 (Chvátal and Thomassen 1978 [5]) *For every bridgeless graph G , there exists an orientation H of G such that*

$$\begin{aligned} \text{rad}(H) &\leq \text{rad}(G)^2 + \text{rad}(G), \\ \text{diam}(H) &\leq 2\text{rad}(G)^2 + 2\text{rad}(G). \end{aligned}$$

Moreover, the above bounds are optimal.

There exists a minor error when they constructed the graph G_d which arrives at the upper bound when d is odd. Kwok, Liu and West gave a slight correction in [11].

They also showed that determining whether an arbitrary graph can be oriented so that its diameter is at most 2 is NP-complete. Bounds for the oriented diameter of graphs have also been studied in terms of other parameters, for example, radius, dominating number [5, 6, 11, 18], etc. Some classes of graphs have also been studied in [6, 7, 8, 9, 14].

Let $\eta(G)$ be the smallest integer such that every edge of G belongs to a cycle of length at most $\eta(G)$. In this paper, we show the following result.

Theorem 2 *For every bridgeless graph G , there exists an orientation H of G such that*

$$\begin{aligned} \text{rad}(H) &\leq \sum_{i=1}^{\text{rad}(G)} \min\{2i, \eta(G) - 1\} \leq \text{rad}(G)(\eta(G) - 1), \\ \text{diam}(H) &\leq 2 \sum_{i=1}^{\text{rad}(G)} \min\{2i, \eta(G) - 1\} \leq 2\text{rad}(G)(\eta(G) - 1). \end{aligned}$$

Note that $\sum_{i=1}^{\text{rad}(G)} \min\{2i, \eta(G) - 1\} \leq \text{rad}(G)^2 + \text{rad}(G)$ and $\text{diam}(H) \leq 2\text{rad}(H)$. So our result implies Chvátal and Thomassen's Theorem 1.

A path in an edge-colored graph G , where adjacent edges may have the same color, is called *rainbow* if no two edges of the path are colored the same. An edge-coloring of a graph G is a *rainbow edge-coloring* if every two distinct vertices of the graph G are connected by a rainbow path. The *rainbow connection number* $rc(G)$ of G is the minimum integer k for which there exists a rainbow k -edge-coloring of G . It is easy to see that $\text{diam}(G) \leq rc(G)$ for any connected graph G . The rainbow connection number was introduced by Chartrand et al. in [4]. It is of great use in transferring information of high security in multicomputer networks. We refer the readers to [3] for details.

Chakraborty et al. [3] investigated the hardness and algorithms for the rainbow connection number, and showed that given a graph G , deciding if $rc(G) = 2$ is NP-complete. Bounds for the rainbow connection number of a graph have also been studied in terms of other graph parameters, for example, radius, dominating number, minimum degree, connectivity, etc. [1, 4, 10]. Cayley graphs and line graphs were studied in [12] and [13], respectively.

A subgraph H of a graph G is called *isometric* if the distance between any two distinct vertices in H is the same as their distance in G . The size of a largest isometric cycle in G is denoted by $\zeta(G)$.

Clearly, every isometric cycle is an induced cycle and thus $\zeta(G)$ is not larger than the chordality, where *chordality* is the length of a largest induced cycle in G . In [1], Basavaraju, Chandran, Rajendraprasad and Ramaswamy got the the following sharp upper bound for the rainbow connection number of a bridgeless graph G in terms of $rad(G)$ and $\zeta(G)$.

Theorem 3 (Basavaraju et al. [1]) *For every bridgeless graph G ,*

$$rc(G) \leq \sum_{i=1}^{rad(G)} \min\{2i + 1, \zeta(G)\} \leq rad(G)\zeta(G).$$

In this paper, we show the following result.

Theorem 4 *For every bridgeless graph G ,*

$$rc(G) \leq \sum_{i=1}^{rad(G)} \min\{2i + 1, \eta(G)\} \leq rad(G)\eta(G).$$

From Lemma 2 of Section 2, we will see that $\eta(G) \leq \zeta(G)$. Thus our result implies Theorem 3.

This paper is organized as follows: in Section 2, we introduce some new definitions and show several lemmas. In Section 3, we prove Theorem 2 and study upper bounds for the oriented radius (resp. diameter) of plane graphs, edge-transitive graphs and general (bipartite) graphs. In Section 4, we prove Theorem 4 and study upper for the rainbow connection number of plane graphs, edge-transitive graphs and general (bipartite) graphs.

2 Preliminaries

In this section, we introduce some definitions and show several lemmas.

Definition 1 For any $x \in V(G)$ and $k \geq 0$, the *k-step open neighborhood* is $\{y \mid d(x, y) = k\}$ and denoted by $N_k(x)$, the *k-step closed neighborhood* is $\{y \mid d(x, y) \leq k\}$ and denoted by $N_k[x]$. If $k = 1$, we simply write $N(x)$ and $N[x]$ for $N_1(x)$ and $N_1[x]$, respectively.

Definition 2 Let G be a graph and H be a subset of $V(G)$ (or a subgraph of G). The edges between H and $G \setminus H$ are called *legs* of H . An *H-ear* is a path $P = (u_0, u_1, \dots, u_k)$ in G such that $V(H) \cap V(P) = \{u_0, u_k\}$. The vertices u_0, u_k are called the *feet* of P in H and $u_0u_1, u_{k-1}u_k$ are called the *legs* of P . The *length* of an *H-ear* is the length of the corresponding path. If $u_0 = u_k$, then P is called a *closed H-ear*. For any leg e of H , denote by $\ell(e)$ the smallest number such that there exists an *H-ear* of length $\ell(e)$ containing e , and such an *H-ear* is called an *optimal (H, e)-ear*.

Note that for any optimal (H, e) -ear P and every pair $(x, y) \neq (u_0, u_k)$ of distinct vertices of P , x and y are adjacent on P if and only if x and y are adjacent in G .

Definition 3 For any two paths P and Q , the joint of P and Q are the common vertex and edge of P and Q . Paths P and Q have *k continuous common segments* if the common vertex and edge are k disjoint paths.

A common segment is trivial if it has only one vertex.

Definition 4 Let P and Q be two paths in G . Call P and Q *independent* if they has no common internal vertex.

Lemma 1 Let $n \geq 1$ be an integer, and let G be a graph, H be a subgraph of G and $e_i = u_i v_i$ be a leg of H and $P_i = P_{u_i w_i}$ be an optimal (G, e_i) -ear, where $1 \leq i \leq n$ and u_i, w_i are the foot of P_i . Then for any leg $e_j = u_j v_j$ such that $e_j \neq e_i$ and $e_j \notin E(P_i)$, where $i \in \{1, 2, \dots, n\}$, there exists an optimal (H, e_j) -ear $P_j = P_{u_j w_j}$ such that either P_i and P_j are independent for any P_i , $1 \leq i \leq n$, or P_i and P_j have only one continuous common segment containing w_j for some P_i .

Proof: Let P_j be an optimal (H, e_j) -ear. If P_i and P_j are independent for any i , then we are done. Suppose that P_i and P_j have m continuous common segments for some i , where $m \geq 1$. When $m \geq 2$, we first

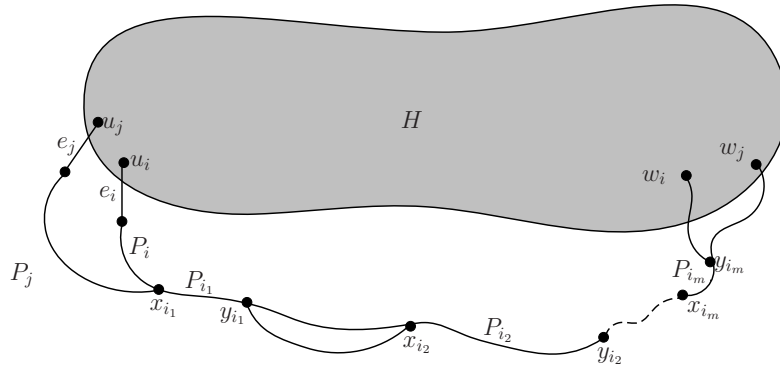


Fig. 1: Two H -ears P_i and P_j

construct an optimal (H, e_j) -ear P_j^* such that P_i and P_j^* has only one continuous common segment. Let $P_{i_1}, P_{i_2}, \dots, P_{i_m}$ be the m continuous common segments of P_i and P_j and they appear in P_i in that order. See Figure 1 for details. Furthermore, suppose that x_{i_k} and y_{i_k} are the two ends of the path P_{i_k} and they appear in P_i successively. We say that the following claim holds.

Claim 1: $\ell(y_k P_i x_{k+1}) = \ell(y_k P_j x_{k+1})$ for any $1 \leq k \leq m - 1$.

If not, that is, there exists an integer k such that $\ell(y_k P_i x_{k+1}) \neq \ell(y_k P_j x_{k+1})$. Without loss of generality, we assume $\ell(y_k P_i x_{k+1}) < \ell(y_k P_j x_{k+1})$. Then we shall get a more shorter path H -ear containing e_j by replacing $y_k P_j x_{k+1}$ with $y_k P_i x_{k+1}$, a contradiction. Thus $\ell(y_k P_i x_{k+1}) = \ell(y_k P_j x_{k+1})$ for any k .

Let P_j^* be the path obtained from P_j by replacing $y_k P_j x_{k+1}$ with $y_k P_i x_{k+1}$, and let $P_j = P_j^*$. If the continuous common segment of P_i and P_j does not contain w_j . Suppose x and y are the two ends of the common segment such that x and y appeared on P starting from u_i to w_i successively. Similar to Claim 1, $\ell(y P_i w_i) = \ell(y P_j w_j)$. Let P_j^* be the path obtained from P_j by replacing $y P_j w_j$ with $y P_i w_i$. Clearly, P_j^* is our desired optimal $(H, u_j v_j)$ -ear. \square

Lemma 2 For every bridgeless graph G , $\eta(G) \leq \zeta(G)$.

Proof: Suppose that there exists an edge e such that the length $\ell(C)$ of the smallest cycle C containing e is larger than $\zeta(G)$. Then, C is not an isometric cycle since the length of a largest isometric cycle is

$\zeta(G)$. Thus there exist two vertices u and v on C such that $d_G(u, v) < d_C(u, v)$. Let P be a shortest path between u and v in G . Then a closed trail C' containing e is obtained from the segment of C containing e between u and v by adding P . Clearly, the length $\ell(C')$ is less than $\ell(C)$. We can get a cycle C'' containing e from C' . Thus there exists a cycle C'' containing e with length less than $\ell(C)$, a contradiction. Therefore $\eta(G) \leq \zeta(G)$. \square

Lemma 3 *Let G be a bridgeless graph and u be a center of G . For any $i \leq \text{rad}(G) - 1$ and every leg e of $N_i(u)$, there exists an optimal $(N_i[u], e)$ -ear with length at most $\min\{2(\text{rad}(G) - i) + 1, \eta(G)\}$.*

Proof: Let P be an optimal $(N_i[u], e)$ -ear. Since e belongs to a cycle with length at most $\eta(G)$, $\ell(P) \leq \eta(G)$. On the other hand, if $\ell(P) \geq 2(\text{rad}(G) - i) + 1$, then the middle vertex of P has distance at least $\text{rad}(G) - i + 1$ from $N_i[u]$, a contradiction. \square

3 Oriented diameter

At first, we have the following observation.

Observation 1 *Let G be a bridgeless graph and H be a bridgeless spanning subgraph of G . Then the oriented radius (resp. diameter) of G is not larger than the oriented radius (resp. diameter) of H .*

Proof of Theorem 2: We only need to show that G has an orientation H such that

$$\text{rad}(H) \leq \sum_{i=1}^{\text{rad}(G)} \min\{2i, \eta(G) - 1\} \leq \text{rad}(G)(\eta(G) - 1).$$

Let u be a center of G and let H_0 be the trivial graph with vertex set $\{u\}$. We assert that *there exists a subgraph G_i of G such that $N_i[u] \subseteq V(G_i)$ and G_i has an orientation H_i satisfying that $\text{rad}(H_i) \leq \text{ecc}_{H_i}(u) \leq \sum_{j=1}^i \min\{2(\text{rad}(G) - j), \eta(G) - 1\}$.*

Basic step: When $i = 1$, we omit the proof since the proof of this step is similar to that of the following induction step.

Induction step: Assume that the above assertion holds for $i - 1$. Next we show that the above assertion also holds for i . For any $v \in N_i(u)$, either $v \in V(H_{i-1})$ or $v \in N_G(V(H_{i-1}))$ since $N_{i-1}[u] \subseteq V(H_{i-1})$. If $N_i(u) \subseteq V(H_{i-1})$, then let $H_i = H_{i-1}$ and we are done. Thus, we suppose $N_i(u) \not\subseteq V(H_{i-1})$ in the following.

Let $X = N_i(u) \setminus V(H_{i-1})$. Pick $x_1 \in X$, let y_1 be a neighbor of x_1 in H_{i-1} and let $P_1 = P_{y_1 z_1}$ be an optimal $(H_{i-1}, x_1 y_1)$ -ear. We orient P such that P_1 is a directed path. Pick $x_2 \in X$ satisfying that all incident edges of x_2 are not oriented. Let y_2 be a neighbor of x_2 in H_{i-1} . If there exists an optimal $(H_{i-1}, x_2 y_2)$ -ear P_2 such that P_1 and P_2 are independent, then we can orient P_2 such that P_2 is a directed path. Otherwise, by Lemma 1 there exists an optimal $(H_{i-1}, x_2 y_2)$ -ear $P_2 = P_{y_2 z_2}$ such that P_1 and P_2 has only one continuous common segment containing z_2 . Clearly, we can orient the edges in $E(P_2) \setminus E(P_1)$ such that P_2 is a directed path. We can pick the vertices of X and oriented optimal H -ears similar to the above method until that for any $x \in X$, at least two incident edges of x are oriented. Let H_i be the graph obtained from H_{i-1} by adding vertices in $V(G) \setminus V(H_{i-1})$, which has at least two new oriented incident edges, and adding the new oriented edges. Clearly, $N_i[u] \subseteq V(H_i) = V(G_i)$.

Now we show that $rad(H_i) \leq \sum_{j=1}^i \min\{2(rad(G) - i), \eta(G) - 1\}$. It suffices to show that for every vertex x of H_i , $d_{H_i}(H_{i-1}, x) \leq \min\{2(rad(G) - i), \eta(G) - 1\}$ and $d_{H_i}(x, H_{i-1}) \leq \min\{2(rad(G) - i), \eta(G) - 1\}$. If $x \in V(H_{i-1})$, then the assertion holds by inductive hypothesis. If $x \notin V(H_{i-1})$. Let P be a directed optimal (H_i, e) -ear containing x , where e is some leg of H_{i-1} (such a leg and such an ear exists by the definition of H_i). By Lemma 3, $\ell(P) \leq \min\{2(rad(G) - i) + 1, \eta(G)\}$. Thus, $d_{H_i}(x, H_{i-1}) \leq \min\{2(rad(G) - i), \eta(G) - 1\}$ and $d_{H_i}(H_{i-1}, x) \leq \min\{2(rad(G) - i), \eta(G) - 1\}$. Therefore, $rad(H_i) \leq \sum_{j=1}^i \min\{2(rad(G) - j), \eta(G) - 1\}$. \square

Remark 1 The above theorem is optimal since it implies Chvátal and Thomassen's optimal Theorem 1. Readers can see [5, 11] for optimal examples.

The following example shows that our result is better than that of Theorem 1.

Example 1 Let F_3 be a triangle with one of its vertices designated as a root. In order to construct F_r , take two copies of F_{r-1} . Let H_r be the graph obtained from the triangle u_0, u_1, u_2 by identifying the root of first (resp. second) copy of F_{r-1} with u_1 (resp. u_2), and u_0 be the root of F_r . Let G_r be the graph obtained by taking two copies of F_r and identifying their roots. See Figure 2 for details. It is easy to check that G_r has radius r and every edge belongs to a cycle of length $\eta(G) = 3$. By Theorem 1, G_r has an orientation H_r such that $rad(H_r) \leq r^2 + r$ and $diam(H_r) \leq 2r^2 + 2r$. But, by Theorem 2, G_r has an orientation H_r such that $rad(G) \leq 2r$ and $diam(G) \leq 4r$. On the other hand, it is easy to check that all the strong orientations of G_r has radius $2r$ and diameter $4r$.

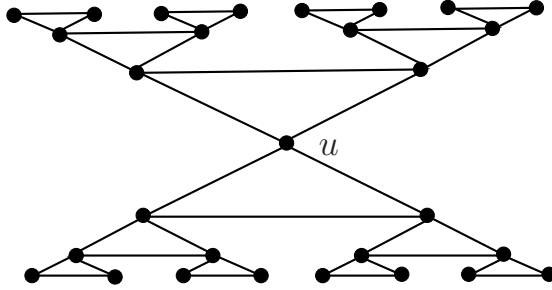


Fig. 2: The graph G_3 which has oriented radius 6 and oriented diameter 12

We have the following result for plane graphs.

Theorem 5 Let G be a plane graph. If the length of the boundary of every face is at most k , then G has an oriented H such that $rad(H) \leq rad(G)(k - 1)$ and $diam(H) \leq 2rad(G)(k - 1)$.

Since every edge of a maximal plane (resp. outerplane) graph belongs to a cycle with length 3, the following corollary holds.

Corollary 1 Let G be a maximal plane (resp. outerplane) graph. Then there exists an orientation H of G such that $rad(H) \leq 2rad(G)$ and $diam(H) \leq 4rad(G)$.

A graph G is *edge-transitive* if for any $e_1, e_2 \in E(G)$, there exists an automorphism g such that $g(e_1) = e_2$. We have the following result for edge-transitive graphs.

Theorem 6 *Let G be a bridgeless edge-transitive graph. Then G has an orientation H such that $\text{rad}(H) \leq \text{rad}(G)(g(G) - 1)$ and $\text{diam}(H) \leq 2\text{rad}(G)(g(G) - 1)$, where $g(G)$ is the girth of G , that is, the length of a smallest induced cycle.*

For general bipartite graphs, the following theorem holds.

Theorem 7 *Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with $|V_1| = n$ and $|V_2| = m$. If $d(x) \geq k > \lceil m/2 \rceil$ for any $x \in V_1$, $d(y) \geq r > \lceil n/2 \rceil$ for any $y \in V_2$, then there exists an orientation H of G such that $\text{rad}(H) \leq 9$.*

Proof: It suffices to show that $\text{rad}(G) \leq 3$ and $\eta(G) \leq 4$ by Theorem 2.

First, we show that $\text{rad}(G) \leq 3$. Fix a vertex x in G , and let y be any vertex different from x in G . If x and y belong to the same part, without loss of generality, say $x, y \in V_1$. Let X and Y be neighborhoods of x and y in V_2 , respectively. If $X \cap Y = \emptyset$, then $|V_2| \geq |X| + |Y| \geq 2k > m$, a contradiction. Thus $X \cap Y \neq \emptyset$, that is, there exists a path between x and y of length two. If x and y belong to different parts, without loss of generality, say $x \in V_1, y \in V_2$. Suppose x and y are nonadjacent, otherwise there is nothing to prove. Let X and Y be the neighborhoods of x and y in G , and let X' be the set of neighbors of X except for x in G . If $X' \cap Y = \emptyset$, then $|V_1| \geq 1 + |Y| + |X'| \geq 1 + r + (r - 1) = 2r > n$, a contradiction (Note that $|X'| \geq r - 1$). Thus $X' \cap Y \neq \emptyset$, that is, there exists a path between x and y of length three in G .

Next we show that $\eta(G) \leq 4$. Let xy be any edge in G . Let X be the set of neighbors of x except for y in G , let Y be the set of neighbors of y except for x in G , let X' be the set of neighbors of X except for x in G . If $X' \cap Y = \emptyset$, then $|V_1| \geq 1 + |Y| + |X'| \geq 1 + (r - 1) + (r - 1) = 2r - 1 > n$, a contradiction (Note that $|X'| \geq r - 1$). Thus $X' \cap Y \neq \emptyset$, that is, there exists a cycle containing xy of length four in G . \square

Remark 2 *The degree condition is optimal. Let m, n be two even numbers with $n, m \geq 2$. Since $K_{n/2, m/2} \cup K_{n/2, m/2}$ is disconnected, the oriented radius (resp. diameter) of $K_{n/2, m/2} \cup K_{n/2, m/2}$ is ∞ .*

For equal bipartition k -regular graph, the following corollary holds.

Corollary 2 *Let $G = (V_1 \cup V_2, E)$ be a k -regular bipartite graph with $|V_1| = |V_2| = n$. If $k > n/2$, then there exists an orientation H of G such that $\text{rad}(H) \leq 9$.*

The following theorem holds for general graphs.

Theorem 8 *Let G be a graph of order n .*

(i) *If there exists an integer $k \geq 2$ such that $|N_k(u)| > n/2 - 1$ for every vertex u in G , then G has an orientation H such that $\text{rad}(H) \leq 4k^2$ and $\text{diam}(H) \leq 8k^2$.*

(ii) *If $\delta(G) > n/2$, then G has an orientation H such that $\text{rad}(H) \leq 4$ and $\text{diam}(H) \leq 8$.*

Proof: Since methods of the proofs of (i) and (ii) are similar, we only prove (i). For (i), it suffices to show that $\text{rad}(G) \leq 2k$ and $\eta(G) \leq 2k + 1$ by Theorem 2.

We first show $\text{rad}(G) \leq 2k$. Fix u in G , for every $v \in V(G)$, if $v \in N_k[u]$, then $d(u, v) \leq k$. Suppose $v \notin N_k[u]$, we have $N_k(u) \cap N_k(v) \neq \emptyset$. If not, that is, $N_k(u) \cap N_k(v) = \emptyset$, then $|N_k(u)| + |N_k(v)| + 2 > n$ (a contradiction). Thus $d(u, v) \leq 2k$.

Next we show $\eta(G) \leq 2k + 1$. Let $e = uv$ be any edge in G . If $N_k(u) \cap N_k(v) = \emptyset$, then $|V(G)| \geq |N_k(u)| + |N_k(v)| + 2 > n$, a contradiction. Thus $N_k(u) \cap N_k(v) \neq \emptyset$. Pick $w \in N_k(u) \cap N_k(v)$, and let P (resp. Q) be a path between u and w (resp. between v and w). Then e belongs a close trial $uPwQvu$ of length $2k + 1$. Therefore, e belongs a cycle of length at most $2k + 1$. \square

Remark 3 *The above condition is almost optimal since $K_{n/2} \cup K_{n/2}$ is disconnected for even n .*

Corollary 3 *Let G be a graph with minimum degree $\delta(G)$ and girth $g(G)$. If there exists an integer k such that $k \leq g(G)/2$ and $\delta(G)(\delta(G) - 1)^{k-1} > n/2 - 1$, then G has an orientation H such that $\text{rad}(H) \leq 4k^2$.*

Proof: Let k be an integer such that $k \leq g(G)/2$ and $\delta(G)(\delta(G) - 1)^{k-1} > n/2 - 1$. For any vertex u of G , let $1 \leq i < k$ be any integer and $x, y \in N_i(u)$. If x and y have a common neighbor z in $N_{i+1}(u)$, then G has a cycle of length at most $2i < 2k \leq g(G)/2$, a contradiction. Thus x and y has no common neighbor in $N_{i+1}(u)$. Therefore, $|N_k(u)| \geq \delta(G)(\delta(G) - 1)^{k-1} > n/2 - 1$. By Theorem 2, G has an orientation H such that $\text{rad}(H) \leq 4k^2$. \square

4 Upper bound for rainbow connection number

At first, we have the following observation.

Observation 2 *Let G be a graph and H be a spanning subgraph of G . Then $rc(G) \leq rc(H)$.*

Proof of Theorem 4: Let u be a center of G and let H_0 be the trivial graph with vertex set $\{u\}$. We assert that *there exists a subgraph H_i of G such that $N_i[u] \subseteq V(H_i)$ and $rc(H_i) \leq \sum_{j=1}^i \min\{2(\text{rad}(G) - j) + 1, \eta(G)\}$.*

Basic step: When $i = 1$, we omit the proof since the proof of this step is similar to that of the following induction step.

Induction step: Assume that the above assertion holds for $i - 1$ and c is a $rc(H_{i-1})$ -rainbow coloring of H_{i-1} . Next we show that the above assertion holds for i . For any $v \in N_i(u)$, either $v \in V(H_{i-1})$ or $v \in N_G(V(H_{i-1}))$ since $N_{i-1}[u] \subseteq V(H_{i-1})$. If $N_i(u) \subseteq V(H_{i-1})$, then let $H_i = H_{i-1}$ and we are done. Thus, we suppose $N_i(u) \not\subseteq V(H_{i-1})$ in the following.

Let $C_1 = \{\alpha_1, \alpha_2, \dots\}$ and $C_2 = \{\beta_1, \beta_2, \dots\}$ be two pools of colors, none of which are used to color H_{i-1} , and there exists no common color in C_1 and C_2 . An edge-coloring of an H -ear $P = (u_0, u_1, \dots, u_k)$ is a *symmetrical coloring* if its edges are colored by $\alpha_1, \alpha_2, \dots, \alpha_{\lceil k/2 \rceil}, \beta_{\lfloor k/2 \rfloor}, \dots, \beta_2, \beta_1$ in that order or $\beta_1, \beta_2, \dots, \beta_{\lfloor k/2 \rfloor}, \alpha_{\lceil k/2 \rceil}, \dots, \alpha_2, \alpha_1$ in that order.

Let $X = N_i(u) \setminus V(H_{i-1})$ and $m = \min\{2(\text{rad}(G) - i) + 1, \eta(G)\}$. Pick $x_1 \in X$. Let y_1 be a neighbor of x_1 in H_{i-1} and P_1 be an optimal (H_{i-1}, x_1y_1) -ear. We can color P symmetrically with colors $\alpha_1, \alpha_2, \dots, \alpha_{\lceil \ell(P)/2 \rceil}, \beta_{\lfloor \ell(P)/2 \rfloor}, \dots, \beta_2, \beta_1$. Pick $x_2 \in X$ satisfying that all the incident edges of x_2 are not colored. Let y_2 be a neighbor of x_2 in H_{i-1} . If there exists an optimal (H_{i-1}, x_2y_2) -ear P_2 such that P_1 and P_2 are independent, then we can color P_2 symmetrically with colors $\alpha_1, \alpha_2, \dots, \alpha_{\lceil \ell(P_2)/2 \rceil}, \beta_{\lfloor \ell(P_2)/2 \rfloor}, \dots, \beta_2, \beta_1$. Otherwise, by Lemma 1, there exists an optimal (H_{i-1}, x_2y_2) -ear $P_2 = P_{y_2z_2}$ such that P_1 and P_2 have only one continuous common segment containing z_2 , where z_2 is the other foot of P_2 . Thus we can color P_2 symmetrically with colors

$\alpha_1, \alpha_2, \dots, \alpha_{\lceil \ell(P_2)/2 \rceil}, \beta_{\lfloor \ell(P_2)/2 \rfloor}, \dots, \beta_2, \beta_1$ by preserving the coloring of P_1 . We can pick the vertices of X and color optimal H_i -ears until that for any $x \in X$, at least two incident edges of x are colored. Since for any leg e of H_{i-1} , $\ell(e) \leq m$ by Lemma 3, we use at most m coloring in the above coloring process.

Let H_i be the graph obtained from H_{i-1} by adding all the vertices in $V(G) \setminus V(H_{i-1})$, which have at least two new colored incident edges, and adding the new colored edges. Clearly, $N_i[u] \subseteq V(H_i)$. It suffices to show that H_i is rainbow connected. Let x and y be two distinct vertices in H_i . If $x, y \in V(H_{i-1})$, then there exists a rainbow path between x and y by inductive hypothesis. If exactly one of x and y belongs to $V(H_{i-1})$, say x . Let P be a symmetrical colored H_{i-1} -ear containing y , and y' be a foot of P . There exists a rainbow path Q between x and y' in H_{i-1} by inductive hypothesis. Thus, $xQy'Py$ is a rainbow path between x and y in H_i .

Suppose none of x and y belongs to H_{i-1} . Let P and Q be symmetrical colored H_{i-1} -ear containing x and y , respectively. Furthermore, let x', x'' be the feet of P and y', y'' be the feet of Q . Without loss of generality, assume that P is colored from x' to x'' by $\alpha_1, \alpha_2, \dots, \alpha_{\lceil \ell(P)/2 \rceil}, \beta_{\lfloor \ell(P)/2 \rfloor}, \dots, \beta_2, \beta_1$ in that order, and Q is colored from y' to y'' by $\alpha_1, \alpha_2, \dots, \alpha_{\lceil \ell(Q)/2 \rceil}, \beta_{\lfloor \ell(Q)/2 \rfloor}, \dots, \beta_2, \beta_1$ in that order. If $\ell(x'Px) \leq \ell(y'Qy)$, let R be a rainbow path between x' and y'' in H_{i-1} . Then $xPx'Ry''Qy$ is a rainbow path between x and y in H_i . Otherwise, $\ell(x'Px) > \ell(y'Qy)$. Let R be a rainbow path between y' and x'' in H_{i-1} . Then $yQy'Rx''Px$ is a rainbow path between x and y in H_i . Thus, there exists a rainbow path between any two distinct vertices in H_i , that is, H_i is $(\sum_{j=1}^i \min\{2(\text{rad}(G) - j) + 1, \eta(G)\})$ -rainbow connected. \square Readers can see [1] for an optimal example. The following example shows that our result is better than that of Theorem 3.

Example 2 Let $r \geq 3, k \geq 2r$ be two integers, and $W_k = C_k \vee K_1$ be a wheel, where $V(C_k) = \{u_1, u_2, \dots, u_k\}$ and $V(K_1) = \{u\}$. Let H be the graph obtained from W_k by inserting $r - 1$ vertices between every edge uu_i , $1 \leq i \leq k$. For every edge $e = xy$ of H , add a new vertex v_e and new edges v_ex, v_ey . Denote by G the resulting graph. It is easy to check that $\text{rad}(G) = r$, $\text{diam}(G) = 2r$, $\eta(G) = 3$ and $\zeta(G) = 2r - 1$. By Theorem 3, we have $rc(G) \leq \sum_{i=1}^r \min\{2i + 1, \zeta(G)\} \leq r^2 + 2r - 2$. But, by Theorem 4 we have $rc(G) \leq 3r$. On the other hand, $rc(G) \geq 2r$ since $\text{diam}(G) = 2r$.

The remaining results are similar to those in Section 3.

Theorem 9 Let G be a plane graph. If the length of the boundary of every face is at most k , then $rc(G) \leq k \text{rad}(G)$.

Corollary 4 Let G be a maximal plane (resp. outerplane) graph. Then $rc(G) \leq 3 \text{rad}(G)$.

Theorem 10 Let G be a bridgeless edge-transitive graph. Then $rc(G) \leq \text{rad}(G)g(G)$, where $g(G)$ is the girth of G .

Theorem 11 Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with $|V_1| = n$ and $|V_2| = m$. If $d(x) \geq k > \lceil m/2 \rceil$ for any $x \in V_1$, $d(y) \geq r > \lceil n/2 \rceil$ for any $y \in V_2$, then $rc(G) \leq 12$.

Remark 4 The degree condition is optimal. Let m, n be two even numbers with $n, m \geq 2$. Since $K_{n/2, m/2} \cup K_{n/2, m/2}$ is disconnected, $rc(K_{n/2, m/2} \cup K_{n/2, m/2}) = \infty$.

Corollary 5 Let $G = (V_1 \cup V_2, E)$ be a k -regular bipartite graph with $|V_1| = |V_2| = n$. If $k > \lceil n/2 \rceil$, then $rc(G) \leq 12$.

The following theorem holds for general graphs.

Theorem 12 *Let G be a graph.*

(i) *If there exists an integer $k \geq 2$ such that $|N_k(u)| > n/2 - 1$ for every vertex u in G , then $rc(G) \leq 4k^2 + 2k$.*

(ii) *If $\delta(G) > n/2$, then $rc(G) \leq 6$.*

Remark 5 *The above condition is almost optimal since $K_{n/2} \cup K_{n/2}$ is disconnected for even n .*

Corollary 6 *Let G be a graph with minimum degree $\delta(G)$ and girth $g(G)$. If there exists an integer k such that $k < g(G)/2$ and $\delta(G)(\delta(G) - 1)^{k-1} > n/2 - 1$, then $rc(G) \leq 4k^2 + 2k$.*

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