One-dimensional skew diffusions: explicit expressions of densities and resolvent kernels

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Abstract: The study of skew diffusion is of primary concern for their implication in the modeling and simulation of diffusion phenomenons in media with interfaces. First, we provide results on one-dimensional processes with discontinuous coefficients and their connections with the Feller theory of generators as well as the oneof stochastic differential equations involving local time. Second, in view of developing new simulation techniques, we give a method to compute the density and the resolvent kernel of skew diffusions which can be extended to Feller processes in general. Explicit closed-form are given for some particular cases.

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1. Introduction

Linked with diffusion phenomena in media with interfaces which are encountered in many domains, from geophysics to finance through population ecology or astrophysics, the so-called *skew diffusions* face challenging theoretical and numerical problems [30, 39]. In particular, not only their dynamics have to be properly understood, but their descriptions should lead to implementable numerical schemes. As a result, despite being introduced in the '60, the topic of skew diffusions becomes now more and more popular.

In one dimensional situations, skew diffusions can be build on the top of the Skew Brownian motion (SBM) which is the solution of a Stochastic Differential Equation (SDE) involving its local

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time [20, 28]. Like for the solutions of classical SDEs, the interplay between partial differential equations (PDE) and SDEs involving local time is fruitful. Besides, the main applications of skew diffusions are deeply motivated by purely deterministic models. In effect, the transition function $(t, x) \mapsto p(t, x, y)$ is the solution in $\mathrm{L}^2([0, T], \mathrm{L}^2(\mathbb{R}))$ for $y \in \mathbb{R}$ fixed of

$$\partial_t p(t, x, y) = L\, p(t, x, y),\ L = \frac{\rho(x)}{2} \frac{d}{dx} a(x) \frac{d}{dx} + b(x) \frac{d}{dx},\ p(0, x, y) = \delta_y(x) \tag{1.1}$$

where a, ρ and b are discontinuous functions. This setting includes the SBM. Indeed, assuming that b is piecewise constant, one could easily derives using integration by part that $p(t, x, y)$ is the solution of the Advection-Diffusion Equation with discontinuous coefficients

$$\partial_t p(t, x, y) = L^\star\, p(t, x, y),\ L^\star = \frac{\rho(x)}{2} \frac{d}{dy} a(y) \frac{d}{dy} - b(y) \frac{d}{dy},\ p(0, x, y) = \delta_x(y).$$

since $p(t, x, y)$ is also the solution of diffraction problems [25].

Concerning numerical schemes, due to the discontinuities, the particles replicating the dynamics of a skew diffusion should be moved over a time step Δt with a density as close as possible to $p(\Delta t, x, \cdot)$ when at (t, x), as relying on a simple Gaussian step does not work. Even without a drift term b, some of the already proposed schemes could lead to non-negligible errors [31]. However, an accurate simulation of these process gives an understanding of the microscopic dynamic of the Advection-Diffusion Equation and provides us with useful schemes which do not suffer from artificial diffusion.

The density of the Skew Brownian motion was computed by J. Walsh in [44]. Recently, this density allowed to build a simple exact simulation schemes [30] when a and ρ are piecewise constant and $b = 0$. After [3], P. Étoré and M. Martinez [11] proposed an exact rejection-acceptance scheme bases for skew diffusion involving drift $b \neq 0$. The computations of [3, 11, 10] and the related ones of [14, 16] heavily rely on stochastic analysis. Most of these results actually deals with a piecewise constant but only with a constant drift b. In addition, they were obtained for one point of discontinuity and appears not trivial to extend.

The present paper consists of three interconnected parts. The main purpose is to provide an analytical method to compute the resolvent density by giving an explicit representation of this functional. Subsequently, approximations, fine bounds and Feynman-Kac formulas can be easily derived. Moreover, adaptation to deal with any classical boundary conditions is relatively straightforward. Now the drawback of this method is the intermediate Sturm-Liouville problems involved which can be sometimes tricky and led to heavy computation when one wants a particular closed form of the resolvent density. Nevertheless, for the case of piecewise constant coefficients which remains an open case when $b \neq 0$, we obtain a simple closed form which allows to recover known transition functions and to get new ones.

First it is necessary to clearly point out the link between processes generated by divergence form operators, those defined in the sense of Feller through their scale functions and their speed measures, and SDE with local time. In fact, there are several ways to construct a one-dimensional skew diffusion and one does not always see the link with partial differential equation. For example, one could define a skew diffusion through a scale function and a speed measure. This construction was given for the Skew Brownian motion (SBM) by K. Itô and H.P. McKean [22]. One could also consider a skew diffusion as the unique strong solutions of SDEs involving local time. This was done for the SBM by J.M. Harrison and L.A. Shepp [20], and then extended to more general diffusion by J. F. Le Gall [26]. In the same time, J. Groh linked this type of SDEs through scale functions and speed measures [18]. In the analytic side, M. Fukushima linked diffusions with scale functions and speed measure to diffusions with Dirichlet forms [15]. Alternatively, N. Portenko constructed through a parametrix technique the density SBM as a process with singular drift [37]. A series of works [7, 11, 34] exploited the link between SDEs and PDE with the aim at solving them through Monte Carlo methods.

We recall briefly in §2 the interconnection between them, with the aim of simplifying a few results encountered in [7, 29, 34] by taking profit of the properties of the Sturm-Liouville equation. More precisely, the Sturm-Liouville problem provides us with an alternative formulation and a straightforward proof to the so-called *diffraction problem* [24] with minimal regularity assumption on the coefficients. We also show a stability property of this class of functions under a family of piecewise regular one-to-one space transforms. Such a result was also obtained using the Itô-Tanaka formula in [8]. Beyond that, the Sturm-Liouville problem constitutes the key of the proposed computation method. In particular, we focus on some holomorphic character of some solutions with the aim at inverting Laplace transforms. At the notable exception of [36], this aspect is hardly used when dealing with the approach of W. Feller [12, 13]. It enlightens the relationship between the Weyl-Titchmarsh-Kodaira theory [23, 43, 45] and probability.

Using the above results, we provide in §3, the aforementioned analytic machinery for the computation of the resolvent kernels of operators in free space. This approach relies on a appropriate combination of known resolvents in free space resulting from the solutions of the Sturm-Liouville problem developed above. It can be easily generalized to deal with boundary conditions and solve related problems such as the elastic skew Brownian motion and the Feynman-Kac formula. This also leads to expressions for the Laplace transform of the local time at 0 or the occupation time of some segment or half-line.

As shown in §4, the resolvent can be inverted for a wide range of cases. For example, we provide new very short proof of some results contained in [3, 11, 14, 16]. We also give a new result concerning the skew Brownian motion with a possibly piecewise constant drift which takes two values b_+ on $\mathbb{R}_+$ and b_- on $\mathbb{R}_-$. For several practical situations, we give an explicit closed-form of the density. Although it seems hard to perform this inversion for general values of b_+ and b_-, we provide an explicit expression for a new case, called *constant Péclet*, which possesses a natural interpretation in fluid mechanics.

Our method can be considered as a first incursion in alternative methods to overcome the difficulties raised by approaches based on stochastic calculus [3, 11, 14, 16]. It gives the resolvent density for any piecewise constant coefficients.

Although our framework presents similarities with the one developed by B. Gaveau *et al.* [17], we differ by our application of the Weyl-Titchmarsh-Kodaira theory. Besides, the drift term, which was our primarily purpose, can only be taken into account in [17] when it is continuous and require limiting arguments on complex formulas.

2. Operators, semi-groups and diffusion processes

This section is mostly theoretical and pursues multiple goals:

1. In view of our general purpose to solve PDE through a Monte Carlo method, we complement and extend the results obtained in [29] on the relationship between the process generated by a divergence form operator (in the sense defined by D. Stroock in [42]) and the one defined through its scale function and speed measure. Through a formulation of type Sturm-Liouville, many analytic properties are obtained from the latter (See 2. and 3. below).
2. For coefficients that are piecewise smooth with discontinuities of the first kind, to identify the domain of the generator of $X = (X_t)_{t\geq 0}$ and state a result on the regularity of the function given by the probabilistic representation $(t, x) \mapsto \mathbf{E}_x[f(X_t)]$, providing both a simplification and an extension to the results provided in [35].
3. To obtain some analytic properties on the resolvent kernel associated with the process X, on particular regarding their holomorphic character on a suitable basis of solutions. Therefore, one could pass in some situations from close formula on the resolvent kernel to close formula on the density.
4. To relate the process X when its coefficients are as in 2. above with an SDE involving local time in the sense defined by J.-F. Le Gall in [26], which allows to use our analytic machinery for solving problems raised at the level of stochastic differential equations.

When I is a union of disjoint open abutting intervals, we mean by f in $\mathcal{C}^k(I, \mathbb{R})$ a function of class $\mathcal{C}^k$ in each interval composing I such that f and its derivatives up to order k have limits at the inner separation of the intervals.

We set $\overline{\mathbb{R}} = [-\infty, \infty]$ and for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$,

$$\mathcal{C}_{\mathrm{b}}(\overline{\mathbb{R}}, \mathbb{K}) = \{f : \overline{\mathbb{R}} \mapsto \mathbb{K} \mid f \text{ is continuous and bounded}\},$$
$$\mathcal{C}_0(\overline{\mathbb{R}}, \mathbb{K}) = \{f \in \mathcal{C}_{\mathrm{b}}(\overline{\mathbb{R}}, \mathbb{K}) \mid f(\pm\infty) = 0\}.$$

Any continuous function $f : \mathbb{R} \to \mathbb{K}$ vanishing at infinity is naturally identified with a function in $\mathcal{C}_0(\overline{\mathbb{R}}, \mathbb{K})$.

Let $0 < \lambda < \Lambda$ be two fixed real numbers. We consider the sets

$$\mathfrak{C} = \{\mathbf{a} = (a, \rho, b) \text{ measurable from } \mathbb{R}^3 \to \mathbb{R} \mid \lambda \leq a, \rho \leq \Lambda \text{ and } |b| \leq \Lambda\}$$
$$\text{and } \mathfrak{P} = \big\{(a, \rho, b) \in \mathfrak{C} \,\big|\, \text{For a set } J \text{ without cluster points,} a \in \mathcal{C}^1(\mathbb{R} \setminus J, \mathbb{R}), b, \rho \in \mathcal{C}(\mathbb{R} \setminus J, \mathbb{R})\big\}.$$

For a piecewise smooth function $f : \mathbb{R} \to \mathbb{R}^d$, we denote by $\mathrm{Dis}(f)$ the set of discontinuities of the first kind. When we extend this notation to $\mathbf{a} = (a, \rho, b)$, we obtain $\mathrm{Dis}(\mathbf{a}) = \mathrm{Dis}(a) \cup \mathrm{Dis}(\rho) \cup \mathrm{Dis}(b)$.

2.1. Diffusion process associated with a divergence-form operator

Here we recall the construction of a Feller process X associated with a divergence form operator $\frac{\rho}{2}\nabla(a\nabla\cdot) + b\nabla\cdot$ when a is discontinuous. Unlike the solutions of SDEs, such a process is not always a semi-martingale. In Subsection 2.6, we however characterize the situations where X solves a SDE involving local time.

This section relies on [42] (See also [29]). We use the framework of the variational theory of PDE where the solutions are primary sought in subspaces of $\mathrm{L}^2(\mathbb{R})$, the space of square integrable functions on $\mathbb{R}$. We base our construction on a Gaussian estimate on the fundamental solution.

To each $\mathbf{a}=(a,\rho,b)\in\mathfrak{C}$ is associated a family of bilinear form

$$E_\alpha(u,v)=\frac{1}{2}\int a(x)\nabla u(x)\nabla v(x)\,\mathrm{d}x-\int b(x)\nabla u(x)v(x)\frac{\mathrm{d}x}{\rho(x)}\\ +\alpha\int u(x)v(x)\frac{\mathrm{d}x}{\rho(x)},\quad \forall(u,v)\in\mathrm{H}^1(\mathbb{R})\times\mathrm{H}^1(\mathbb{R}),$$

where $\mathrm{H}^1(\mathbb{R})$ is the Sobolev space of square of functions f with weak derivatives ∇f such that $f,\nabla f\in\mathrm{L}^2(\mathbb{R})$.

Clearly, E_α is a Gårding form for $\alpha\geq\alpha_0$ large enough (depending only on λ and Λ). By an application of the Lax-Milgram theorem, one may associate with $(E_\alpha)_{\alpha>0}$ a resolvent $(G_\alpha)_{\alpha\geq\alpha_0}$ on $\mathrm{L}^2(\mathbb{R})$, as well as a linear operator $(L,\mathrm{Dom}(L))$ with $\mathrm{Dom}(L)=G_\alpha(\mathrm{L}^2(\mathbb{R}))$ such that $G_\alpha=(\alpha-L)^{-1}$ and $E_\alpha(f,g)=\int_{\mathbb{R}}(\alpha-L)u(x)v(x)\,\mathrm{d}x/\rho(x)$.

From the Hille-Yosida theorem, the operator L is the infinitesimal generator of a contraction semi-group $(P_t)_{t\geq0}$. A result from J. Nash and J. Moser extended by D. Aronson [5] and D. Stroock [42] shows that there exists a density q with respect to the measure $\mathrm{d}x/\rho(x)$ such that for any $f\in\mathrm{L}^2(\mathbb{R})\cap\mathrm{L}^\infty(\mathbb{R})$,

$$u(t,x)=\int_{\mathbb{R}}q(t,x,y)f(y)\frac{\mathrm{d}y}{\rho(y)},\ t>0,\ x\in\mathbb{R}$$

is a version of $P_tf(x)\in\mathrm{L}^2(\mathbb{R})$. Moreover, q is Hölder continuous and satisfies a Gaussian upper and lower bounds. The main feature is that the constants appearing in the bound and in the Hölder modulus of continuity depend only on λ and Λ. This proves the existence of a diffusion as well as some stability properties (See [9, 29, 41] where similar arguments are deployed).

Let us set

$$\mathbb{p}=\{(t,x,y)\in\mathbb{R}^3\mid t>0,\ x\in\mathbb{R},y\in\mathbb{R}\}.\tag{2.1}$$

Proposition 2.1. *To each element $\mathbf{a}=(a,\rho,b)\in\mathfrak{C}$ is associated a conservative Feller process $(X^{\mathbf{a}},(\mathcal{F}_t),\mathbf{P}_x)_{t\geq0,x\in\mathbb{R}}$ with continuous paths. Its transition function admits a density with respect to the Lebesgue measure denoted by $p(t,x,y)$ which satisfies a Gaussian upper bound and is jointly continuous on $\mathbb{p}$ defined by* (2.1)*. Moreover, if a sequence $\{\mathbf{a}_n\}_{n\in\mathbb{N}}$ of elements of $\mathfrak{C}$ converges pointwise to $\mathbf{a}\in\mathfrak{C}$, then $X^{\mathbf{a}_n}$ converges in distribution to $X^{\mathbf{a}}$ under $\mathbf{P}_x$ for any starting point x.*

Of course, for this diffusion process $X^{\mathbf{a}}$,

$$P_t^{X^{\mathbf{a}}}f(x)\stackrel{\text{def}}{=}\mathbf{E}_x[f(X_t^{\mathbf{a}})]=\int_{\mathbb{R}}p(t,x,y)f(y)\,\mathrm{d}y\stackrel{\text{a.e.}}{=}P_tf(x),\ \forall t\geq0,\tag{2.2}$$

$$\text{and } G_\alpha^{X^{\mathbf{a}}}f(x)\stackrel{\text{def}}{=}\int_0^{+\infty}e^{-\alpha t}\mathbf{E}_x[f(X_t^{\mathbf{a}})]\,\mathrm{d}t\stackrel{\text{a.e.}}{=}G_\alpha f(x),\ \forall\alpha>0,\tag{2.3}$$

for any $x\in\mathbb{R}$ and $f\in\mathrm{L}^2(\mathbb{R})$. Even if P_t and G_α are only operators from $\mathrm{L}^2(\mathbb{R})$ to $\mathrm{L}^2(\mathbb{R})$, we now identify P_t and G_α to $P_t^{X^{\mathbf{a}}}$ and $G_\alpha^{X^{\mathbf{a}}}$. These operators $P_t^{X^{\mathbf{a}}}$ and $G_\alpha^{X^{\mathbf{a}}}$ also act on $\mathcal{C}_{\mathrm{b}}(\overline{\mathbb{R}},\mathbb{R})$. Since $X^{\mathbf{a}}$ is a Feller process, $P_t^{X^{\mathbf{a}}}$ and $G_\alpha^{X^{\mathbf{a}}}$ also map $\mathcal{C}_0(\overline{\mathbb{R}},\mathbb{R})$ to $\mathcal{C}_0(\overline{\mathbb{R}},\mathbb{R})$.

2.2. *The infinitesimal generator of $X^{\mathbf{a}}$*

In the previous Subsection, a process $X^{\mathbf{a}}$ has been associated with $\mathbf{a} \in \mathfrak{C}$. The domain of its infinitesimal generator may be described as the range of the resolvent $G_\alpha(\mathrm{L}^2(\mathbb{R}))$. Here we give an alternative description of the infinitesimal generator of $X^{\mathbf{a}}$ using the approach from W. Feller on scale functions and speed measures where the underlying Banach space is the one of continuous bounded functions [12, 13]. This characterization has over the previous one of giving a simple way to understand the transmission condition and the probabilistic representation of the solution.

We point out that every one-dimensional regular diffusion process Y is characterized by its scale function S and its speed measure M (for classical references on this subject, see *e.g.* [6, 40, 33]). We also recall that the *speed measure* M is a right-continuous, increasing function, which can be identified with a measure on $\mathbb{R}$ and the *scale function* S, which is unique up to an affine transformation, is a convex, continuous and increasing function.

For a continuous, increasing function $g : \mathbb{R} \to \mathbb{R}$, let us define

$$D_g^+ f(x) = \lim_{y \searrow x} \frac{f(y+) - f(x-)}{g(y) - g(x)} \text{ and } D_g^- f(x) = \lim_{y \nearrow x} \frac{f(x+) - f(y-)}{g(y) - g(x)}. \tag{2.4}$$

From now, we only consider the case of a diffusion Y having $\mathbb{R}$ as a state of space and possessing a continuous speed measure M. The infinitesimal generator of such a process Y associated with (S, M) is

$$Af = \frac{1}{2} D_M D_S f \text{ where } D_S f(x) = D_S^+ f(x) = D_S^- f(x) \text{ and } D_M = D_M^+ = D_M^-$$

whose domain is [12, Theorem 8.2] for $\overline{\mathbb{R}} = [-\infty, \infty]$,

$$\mathrm{Dom}(A) = \left\{ f \in \mathcal{C}_{\mathrm{b}}(\overline{\mathbb{R}}, \mathbb{R}) \;\middle|\; D_M D_S f \in \mathcal{C}_{\mathrm{b}}(\overline{\mathbb{R}}, \mathbb{R}) \right\}. \tag{2.5}$$

To $\mathbf{a} = (a, \rho, b) \in \mathfrak{C}$, we associate

$$h(x) = 2 \int_0^x \frac{b(y)}{a(y)\rho(y)} \,\mathrm{d}y,\ S(x) = \int_0^x s(y) \,\mathrm{d}y \text{ where } s(x) = \frac{\exp(-h(x))}{a(x)} \tag{2.6}$$

$$\text{and } M(x) = \int_0^x m(y) \,\mathrm{d}y \text{ where } m(x) = \frac{\exp(h(x))}{\rho(x)}. \tag{2.7}$$

Thus $D_M f(x) = m(x)^{-1} D^+ f(x)$ and $D_S f(x) = a(x) \exp(h(x)) D^+ f(x)$, where $D^+ f$ is the right derivative operator.

The following result is then obtain through a regularization argument over (a, ρ, b) [29].

Proposition 2.2. *For $a \in \mathfrak{C}$, the infinitesimal generator of $X^{\mathbf{a}}$ is $A = \frac{1}{2} D_M D_S$ with (S, M) defined by* (2.6)-(2.7) *and* $\mathrm{Dom}(A)$ *given by* (2.5).

Remark 2.1. The infinitesimal generator $\frac{1}{2} D_M D_S$ is the same for $(a, \rho, b) \in \mathfrak{C}$ as well as for $(\widetilde{a}, \widetilde{\rho}, 0)$ with $\widetilde{a}(x) = a(x) \exp(h(x))$ and $\widetilde{\rho}(x) = \exp(-h(x))\rho(x)$. However $(\widetilde{a}, \widetilde{\rho}, 0)$ does not necessarily belong to $\mathfrak{C}$ as the coefficients could decrease to 0 or increase to ∞ for large values of $|x|$. But, since h is continuous whatever b, this explains why the domain of A does not depend on the regularity of b.

2.3. The resolvent kernel

Let us assume that $\mathbf{a} \in \mathfrak{C}$. Throughout this section, we consider the scale function S and the speed measure M given by (2.6) and (2.7), and A the associated operator $A = \frac{1}{2} D_M D_S$ with domain given by (2.5).

For an interval I of $\mathbb{R}$, let $\mathcal{AC}(I, \mathbb{C})$ be the space of absolutely continuous functions from I to $\mathbb{C}$ and $\mathcal{S}_{\mathbf{a}}(I, \mathbb{C})$ the space of functions u in $\mathcal{AC}(I, \mathbb{C})$ with $D_S u \in \mathcal{AC}(I, \mathbb{C})$. The next proposition concerns the existence and the regularity of solutions to $(\lambda - A)u = 0$. This proposition is the cornerstone of the Sturm-Liouville theory and its extension by H. Weyl, K. Kodaira and E.C. Titchmarsh using complex analysis.

Proposition 2.3. *For any $(\lambda, \alpha, \beta) \in \mathbb{C}^3$, there exists a unique function $u \in \mathcal{S}_{\mathbf{a}}(\mathbb{R}, \mathbb{C})$ that solves $(\lambda - A)u = 0$ with $u(0) = \alpha$ and $D_S u(0) = \beta$. Besides, $(\lambda, \alpha, \beta) \mapsto (u(x), D_S u(x))$ is holomorphic for any $x \in \mathbb{R}$ on $\mathbb{C}^3$.*

Proof. For any $\lambda \in \mathbb{C}$, any solution $Z : \mathbb{R} \to \mathbb{C}^2$ to the first order differential equation

$$Z'(x) = \begin{bmatrix} 0 & s(x) \\ 2\lambda m(x) & 0 \end{bmatrix} Z(x),\ Z(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{C}^2 \text{ for any } x \in \mathbb{R} \tag{2.8}$$

is such that $(\lambda - A)u(x) = 0$ when $Z(x) = (u(x), D_S u(x))$. Regarding the spatial regularity of the solution, let u and f be continuous functions from $\mathbb{R}$ to $\mathbb{C}$ such that $(\lambda - A)u = f$. Then integrating between x and y against the measure M leads to

$$\lambda \int_x^y u(z) m(z)\, \mathrm{d}z - D_S u(x) + D_S u(y) = 2 \int_x^y f(z) m(z)\, \mathrm{d}z. \tag{2.9}$$

In particular, $D_S u \in \mathcal{AC}(\mathbb{R}, \mathbb{C})$ and $u \in \mathcal{S}_{\mathbf{a}}(\mathbb{R}, \mathbb{C})$. $\square$

The *Wronskian* of two functions u and v in $\mathcal{S}_{\mathbf{a}}(\mathbb{R}, \mathbb{C})$ is

$$\mathrm{Wr}[u, v](x) \stackrel{\text{def}}{=} u(x) D_S v(x) - v(x) D_S u(x).$$

If u and v solve $(\lambda - A)u = 0$, then $\mathrm{Wr}[u, v](x)$ is constant over x (See (2.13) below). The solutions u and v are independent if and only $\mathrm{Wr}[u, v](x) \neq 0$. In this case, any solution to $(\lambda - A)w = 0$ on some interval I is a linear combination of u and v on I.

We introduce

$$\mathbb{H}^+ = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) > 0\},$$
$$\mathbb{h}^+ = \{\lambda \in \mathbb{C} \mid \mathrm{Im}(\lambda) = 0,\ \mathrm{Re}(\lambda) > 0\},\ \text{and}\ \overline{\mathbb{h}}^- = \{\lambda \in \mathbb{C} \mid \mathrm{Im}(\lambda) = 0,\ \mathrm{Re}(\lambda) \leq 0\}.$$

Lemma 2.1. *Let $\tau_x = \inf\{t > 0 \mid X_t^{\mathbf{a}} = x\}$ be the first hitting time of $x \in \mathbb{R}$ of the process $X^{\mathbf{a}}$. For any $y < x$, the function*

$$\phi(\lambda, x) = \mathbf{E}_x[\exp(-\lambda \tau_y)]$$

is holomorphic on $\mathbb{H}^+$. Besides, for any $\lambda \in \mathbb{H}^+$, $\phi(\lambda, x)$ is solution to $(\lambda - A)\phi(\lambda, x) = 0$ for $x \geq y$ and $\phi(\lambda, y) = 1$.

Proof. Whatever $\lambda \in \mathbb{H}^+$, $0 \leq |\phi(\lambda, x)| \leq \phi(\mathrm{Re}(\lambda), x) \leq 1$ so that ϕ is well defined on $\mathbb{H}^+$. In addition, from the Feynman-Kac formula [22, § 2.6],

$$(\lambda - A)\phi(\lambda, x) = 0, \quad \forall x \geq y.$$

We decide to show that $\phi(\lambda, x)$ can be developed as a power series on a neighborhood of any $\lambda \in \mathbb{H}^+$ since it is equivalent of being holomorphic for complex function.

Notice that

$$\phi(\lambda, x) = 1 - \lambda \mathbf{E}_x \left[\int_0^{\tau_y} \exp(-\lambda s) \,\mathrm{d}s \right],$$

Now using the series representation of the exponential, for $\sigma \in \mathbb{C}$,

$$\phi(\lambda + \sigma, x) = \phi(\lambda, x) - \sum_{k \geq 1} (-1)^k \sigma^k c_k(\lambda, x) \tag{2.10}$$

with $c_k(\lambda, x) = \mathbf{E}_x \left[\int_0^{\tau_y} \left(\lambda \frac{s^k}{k!} - \frac{s^{k-1}}{(k-1)!} \right) \exp(-\lambda s) \,\mathrm{d}s \right]$. Since $\int_0^{+\infty} s^k \exp(-\lambda s) \,\mathrm{d}s = k!/\lambda^{k+1}$ and $\tau_y \geq 0$, $|c_k(\lambda, x)| \leq 1 + 1/|\lambda|$ for each k. Hence, the series in the right-hand side of (2.10) converges absolutely for any $\sigma \in \mathbb{H}^+$, $|\sigma| < \min 1, \mathrm{Re}\,\lambda$. The result is proved. □

The function $\lambda \mapsto \phi(\lambda, x)$ is not necessarily analytic around $\lambda = 0$. Otherwise, it would mean that τ_y has finite moments of all orders. This is not true for example for the Brownian motion.

Proposition 2.4. *For any $\lambda \in \mathbb{H}^+$, the functions*

$$u^{\nearrow}(\lambda, x) = \mathbf{E}_x[\exp(-\lambda \tau_0)],\ u^{\searrow}(\lambda, x) = \frac{1}{\mathbf{E}_0[\exp(-\lambda \tau_x)]} \text{ if } x < 0, \tag{2.11}$$

$$u^{\nearrow}(\lambda, x) = \frac{1}{\mathbf{E}_0[\exp(-\lambda \tau_x)]},\ u^{\searrow}(\lambda, x) = \mathbf{E}_x[\exp(-\lambda \tau_0)] \text{ if } x > 0 \tag{2.12}$$

are solutions to $(\lambda - A)u = 0$. For λ on the half-line $\mathbb{h}^+$, $x \mapsto u^{\nearrow}(\lambda, x)$ is increasing from 0 to $+\infty$, $x \mapsto u^{\searrow}(\lambda, x)$ is decreasing from $+\infty$ to 0. Moreover, $u^{\nearrow}(\lambda, 0) = u^{\searrow}(\lambda, 0) = 1$. For $\lambda \in \mathbb{H}^+$, $|u(\lambda, x)|$ converges to 0 as $|x| \to \infty$. Besides, for any $x \in \mathbb{R}$, $\lambda \mapsto u(\lambda, x)$ and $\lambda \mapsto D_S u(\lambda, x)$ for $u = u^{\nearrow}, u^{\searrow}$ are holomorphic on $\mathbb{H}^+$.

Proof. We follow the classical probabilistic construction of $u^{\nearrow}$ and $u^{\searrow}$ [6, 40] for $\lambda \in \mathbb{R}$, which extends to $\lambda \in \mathbb{H}^+$. In the order,

1. From the Feynman-Kac formula, $u^{\nearrow}$ and $u^{\searrow}$ are solutions to $(\lambda - A)u = 0$ with $u(0) = 1$.
2. Using the strong Markov property, $\mathbf{E}_x[e^{-\lambda \tau_y}] = \mathbf{E}_x[e^{-\lambda \tau_z}]\mathbf{E}_z[e^{-\lambda \tau_y}]$ for any $x \leq y \leq z$. From this, one easily checks that $\mathbf{E}_x[e^{-\lambda \tau_y}] = u(x)/u(z)$ for $u = u^{\nearrow}(\lambda, \cdot)$ or $u = u^{\searrow}(\lambda, \cdot)$. Since these functions are positive, Lemma 2.1 implies that they are holomorphic on $\mathbb{H}^+$.
3. Using for a basis of solutions for $(\lambda - A)u = 0$ the functions u_1 and u_2 with $u_1(\lambda, 0) = D_S u_2(\lambda, 0) = 1$ and $D_S u_1(\lambda, 0) = u_2(\lambda, 0) = 0$, $u^{\nearrow}$ may be written $u^{\nearrow}(\lambda, x) = u_1(\lambda, x) + \alpha(\lambda) u_2(\lambda, x)$. Since u_1 and u_2 are holomorphic from Proposition 2.3, $\alpha(\lambda)$ is meromorphic. Since $u^{\nearrow}(\cdot, x)$ is holomorphic on $\mathbb{H}^+$ from Lemma 2.1, α is necessarily holomorphic on $\mathbb{H}^+$. With Proposition 2.3, $D_S u^{\nearrow}(\cdot, x)$ is holomorphic on $\mathbb{H}^+$ for any $x \in \mathbb{R}$ since α, $D_S u_1(\cdot, x)$ and $D_S u_2(\cdot, x)$ are holomorphic on $\mathbb{H}^+$ (actually, $\alpha(\lambda) = D_S u(\lambda, 0)$ from our choice of u_1 and u_2). The same argument holds for $u^{\searrow}$. We have then proved that $D_S u^{\nearrow}$ and $D_S u^{\searrow}$ are holomorphic on $\mathbb{H}^+$.

4. For $\lambda \in \mathbb{h}^+$, it is easily checked that the functions $u^\nearrow$ and $u^\searrow$ defined by (2.11) and (2.12) are positive, monotone, and range respectively from 0 to ∞ and from ∞ to 0.

$\square$

Corollary 2.1. *For $\lambda \in \mathbb{H}^+$, let u in $\mathcal{S}_{\mathbf{a}}(\mathbb{R}_+, \mathbb{C})$ (resp. $\mathcal{S}_{\mathbf{a}}(\mathbb{R}_-, \mathbb{C})$) be a solution to $(\lambda - A)u = 0$ on $\mathbb{R}_+$ (resp. on $\mathbb{R}_-$) which vanishes at $+\infty$ (resp. $-\infty$) and satisfies $u(0) = 0$. Then $u = 0$.*

Proof. For a continuous, bounded function f on $\mathbb{R}_+$, set

$$H_\lambda f(x) \stackrel{\text{def}}{=} \mathbf{E}_x \left[\int_0^{\tau_0} e^{-\lambda s} f(X_s^{\mathbf{a}}) \, \mathrm{d}s \right].$$

Using the strong Markov property of $X^{\mathbf{a}}$,

$$H_\lambda f(x) = G_\lambda^{X^{\mathbf{a}}} f(x) - u^\searrow(\lambda, x) G_\lambda^{X^{\mathbf{a}}} f(0).$$

With the Feynman-Kac formula, $(\lambda - A) H_\lambda f(x) = f(x)$ on $\mathbb{R}_+$. In addition, $H_\lambda f(0) = 0$. Finally, H_λ maps bounded functions to bounded functions. Besides, if f vanishes at infinity, then $G_\lambda^{X^{\mathbf{a}}} f$ and $u^\searrow(\lambda, \cdot)$ vanish at infinity, so does $H_\lambda f$. This proves that H_λ is a bounded inverse of $\lambda - A$ for $\lambda \in \mathbb{H}^+$. Therefore, there exists a unique solution to $(\lambda - A)u = f$ when f is bounded, continuous on $\mathbb{R}_+$, and this solution vanishes at infinity when f does. Hence, for $f = 0$, $u = 0$. $\square$

Remark 2.2. Proposition 2.4 and Corollary 2.1 could have been treated by a purely analytic method with the theory of limit circle and limit points that was initiated by H. Weyl [45] (See also [23, 43]). It could be proved that $u^\nearrow$ and $u^\searrow$ are actually meromorphic on $\mathbb{C} \setminus \overline{\mathbb{h}}^-$.

The functions $u^\nearrow$ and $u^\searrow$ serve to build the resolvent. For our purpose, it is important that they are holomorphic on $\mathbb{H}^+$ so that we could identify the resolvent kernel with another analytic expression to recover the densities through Laplace inversion (See Proposition 3.1 below).

The holomorphic property is the key to prove results such as Proposition 2.5 below on the regularity of the density (See also [21]) as well as the existence of a diffusion process. This is not our purpose here. Therefore, we took benefit of the known probabilistic representation of the resolvent to refine our knowledge on the resolvent kernel.

Corollary 2.2. *The map $\lambda \mapsto \mathrm{Wr}[u^\searrow, u^\nearrow](\lambda, x)$ is holomorphic on $\mathbb{H}^+$ and vanishes only on the closed half-line $\overline{\mathbb{h}}^-$.*

Proof. First, $\mathrm{Wr}[u^\searrow, u^\nearrow](\lambda, x)$ does not depend on x. Second, from the very definition of $u^\nearrow$ and $u^\searrow$, $\mathrm{Wr}[u^\searrow, u^\nearrow](\lambda, x) \neq 0$ for $x \in \mathbb{h}^+$ and is holomorphic in λ. Third, $\mathrm{Wr}[u^\searrow, u^\nearrow](\lambda, x) = 0$ for some $\lambda \in \mathbb{C}$ if and only of $u^\nearrow = \alpha u^\searrow$ for $\alpha \in \mathbb{C} \setminus \{0\}$. Necessarily, $\alpha = 1$ since $u^\nearrow(\lambda, 0) = u^\searrow(\lambda, 0)$. Using an integration by parts on an interval $[a, b]$, for any $\phi, \psi \in \mathrm{Dom}(A)$,

$$\int_a^b [\phi(x) A\psi(x) - \psi(x) A\phi(x)] m(x) \, \mathrm{d}x = \frac{1}{2} \mathrm{Wr}[\phi, \psi](b) - \frac{1}{2} \mathrm{Wr}[\phi, \psi](a). \tag{2.13}$$

Since $u^\searrow(\lambda, 0)$ is solution to $(\overline{\lambda} - A)u = 0$, and when $u^\nearrow(\lambda, \cdot) = u^\searrow(\lambda, \cdot)$, (2.13) with $\phi = u^\nearrow$ and $\psi = \overline{u^\nearrow}$ yields

$$2i \,\mathrm{Im} \left(\int_a^b \lambda u^\nearrow(\lambda, x) \overline{u^\searrow(\lambda, x)} m(x) \, \mathrm{d}x \right) = 0 \text{ or } \mathrm{Im}(\lambda) = 0.$$

Therefore, $\mathrm{Wr}[u^{\searrow}, u^{\nearrow}](\lambda, x) = 0$ implies that $\mathrm{Im}(\lambda) = 0$. On the other hand, since $u^{\nearrow}(\lambda, \cdot)$ increases whereas $u^{\searrow}(\lambda, \cdot)$ decreases for $\lambda \in \mathbb{h}^+$ while both functions are positive implies that $\mathrm{Wr}[u^{\searrow}, u^{\nearrow}](\lambda, x) = 0$ only on $\overline{\mathbb{h}}^-$. □

We give two examples that are of great importance later.

Example 2.1. For $s(x) = m(x) = 1$ so that $A = \frac{1}{2}D_x^2$,

$$u^{\nearrow}(\lambda, x) = \exp(\sqrt{2\lambda}x) \text{ and } u^{\searrow}(\lambda, x) = \exp(-\sqrt{2\lambda}x).$$

They are holomorphic on $\mathbb{C} \setminus \overline{\mathbb{h}}^-$. Besides, $\mathrm{Wr}[u^{\searrow}, u^{\nearrow}](\lambda, x) = 0$ if and only if $\lambda = 0$.

Example 2.2. For $s(x) = \exp(-\gamma x)$ and $m(x) = \exp(\gamma x)$ so that $A = \frac{1}{2}D_x^2 + \gamma D_x$,

$$\begin{aligned} u^{\nearrow}(\gamma, \lambda, x) &= \exp((-\gamma + \sqrt{\gamma^2 + 2\lambda})x) \\ \text{and } u^{\searrow}(\gamma, \lambda, x) &= \exp((-\gamma - \sqrt{\gamma^2 + 2\lambda})x). \end{aligned} \tag{2.14}$$

They are holomorphic on $\mathbb{C} \setminus \{\lambda \in \mathbb{C} \mid \mathrm{Im}(\lambda) = 0, \mathrm{Re}(\lambda) < -|\gamma|/2\}$. Besides,

$$\mathrm{Wr}[u^{\searrow}, u^{\nearrow}](\lambda, x) = \sqrt{\gamma^2 + 2\lambda} \text{ and } \mathrm{Wr}[u^{\searrow}, u^{\nearrow}](\lambda, x) = 0$$

if and only if $\lambda = -\gamma^2/2$.

Let us set for $\lambda \in \mathbb{H}^+$ and the density m of the speed measure M,

$$r(\lambda, x, y) = \frac{m(y)}{\mathrm{Wr}[u^{\searrow}, u^{\nearrow}](\lambda, y)} \begin{cases} u^{\nearrow}(\lambda, x)u^{\searrow}(\lambda, y) & \text{if } x < y, \\ u^{\searrow}(\lambda, x)u^{\nearrow}(\lambda, y) & \text{if } x > y. \end{cases}$$

The resolvent $R_\lambda = G_\lambda^{X^{\mathbf{a}}}{}_{|\mathcal{C}_0(\overline{\mathbb{R}}, \mathbb{R})}$ can be extended for $\lambda \in \mathbb{H}^+$ as a bounded (by $1/\mathrm{Re}(\lambda)$) operator. The resolvent is actually $(\lambda - A)^{-1}$ for $\lambda \in \mathbb{H}^+$. From standard results (See *e.g.* [6, Theorem 16.75] or [12]), for any $\lambda \in \mathbb{H}^+$,

$$R_\lambda f(x) = \int_{\mathbb{R}} r(\lambda, x, y) f(y) \,\mathrm{d}y, \; f \in \mathcal{C}_0(\overline{\mathbb{R}}, \mathbb{R}).$$

Exploiting (2.9), we easily see that $r(\lambda, x, y)$ satisfies

$$D_S r(\lambda, y-, y) - D_S r(\lambda, y+, y) = 2m(y), \text{ for any } y \in \mathbb{R}, \; \lambda \in \mathbb{H}^+. \tag{2.15}$$

The density $p(t, x, y)$ of the process $X^{\mathbf{a}}$ generated by $(A, \mathrm{Dom}(A))$ or equivalently by $(L, \mathrm{Dom}(L))$ is related to $r(\lambda, x, y)$ by the Laplace transform $\mathcal{L}$. The next proposition is due to H.P. McKean. It is actually valid for any process with generators of type $\frac{1}{2}D_M D_S$. Proposition's proof exploits the fact that $(A, \mathrm{Dom}(A))$ is a self-adjoint operator to provide the existence of a density p which is analytic[1].

[1]approach, L is self-adjoint and the semi-group $(P_t)_{t>0}$ is analytic and maps $\mathrm{L}^2(\mathbb{R})$ to $\mathrm{Dom}(L)$. The non-trivial part is the existence of a density. In [36], H.P. McKean uses a spectral representation on the kernel of the resolvent to show the existence of a density transition function and analyse its properties, in particular to show that $p(t, \cdot, y)$ belongs to $\mathrm{Dom}(A^n)$ for any $n \in \mathbb{N}$. These two approaches are parallel, one at the level of the operators, the second on their kernels. Many of the properties of p exhibited in [36] are not quoted in the subsequent literature such as [6, 22].

Proposition 2.5 (H.P. McKean [36]). *The density $p(t,x,y)$ of the stochastic process $X^{\mathbf{a}}$ for $\mathbf{a} \in \mathfrak{C}$ satisfies $p(t,x,y) = \mathcal{L}^{-1}(r(\lambda,x,y))$. In addition, for any $n \geq 1$, $\partial_t^n p$ exists and is continuous on $\mathbb{p}$ defined by* (2.1)*. In addition, it satisfies $\partial_t^n p = A^n p$ on $\mathbb{p}$ and $\partial_t^n p(t,\cdot,y)$ vanishes at infinity for any $t > 0$, $y \in \mathbb{R}$.*

Remark 2.3. The above theorem is done with $S(x) = x$. Nevertheless we could always reduce to this case (see also the results in Section 2.5).

2.4. Diffusion with piecewise regular coefficients

We now focus on the process associated with $\mathbf{a} \in \mathfrak{P}$, that is a Feller process with piecewise regular coefficients.

Notation 2.1. *For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, given $\mathbf{a} \in \mathfrak{P}$ and I an interval of $\mathbb{R}$, we denote by $\mathcal{A}_{\mathbf{a}}(I,\mathbb{K})$ the set of functions $u \in \mathcal{C}(I,\mathbb{K}) \cap \mathcal{C}^2(I \setminus \mathrm{Dis}(\mathbf{a}),\mathbb{K})$ such that*

$$a(x-)u'(x-) = a(x+)u'(x+),\ \forall x \in I \cap \mathrm{Dis}(\mathbf{a}). \tag{2.16}$$

We show how close the domains of A and L are.

Proposition 2.6. *For $\mathbf{a} \in \mathfrak{P}$,*

$$\mathrm{Dom}(A) \subset \mathcal{A}_{\mathbf{a}}(\mathbb{R},\mathbb{R}) \text{ and } \mathcal{A}_{\mathbf{a}}(\mathbb{R},\mathbb{R}) \cap \{f \in \mathrm{L}^2(\mathbb{R}) \mid Af \in \mathrm{L}^2(\mathbb{R})\} \subset \mathrm{Dom}(L).$$

Besides, the latter inclusion is dense in $\mathrm{Dom}(L)$ with respect to the L^2-norm.

Proof. The inclusion $\mathrm{Dom}(A) \subset \mathcal{A}_{\mathbf{a}}(\mathbb{R},\mathbb{R})$ is straightforward using the fact that $\mathrm{Dom}(A) \subset \mathcal{S}_{\mathbf{a}}(\mathbb{R},\mathbb{R})$ and the continuity of m and s out of $\mathrm{Dis}(\mathbf{a})$.

Let $u \in \mathcal{A}_{\mathbf{a}}(\mathbb{R},\mathbb{R})$. The set $\mathbb{R} \setminus \mathrm{Dis}(\mathbf{a})$ may be expressed as the disjoint union of intervals of type (x_i, x_{i+1}) with $\mathrm{Dis}(\mathbf{a}) = \{x_i\}_{i \in J}$ with $x_i < x_{i+1}$, $i \in J \subset \mathbb{Z}$. Thus, for $\phi \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R},\mathbb{R})$,

$$\begin{aligned} 2\int_{\mathbb{R}} Au(x)\phi(x)\frac{\mathrm{d}x}{\rho(x)} &= \sum_{i\in\mathbb{Z}} \int_{x_i}^{x_{i+1}} e^{-h(x)} \left(\frac{1}{s(x)}u'(x)\right)' \phi(x)\,\mathrm{d}x \\ &= \sum_{i\in\mathbb{Z}} \left(\frac{1}{s(x_{i+1}-)}u'(x_{i+1}-)\phi(x_{i+1})e^{-h(x_{i+1})} - \frac{1}{s(x_i)}u'(x_i)\phi(x_i)e^{-h(x_i)} \right) \\ &\qquad - \int_{\mathbb{R}} a(x)u'(x)\phi'(x)\,\mathrm{d}x + 2\int_{\mathbb{R}} b(x)u'(x)\phi(x)\frac{\mathrm{d}x}{\rho(x)}. \end{aligned}$$

Using the interface condition (2.16) at x_i,

$$\int_{\mathbb{R}} (\alpha - A)u(x)\phi(x)\frac{\mathrm{d}x}{\rho(x)} = E_\alpha(u,\phi),\ \forall \phi \in \mathcal{C}^1_{\mathrm{c}}(\mathbb{R},\mathbb{R}). \tag{2.17}$$

If $u \in \mathrm{L}^2(\mathbb{R})$ is such that $Au \in \mathrm{L}^2(\mathbb{R})$, it follows from (2.17) that $u = (\alpha - L)^{-1}(\alpha - A)u \in \mathrm{Dom}(L)$.

The space $\mathcal{C}_{\mathrm{c}}(\mathbb{R},\mathbb{R})$ is dense in $\mathrm{L}^2(\mathbb{R})$. For $f \in \mathcal{C}_{\mathrm{c}}(\mathbb{R},\mathbb{R})$, $u_A = (\alpha - A)^{-1}F \in \mathrm{Dom}(A) \subset \mathcal{A}_{\mathbf{a}}(\mathbb{R},\mathbb{R})$ and $u_L = (\alpha - L)^{-1}f \in \mathrm{Dom}(L)$. With (2.17), we deduce that u_A is a version of u_L. This proves that $(\alpha - A)^{-1}(\mathcal{C}_{\mathrm{c}}(\mathbb{R},\mathbb{R})) \subset \mathrm{Dom}(A) \subset \mathcal{A}_{\mathbf{a}}(\mathbb{R},\mathbb{R})$ is dense in $\mathrm{Dom}(L)$ since $\mathcal{C}_{\mathrm{c}}(\mathbb{R},\mathbb{R})$ is. $\square$

We characterize the resolvent kernel.

Proposition 2.7. *The resolvent kernel r is such that*

1. *For any $\lambda \in \mathbb{H}^+$, $y \in \mathbb{R}$, $x \mapsto r(\lambda, x, y)$ belongs to $\mathcal{A}_{\mathbf{a}}(\mathbb{R} \setminus \{y\}, \mathbb{C})$ and vanishes at infinity.*
2. *The map $r(\lambda, \cdot, \cdot)$ is continuous from $\mathbb{R}^2$ to $\mathbb{C}$ for any $\lambda \in \mathbb{H}^+$ and $r(\cdot, x, y)$ is holomorphic on $\mathbb{H}^+$ for any $(x, y) \in \mathbb{R}^2$.*
3. *For any $y \notin \mathrm{Dis}(\mathbf{a})$,*
$$D_x r(\lambda, y-, y) - D_x r(\lambda, y+, y) = \frac{2}{a(y)\rho(y)}. \tag{2.18}$$

Besides, if for some $\lambda \in \mathbb{H}^+$ and $y \in \mathbb{R}$, $g \in \mathcal{A}_{\mathbf{a}}(\mathbb{R} \setminus \{y\}, \mathbb{C})$ vanishes at infinity and satisfies $(\lambda - A)g(x) = 0$ for $x \neq y$ and (2.18) *at y, then $g(x) = r(\lambda, x, y)$ for any $x \in \mathbb{R}$.*

Proof. The direct part is an immediate consequence of Proposition 2.6 and (2.15).

Concerning the uniqueness, for any $y \notin \mathrm{Dis}(\mathbf{a})$, the difference $u(x) = g(x) - r(\lambda, x, y)$ is solution to $(\lambda - A)u(x) = 0$ for any $x \notin \mathrm{Dis}(\mathbf{a}) \cup \{y\}$ and vanishes at infinity. Because both g and $r(\lambda, \cdot, y)$ satisfy (2.18), u belongs to $\mathcal{A}_{\mathbf{a}}(\mathbb{R}, \mathbb{C})$. It also satisfies $(\lambda - A)u(x) = 0$ for $x \notin \mathrm{Dis}(\mathbf{a})$. Since $\mathrm{Dis}(\mathbf{a})$ is discrete and due to (2.9), $(\lambda - A)u(x) = 0$ for every $x \in \mathbb{R}$. As R_λ is a bounded operator for $\lambda \in \mathbb{H}^+$, u vanishes everywhere because it vanishes at infinity. □

Using Proposition 2.5, we weaken the conditions of applications of Theorem 3.1 in [35].

Corollary 2.3. *If $f \in \mathrm{L}^\infty(\mathbb{R})$, let us set*

$$u(t, x) = \int_{\mathbb{R}} p(t, x, y) f(y)\,\mathrm{d}y = \mathbf{E}_x[f(X_t^{\mathbf{a}})].$$

For each $t > 0$, $u(t, x) \in \mathcal{A}_{\mathbf{a}}(\mathbb{R})$. Hence, it is a classical solution to $\partial_t u(t, x) = Lu(t, x)$ for $t > 0$, $x \in \mathbb{R} \setminus \mathrm{Dis}(\mathbf{a})$ satisfying the transmission condition (2.16) *at any point of* $\mathrm{Dis}(\mathbf{a})$. *If in addition, $f \in \mathrm{L}^2(\mathbb{R})$, then $u(t, x)$ is a variational solution in $\mathrm{L}^2([0, T], \mathrm{H}^1(\mathbb{R})) \cap \mathcal{C}([0, T], \mathrm{L}^2(\mathbb{R}))$ to $\partial_t u(t, x) = Lu(t, x)$.*

2.5. Stability under space transforms

Let us consider a continuous, increasing one-to-one function ϕ and define for every $f : \mathbb{R} \to \mathbb{R}$ the transform $\phi_* f = f \circ \phi^{-1}$.

For a continuous increasing function g, it is easily checked that $D_{\phi_* g}(\phi_* f) = \phi_*(D_g f)$ when $D_g f$ or $D_{\phi_* g}(\phi_* f)$ exist and are continuous.

Now let us introduce the operator $A^\phi = \frac{1}{2} D_{\phi_* M} D_{\phi_* S}$ associated to the scale function $\phi_* S$ and the speed measure $\phi_* M$. The proof of the next proposition is straightforward.

Proposition 2.8. *Let X be the process with infinitesimal generator A. Then $X^\phi = \phi(X)$ has infinitesimal generator A^ϕ.*

As a result, a large class of transforms keep the family $\{X^{\mathbf{a}}\}_{\mathbf{a} \in \mathfrak{C}}$ invariant.

Proposition 2.9. *Assume that ϕ is continuous, locally absolutely continuous on $\mathbb{R}$ and that for some $\eta > 0$, $\eta < \phi'(x) < \eta^{-1}$ for almost every $x \in \mathbb{R}$. Let us set $\phi_\sharp g(x) = g \circ \phi^{-1} \cdot \phi' \circ \phi^{-1}$. Then for $\mathbf{a} = (a, \rho, b) \in \mathfrak{C}$, $\phi_\sharp \mathbf{a} \in \mathfrak{C}$ (the constants λ and Λ may be changed) and $\phi(X^{\mathbf{a}}) = X^{\phi_\sharp \mathbf{a}^\phi}$ where $\phi_\sharp \mathbf{a} = (\phi_\sharp a, \phi_\sharp \rho, \phi_\sharp b)$.*

Remark 2.4. This choice is such that $\widehat{g}(x)\nabla(\phi_* f(x)) = \phi_*(g\nabla f)(x)$ for almost every $x \in \mathbb{R}$.

Proof. From the conditions on ϕ, $\lim_{x\to\pm\infty} \phi(x) = \pm\infty$ so that ϕ is one-to-one. Moreover, we have seen in Proposition 2.8 that the scale function and the speed measure of X^ϕ are $\phi_* S$ and $\phi_* M$. From a change of variable, $\phi_* S$ and $\phi_* M$ are indeed identified with the scale function and speed measure of $\phi_\sharp \mathbf{a}$. Thus, the result follows from Proposition 2.2. □

Corollary 2.4. *If ϕ is continuous and piecewise of class $\mathcal{C}^2$, where the points of discontinuity of ϕ' have no cluster points (hence absolutely continuous), and $0 < \eta < \phi'(x) < \eta^{-1}$ for almost every x. If $\mathbf{a} \in \mathfrak{P}$, then $\phi_\sharp \mathbf{a} \in \mathfrak{P}$.*

Remark 2.5. We could have also prove the Proposition 2.9 using the Itô-Tanaka formula as in [7, 8].

Remark 2.6. Using a well chosen transform is actually the key for reducing the problem to a simpler one. It was heavily used with different transforms in [29, 30] and in [34, 35].

The case we consider in the sequel is the following: if $\mathbf{a} \in \mathfrak{P}$ is piecewise continuous, and ϕ is piecewise linear, then $\mathbf{a}^\phi$ is also piecewise continuous.

2.6. Stochastic differential equations

Here we prove that a diffusion $X^{\mathbf{a}}$ associated with $\mathbf{a} \in \mathfrak{P}$ is the unique strong solution to a SDE involving local time. For this purpose, we introduce

$$\mathfrak{G} = \left\{ (\sigma,\nu) \;\middle|\; \begin{array}{c} \sigma \text{ is of bounded variation over } \mathbb{R} \text{ and } \lambda \leq \sigma(x) \leq \Lambda,\ \forall x \in \mathbb{R} \\ \nu \text{ is a finite measure over } \mathbb{R} \text{ with } |\nu(\{x\})| < 1, \forall x \in \mathbb{R} \end{array} \right\}.$$

The measure ν is associated with a function s of bounded variation through

$$\nu(\,\mathrm{d}x) = \frac{s'(\,\mathrm{d}x)}{s(x) + s(x-)}$$

where $s'(\,\mathrm{d}x)$ is the measure associated with s.

Now let us consider the SDE given by

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}B_s + \int_{\mathbb{R}} \nu(\,\mathrm{d}y) L_t^y(X) \tag{2.19}$$

where B is a Brownian motion and $L^x(X)$ the symmetric local time of X at x.

From [26, Theorem 2.3], this SDE has a unique strong solution. Now using [26, Proposition 2.2], we easily deduce that X is a diffusion process with scale function $S = \int s$ and speed measure with density $m(\,\mathrm{d}x) = s(x)^{-1}\sigma^{-2}(x)\,\mathrm{d}x$.

Considering $\mathbf{a} = (a, \rho, b) \in \mathfrak{P}$, the expressions (2.6)-(2.7) for the scale function and the speed measure leads us to associate to $\mathbf{a}$ the function $\sigma(x) = \sqrt{a(x)\rho(x)}$ as well as the measure

$$\nu(\,\mathrm{d}x) = \frac{b(x)}{a(x)\rho(x)}\,\mathrm{d}x + \frac{a'(x)}{2a(x)}\,\mathrm{d}x + \sum_{x \in \mathrm{Dis}(\mathbf{a})} \frac{a(x+) - a(x-)}{a(x+) + a(x-)}, \tag{2.20}$$

where $a'(x)$ is the derivative of a on the intervals on which a is $\mathcal{C}^1$ and $a' = 0$ on the set of discontinuities of a of zero Lebesgue measure. Because ν is not necessarily a finite measure over $\mathbb{R}$, σ and ν do not necessarily belong to $\mathfrak{G}$. However, using a localization argument, there is no problem to set $(a, \rho, b) = (1, 1, 0)$ outside a given compact K so that ν becomes a finite measure over $\mathbb{R}$ and then to let K grow.

Since a one-dimensional diffusion process is uniquely characterized by its scale function and speed measure, $X^{\mathbf{a}}$ is a semi-martingale solution to (2.19).

Proposition 2.10. *When $\mathbf{a} \in \mathfrak{P}$, $X^{\mathbf{a}}$ is the unique strong solution to* (2.19) *with ν given by* (2.20) *and $\sigma = \sqrt{a\rho}$.*

Remark 2.7. For $\mathbf{a} = (a, \rho, b)$ and $\kappa > 0$, let us set $\mathbf{a}_\kappa = (\kappa a, \kappa^{-1}\rho, b)$. With (2.20), it is easily seen that $X^{\mathbf{a}}$ and $X^{\mathbf{a}_\kappa}$ are solutions to the same SDE and equal in distribution.

For $\mathbf{a} \in \mathfrak{C}$, the diffusion $X^{\mathbf{a}}$ has been characterized as a diffusion with infinitesimal generator $(L, \mathrm{Dom}(L))$ obtained through the quadratic form $(E_\alpha, \mathrm{H}^1(\mathbb{R}))$ as well as a diffusion $(A, \mathrm{Dom}(A))$ with $A = \frac{1}{2} D_M D_S$ with (S, M) given by (2.6)-(2.7).

Proposition 2.11. *Let σ be piecewise $\mathcal{C}^1$ and ν be a finite measure of type*

$$\nu(\,\mathrm{d}x) = d(x)\,\mathrm{d}x + \sum_{i \in \mathbb{Z}} \alpha_{x_i}\delta_{x_i},\ |\alpha_{x_i}| \leq 1 - \epsilon,\ i \in J \subset \mathbb{Z},$$

for an increasing family $\{x_i\}_{i \in J}$ with no cluster points, some $\epsilon > 0$, and a bounded function d. Then there exists $\mathbf{a} = (a, \rho, b) \in \mathfrak{P}$ such that $X^{\mathbf{a}}$ is solution to (2.19).

Proof. Select an interval $[x_i, x_{i+1}]$. Set $a_i(x) = \beta_i = 1$, $\rho_i(x) = \sigma(x)$ and $b(x) = d(x)a_i(x)\rho_i(x)$ for $x \in (x_i, x_{i+1})$. Now, at the endpoints x_{i+1}, find β_{i+1} such that

$$\frac{\beta_{i+1} - \beta_i}{\beta_{i+1} + \beta_i} = \alpha_i.$$

Set $a_{i+1}(x) = \beta_i$, $\rho_i(x) = \sigma(x)/\beta_i$ and $b(x) = d(x)a_i(x)\rho_i(x)$ for $x \in (x_{i+1}, x_{i+2})$. Hence, we construct (a, ρ, b) iteratively on the intervals (x_j, x_{j+1}) for $j > i$. The same construction could be performed for $j < i$ by going downward instead of upward. It is easily checked that $\mathbf{a} = (a, \rho, b) \in \mathfrak{P}$ with $\mathrm{Dis}(\mathbf{a}) = \{x_i\}_{i \in J}$. □

3. Resolvent in the presence of one discontinuity

3.1. A computation method

We now assume that the coefficients $\mathbf{a} = (a, \rho, b) \in \mathfrak{P}$ are discontinuous only at one point, say 0. We fix $y \neq 0$ and $\lambda \in \mathbb{C}$.

We denote by I the intervals $(-\infty, 0 \wedge y)$, $(0 \wedge y, y \vee 0)$ and $(y \vee 0, +\infty)$ and introduce the operator A_I whose coefficients $\mathbf{a}_I$ are such that $\mathbf{a}_I = \mathbf{a}$ on I.

Let $u_I^{\nearrow}$ and $u_I^{\searrow}$ be the two functions on $\mathbb{C} \times \mathbb{R}$ given by Proposition 2.4 for each A_I. When $y > 0$, we set

$$q_1(\lambda, x) = u^{\nearrow}_{(-\infty,0)}(\lambda, x),\ q_2(\lambda, x) = u^{\searrow}_{(0,y)}(\lambda, x),$$
$$q_3(\lambda, x) = u^{\nearrow}_{(0,y)}(\lambda, x) \text{ and } q_4(\lambda, x) = u^{\searrow}_{(y,\infty)}(\lambda, x),$$

whereas, for $y < 0$,

$$q_1(\lambda, x) = u^{\nearrow}_{(-\infty,y)}(\lambda, x),\ q_2(\lambda, x) = u^{\searrow}_{(y,0)}(\lambda, x),$$
$$q_3(\lambda, x) = u^{\nearrow}_{(y,0)}(\lambda, x) \text{ and } q_4(\lambda, x) = u^{\searrow}_{(0,\infty)}(\lambda, x).$$

Proposition 3.1. *When* $\mathbf{a} \in \mathfrak{P}$ *is only discontinuous at* 0*, for each* $y \in \mathbb{R} \setminus \{0\}$*,*

$$\begin{aligned} r(\lambda, x, y) = k_1(\lambda, y)\, q_1(\lambda, x)\mathbb{1}(x < 0) + k_2(\lambda, y)\, q_2(\lambda, x)\mathbb{1}(x \in (0, y)) \\ + k_3(\lambda, y)\, q_3(\lambda, x)\mathbb{1}(x \in (0, y)) + k_4(\lambda, y)\, q_4(\lambda, x)\mathbb{1}(x \geq y), \end{aligned} \tag{3.1}$$

for any $\lambda \in \mathbb{H}^+$ *and any* $x \in \mathbb{R}$*, with the function* $k = (k_1, k_2, k_3, k_4)$ *solution to*

$$M(\lambda, y)k(\lambda, y)^T = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T \tag{3.2}$$

where for $y > 0$*,*

$$M(\lambda, y) = \begin{bmatrix} q_1(\lambda, 0) & -q_2(\lambda, 0) & -q_3(\lambda, 0) & 0 \\ a(0-)q_1'(\lambda, 0) & -a(0+)q_2'(\lambda, 0) & -a(0+)q_3'(\lambda, 0) & 0 \\ 0 & q_2(\lambda, y) & q_3(\lambda, y) & -q_4(\lambda, y) \\ 0 & \frac{\rho(y)a(y)}{2}q_2'(\lambda, y) & \frac{\rho(y)a(y)}{2}q_3'(\lambda, y) & -\frac{\rho(y)a(y)}{2}q_4'(\lambda, y) \end{bmatrix}$$

while for $y < 0$*,*

$$M(\lambda, y) = \begin{bmatrix} 0 & q_2(\lambda, 0) & q_3(\lambda, 0) & -q_4(\lambda, 0) \\ 0 & a(0-)q_2'(\lambda, 0) & a(0-)q_3'(\lambda, 0) & a(0+)q_4'(\lambda, 0) \\ q_1(\lambda, y) & -q_2(\lambda, y) & -q_3(\lambda, y) & 0 \\ \frac{\rho(y)a(y)}{2}q_1'(\lambda, y) & -\frac{\rho(y)a(y)}{2}q_2'(\lambda, y) & -\frac{\rho(y)a(y)}{2}q_3'(\lambda, y) & 0 \end{bmatrix}.$$

Proof. When $\lambda \in \mathbb{C}$, any solution to $(\lambda - A)u = 0$ on some interval I should be sought as a linear combination of $u^{\nearrow}(\lambda, \cdot)$ and $u^{\searrow}(\lambda, \cdot)$ provided that $\mathrm{Wr}[u^{\searrow}, u^{\nearrow}](\lambda) \neq 0$. This is the case for $\lambda \in \mathbb{H}^+$ (See Corollary 2.2). Our choice of $q_i(\lambda, \cdot)$ in the decomposition 3.1 ensures the correct vanishing behaviour at infinity for $\lambda \in \mathbb{H}^+$.

The system (3.2) is an algebraic transcription of the compatibility condition at 0 given by (2.16) and y given by (2.18) in Proposition 2.7. Hence, with any solution $k(\lambda, y)$ to (3.2), the right-hand-side of (3.1) is indeed equal to $r(\lambda, x, y)$ for any $\lambda \in \mathbb{H}^+$.

When $M(\lambda, y)$ is invertible for $y \in \mathbb{R}$ and λ in a domain $\mathbb{H}^+$ then $\lambda \mapsto M(\lambda, y)^{-1}$ is holomorphic on $\mathbb{H}^+$ since it is composed of sums, products and ratios of the terms of $M(\cdot, y)$ which are themselves holomorphic on $\mathbb{H}^+$.

Let us prove now that $M(\lambda, y)$ is invertible for $\lambda \in \mathbb{H}^+$. Since $q_2(\lambda, y) = q_4(\lambda, y)$ and $q_2'(\lambda, y) = q_4'(\lambda, y)$ for any $\lambda \in \mathbb{C}$, the determinant of $M(\lambda, y)$ is for $y > 0$

$$\begin{aligned} \det M(\lambda, y) = \frac{\rho(y)\exp(-h(y))}{2} \mathrm{Wr}[q_2, q_3](\lambda, y) \\ \times \{a(0-)q_1'(\lambda, 0)q_2(\lambda, 0) - a(0+)q_1(\lambda, 0)q_2'(\lambda, 0)\} \end{aligned}$$

where $\mathrm{Wr}[q_2, q_3](\lambda, \cdot)$ denotes the Wronskian of $q_2(\lambda, \cdot)$ and $q_3(\lambda, \cdot)$ in x. The above equality follows from the relation $a(x)D_x = \exp(-h(x))D_S$.

From Corollary 2.2, $\mathrm{Wr}[q_2, q_3](\lambda, y) \neq 0$ for any $\lambda \in \mathbb{H}^+$.

Let us consider the solutions $u^{\nearrow}$ and $u^{\searrow}$ associated with the coefficients $\mathbf{a}$ given by Proposition 2.4. From our choice of q_1 and q_2, $q_1(\lambda, 0) = q_2(\lambda, 0) = u^{\nearrow}(\lambda, 0) = u^{\searrow}(\lambda, 0) = 1$ and the functions q_1 and $u^{\nearrow}$ vanish at $-\infty$ whereas q_2 and $u^{\searrow}$ vanish at $+\infty$. With Corollary 2.1, then $u^{\nearrow}(\lambda, x) = q_1(\lambda, x)$ for $x \leq 0$ and $u^{\searrow}(\lambda, x) = q_2(\lambda, x)$ for $x \geq 0$. Thus,

$$a(0-)q_1'(\lambda, 0)q_2(\lambda, 0) - a(0+)q_1(\lambda, 0)q_2'(\lambda, 0) = \exp(-h(0))\,\mathrm{Wr}[u^{\searrow}, u^{\nearrow}](\lambda, 0).$$

This quantity cannot vanish on $\mathbb{H}^+$ according to Corollary 2.2. Thus, $\det M(\lambda, y) \neq 0$ for $y > 0$, $\lambda \in \mathbb{H}^+$ and $M(\lambda, y)$ is holomorphic on $\mathbb{H}^+$. The case $y < 0$ is similar. □

Remark 3.1. Of course, this approach is also feasible when the state space is bounded, one should consider up to 6 functions with the appropriate boundary conditions. This is useful for example to provide formulae related to the first exit time or first passage time as in [27, 2]. One could also use an expansion relying on eigenvalues and eigenfunctions. For practical purposes, especially for numerical simulation which was our original goal, a large number of eigenfunctions should be considered (See [19]). Actually, the eigenvalue formulation is the best for estimating the density for large times.

3.2. Skew Brownian motion with a piecewise constant drift

We now apply the computation method developed in the previous section to skew diffusions with a piecewise constant drift. Therefore, let us consider the operator

$$L = \frac{\rho(x)}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(a(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + b(x)\frac{\mathrm{d}}{\mathrm{d}x} \tag{3.3}$$

with for some $\beta \in (0, 1)$, $\mathbf{a} = (a, \rho, b) = \widehat{\mathbf{a}}(\beta, b_+, b-)$ defined in Table 1.

	for $x \geq 0$	for $x < 0$
$a(x)$	β	$1-\beta$
$\rho(x)$	β^{-1}	$(1-\beta)^{-1}$
$b(x)$	b_+	b_-

TABLE 1
Coefficients $\widehat{\mathbf{a}}(\beta, b_+, b_-) \in \mathfrak{P}$.

Remark 3.2. Out of 0, $\rho(x)a(x) = 1$. Thus, from (2.19) and (2.20), the associated diffusion process $X^{\mathbf{a}}$ is solution to

$$X_t^{\mathbf{a}} = x + B_t + (2\beta - 1)L_t^0(X^{\mathbf{a}}) + \int_0^t b(X_s^{\mathbf{a}})\,\mathrm{d}s \text{ for } t \geq 0. \tag{3.4}$$

For some constant $\gamma \in \mathbb{R}$, we consider the two functions $u^{\nearrow}$ and $u^{\searrow}$ given by (2.14) in Example 2.2 and remark that the Green function of $L_\gamma = \frac{1}{2}D_x^2 + \gamma D_x$ has a density with respect to the Lebesgue measure containing these functions and which is defined by

$$g(\gamma, \lambda, x, y) = \frac{1}{\sqrt{\gamma^2 + 2\lambda}} \begin{cases} e^{(-\gamma + \sqrt{\gamma^2+2\lambda})(x-y)} & \text{if } x < y, \\ e^{(-\gamma - \sqrt{\gamma^2+2\lambda})(x-y)} & \text{if } x \geq y. \end{cases} \tag{3.5}$$

Proposition 3.2. *The resolvent kernel of L with coefficients $\widehat{\mathfrak{a}}(\beta, b_+, b_-)$ is for $y \geq 0$ and $\lambda \in \mathbb{C} \setminus (-\infty, 0)$,*

$$r(\lambda, x, y) = g(b_+, \lambda, x, y)\mathbb{1}(x \geq 0) + A^{-+}(\lambda, y) g(b_-, \lambda, x, y)\mathbb{1}(x < 0) \\ + A^{++}(\lambda, y) g(b_+, \lambda, x, -y)\mathbb{1}(x \geq 0) \tag{3.6}$$

and for $y < 0$,

$$r(\lambda, x, y) = g(b_-, \lambda, x, y)\mathbb{1}(x < 0) + A^{--}(\lambda, y) g(b_-, \lambda, x, -y)\mathbb{1}(x < 0) \\ + A^{+-}(\lambda, y) g(b_+, \lambda, x, y)\mathbb{1}(x \geq 0) \tag{3.7}$$

with

$$\begin{aligned}
A^{-+}(\lambda, y) &= \Theta(\lambda, b_+, b_-)^{-1} 2\,\beta \sqrt{b_-^2 + 2\lambda}\, e^{(b_+ - b_- + \sqrt{b_-^2+2\lambda} - \sqrt{b_+^2+2\lambda})y}, \\
A^{++}(\lambda, y) &= \Theta(\lambda, b_+, b_-)^{-1} \big(\beta(\sqrt{b_+^2 + 2\lambda} - b_+) - (1-\beta)(\sqrt{b_-^2 + 2\lambda} - b_-)\big) e^{2b_+ y}, \\
A^{--}(\lambda, y) &= \Theta(\lambda, b_+, b_-)^{-1} \big(-\beta\,(\sqrt{b_+^2 + 2\lambda} + b_+) + (1-\beta)\,(\sqrt{b_-^2 + 2\lambda} + b_-)\big) e^{2b_- y}, \\
A^{+-}(\lambda, y) &= \Theta(\lambda, b_+, b_-)^{-1} 2(\,\beta - 1) \sqrt{b_+^2 + 2\lambda} e^{(b_- - b_+ + \sqrt{b_-^2+2\lambda} - \sqrt{b_+^{+}2\lambda})y},
\end{aligned}$$

where the common denominator is

$$\Theta(\lambda, b_+, b_-) = \beta\big(\sqrt{b_+^2 + 2\lambda} + b_+\big) + (1-\beta)\big(\sqrt{b_-^2 + 2\lambda} - b_-\big). \tag{3.8}$$

Remark 3.3. Using the expressions of $g(b_\pm, \lambda, x, y)$ and $A^{\pm,\pm}(\lambda, y)$, the inverse transform $\mathcal{L}^{-1}$ can be applied to each terms of the decompositions (3.6) and (3.7), as they are holomorphic whenever $\Theta(\lambda, b_+, b_-) \neq 0$ and $2\lambda \neq -b_\pm^2$ (only for $\lambda \in \mathbb{R}_-$) and decreasing to 0 as $|\lambda| \to \infty$.

Remark 3.4. When $x = 0$, (3.6) and (3.7) simplify to

$$r(\lambda, 0, y) = \begin{cases} 2\beta\Theta(\lambda, b_+, b_-)^{-1} g(b_+, \lambda, 0, y) & \text{for } y > 0, \\ 2(\beta - 1)\Theta(\lambda, b_+, b_-)^{-1} g(b_-, \lambda, 0, y) & \text{for } y < 0, \end{cases} \tag{3.9}$$

where $\Theta(\lambda, b_+, b_-)$ is given by (3.8).

Proof. It is an application of Proposition 3.1. Since $u^{\nearrow}(\gamma, \lambda, 0) = u^{\searrow}(\gamma, \lambda, 0) = 1$, we set

$$q_1(\lambda, x) = u^{\nearrow}(b_-, \lambda, x),\ q_3(\lambda, x) = u^{\nearrow}(b(y), \lambda, x),$$
$$q_2(\lambda, x) = u^{\searrow}(b(y), \lambda, x) \text{ and } q_4(\lambda, x) = u^{\searrow}(b_+, \lambda, x).$$

The solution of $M(\lambda,y)k(\lambda,y)=\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T$ is for $y\geq 0$,

$$\begin{cases} k_1(\lambda,y) = \frac{-2\beta}{(\beta-1)(-b_-+\sqrt{b_-^2+2\lambda})+\beta(-b_+-\sqrt{b_+^2+2\lambda})}\, e^{-(-b_++\sqrt{b_+^2+2\lambda})y}, \\ k_2(\lambda,y) = \frac{(1-\beta)(-b_-+\sqrt{b_-^2+2\lambda})-\beta(-b_++\sqrt{b_+^2+2\lambda})}{\sqrt{b_+^2+2\lambda}[(\beta-1)(-b_-+\sqrt{b_-^2+2\lambda})+\beta(-b_+-\sqrt{b_+^2+2\lambda})]} e^{-(-b_++\sqrt{b_+^2+2\lambda})y} \\ k_3(\lambda,y) = \frac{1}{\sqrt{b_+^2+2\lambda}} e^{-(-b_++\sqrt{b_+^2+2\lambda})y}, \\ k_4(\lambda,y) = \frac{(1-\beta)(-b_-+\sqrt{b_-^2+2\lambda})-\beta(-b_++\sqrt{b_+^2+2\lambda})}{\sqrt{b_+^2+2\lambda}[(\beta-1)(-b_-+\sqrt{b_-^2+2\lambda})+\beta(-b_+-\sqrt{b_+^2+2\lambda})]} e^{-(-b_++\sqrt{b_+^2+2\lambda})y}, \\ \quad +\frac{1}{\sqrt{b_+^2+2\lambda}} e^{-(-b_+-\sqrt{b_+^2+2\lambda})y}, \end{cases}$$

while for $y<0$,

$$\begin{cases} k_1(\lambda,y) = \frac{(1-\beta)(-b_--\sqrt{b_-^2+2\lambda})-\beta(-b_+-\sqrt{b_+^2+2\lambda})}{\sqrt{b_-^2+2\lambda}[(\beta-1)(-b_-+\sqrt{b_-^2+2\lambda})+\beta(-b_+-\sqrt{b_+^2+2\lambda})]} e^{-(-b_--\sqrt{b_-^2+2\lambda})y}, \\ \quad +\frac{1}{\sqrt{b_-^2+2\lambda}} e^{-(-b_-+\sqrt{b_-^2+2\lambda})y}, \\ k_2(\lambda,y) = \frac{1}{\sqrt{b_-^2+2\lambda}} e^{-(-b_--\sqrt{b_-^2+2\lambda})y}, \\ k_3(\lambda,y) = \frac{(1-\beta)(-b_--\sqrt{b_-^2+2\lambda})-\beta(-b_+-\sqrt{b_+^2+2\lambda})}{\sqrt{b_-^2+2\lambda}[(\beta-1)(-b_-+\sqrt{b_-^2+2\lambda})+\beta(-b_+-\sqrt{b_+^2+2\lambda})]} e^{-(-b_--\sqrt{b_-^2+2\lambda})y} \\ k_4(\lambda,y) = \frac{2(\beta-1)}{(\beta-1)(-b_-+\sqrt{b_-^2+2\lambda})+\beta(-b_+-\sqrt{b_+^2+2\lambda})} e^{-(-b_--\sqrt{b_-^2+2\lambda})y}. \end{cases}$$

Finally, it remains to plug the coefficients at their rightful places to obtain the solutions in each case. After some simplifications, we get (3.6) and (3.7). □

3.3. Skew diffusion with piecewise constant coefficients

Let us now consider the diffusion whose infinitesimal generator is $(L,\mathrm{Dom}(L))$ given by (3.3) and coefficients $\mathbf{a}=(a,\rho,b)$ are given in Table 2.

	for $x\geq 0$	for $x<0$
$a(x)$	a_+	a_-
$\rho(x)$	ρ_+	ρ_-
$b(x)$	b_+	b_-

TABLE 2
Coefficients $\mathbf{a}=(a,\rho,b)$ *constants on* $\mathbb{R}_+$ *and* $\mathbb{R}_-$.

In order to set up the diffusion coefficient $\sqrt{a\rho}$ to 1, the one-to-one transform $\phi(x)=x/\sqrt{a(x)\rho(x)}$ is convenient. If $X^{\mathbf{a}}$ is the diffusion process associated with $\mathbf{a}=(a,\rho,b)$, then $X^{\phi_\sharp\mathbf{a}}$ is the diffusion process associated to

$$\phi_\sharp\mathbf{a}=\left(\frac{\sqrt{a}}{\sqrt{\rho}},\frac{\sqrt{\rho}}{\sqrt{a}},\frac{b}{\sqrt{a\rho}}\right).$$

With Remark 2.7, we can consider $\kappa=(\sqrt{a_+/\rho_+}+\sqrt{a_-/\rho_-})^{-1}$ and see that $X^{\phi_\sharp\mathbf{a}}$ has the same distribution as the SBM with parameter $\beta=\kappa\sqrt{a_+/\rho_+}\in(0,1)$ and piecewise constant drift

$\overline{b}(x) = b(x)/\sqrt{a(x)\rho(x)}$, that is $\phi_\sharp \mathbf{a} = \widehat{\mathbf{a}}(\beta, b_+/\sqrt{a_+\rho_+}, b_-/\sqrt{a_-\rho_-})$. Hence, the computations of Section 3.1 could be applied to get the resolvent of $X^{\phi_\sharp \mathbf{a}}$. Now using a change of variable,

$$r_{\mathbf{a}}(\lambda, x, y) = \frac{r_{\phi_\sharp \mathbf{a}}(\lambda, \phi(x), \phi(y))}{\sqrt{a(y)\rho(y)}},$$

where $r_{\mathbf{a}}$ (resp. $r_{\phi_\sharp \mathbf{a}}$) is the resolvent kernel of $X^{\mathbf{a}}$ (resp. $X^{\phi_\sharp \mathbf{a}}$).

4. Some explicit expressions for the density

We introduce the functions

$$H(y) = \mathbb{1}(y \geq 0),\ H^-(y) = \mathbb{1}(y < 0) = \mathbb{1}(y \in \mathbb{R}) - H(y),$$
$$\operatorname{sgn}(y) = \mathbb{1}(y \geq 0) - \mathbb{1}(y < 0),$$
$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-v^2}\,\mathrm{d}v \text{ and } \operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-v^2}\,\mathrm{d}v.$$

4.1. The Skew Brownian Motion

The SBM of parameter β is given by the choice of the coefficients $\widehat{\mathbf{a}}(\beta, 0, 0)$. Thus,

$$A^{-+}(\lambda, y) = 2\beta,\ A^{++}(\lambda, y) = 2\beta - 1, A^{--}(\lambda, y) = 1 - 2\beta,\ A^{--}(\lambda, y) = 1 - 2\beta.$$

To recover the density of the SBM, we need to invert the Green function of the Laplace operator since none of the $A^{\pm\pm}(\lambda, y)$ depend on λ. It is well known that it gives the Gaussian kernel.

More precisely, from [1],

$$\mathcal{L}^{-1}\left(\frac{1}{\sqrt{\lambda}}\, e^{-k\sqrt{\lambda}}\right) = \frac{1}{\sqrt{\pi t}}\, e^{-\frac{k^2}{4t}}, \quad k \geq 0, \tag{4.1}$$

and

$$\mathcal{L}^{-1}(f(c\lambda + d)) = \frac{1}{c}\, e^{-\frac{d}{c}t}\, \mathcal{L}^{-1}(f)\left(\frac{t}{c}\right), \quad c > 0. \tag{4.2}$$

Thus, after an application of (4.1)-(4.2) to (3.6) and (3.7) as well as some rewriting

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} + \operatorname{sgn}(y)(2\beta - 1)\frac{1}{\sqrt{2\pi t}} e^{-\frac{(|y|+|x|)^2}{2t}}.$$

This density was obtained in [44] through a probabilistic argument. Some plots are given in Figure 1.

The cumulative distribution function $P : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \mapsto (0, 1)$ of p is

$$P(t, x, y) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{y - x}{\sqrt{2t}}\right)\right] - \frac{2\beta - 1}{2}\left[1 - \operatorname{sgn}(y) \operatorname{erf}\left(\frac{y + \operatorname{sgn}(y)|x|}{\sqrt{2t}}\right)\right].$$

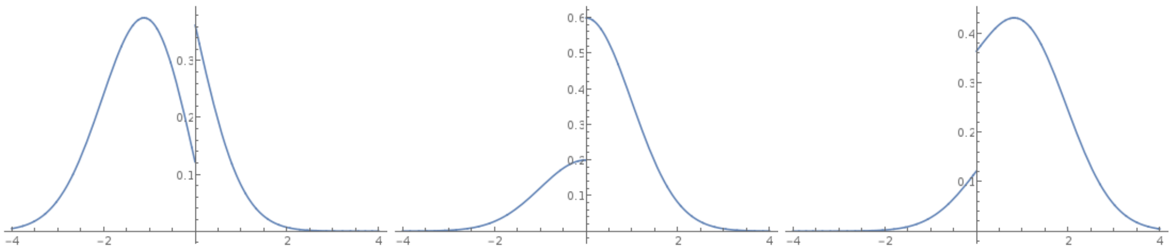

FIG 1. *The density $p(t,x,y)$ of the SBM for $\beta = \frac{3}{4}$ at time $t = 1$ for the initial positions: $x = -1$, $x = 0$ and $x = 1$ (from left to right).*

4.2. The Skew Brownian Motion with a constant drift

The density of the SBM with a constant drift, which corresponds to $\widehat{\mathbf{a}}(\beta, b, b)$ for some $b \neq 0$ was computed in [3, 4, 11, 16] with different probabilistic arguments.

If b is constant, then in Proposition 3.2,

$$A^{-+}(\lambda, y) = \frac{2\beta\sqrt{b^2+2\lambda}}{\sqrt{b^2+2\lambda}+b(2\beta-1)},\ A^{++}(\lambda, y) = e^{2by}\frac{(2\beta-1)(\sqrt{b^2+2\lambda}-b)}{\sqrt{b^2+2\lambda}+b(2\beta-1)},$$
$$A^{--}(\lambda, y) = e^{2by}\frac{(1-2\beta)(\sqrt{b^2+2\lambda}+b)}{\sqrt{b^2+2\lambda}+b(2\beta-1)},$$
$$\text{and } A^{+-}(\lambda, y) = \frac{2(\beta-1)\sqrt{b^2+2\lambda}}{\sqrt{b^2+2\lambda}+b(2\beta-1)}.$$

Let us compute the Laplace inverse of $A^{\pm\pm}(\lambda, y)g(b, \lambda, x, y)$. From [1],

$$\mathcal{L}^{-1}\left(\frac{e^{-k\sqrt{\lambda}}}{d+\sqrt{\lambda}}\right) = \frac{1}{\sqrt{\pi t}}\,e^{-\frac{k^2}{4t}} - de^{dk}\,e^{d^2t}\,\mathrm{erfc}\left(d\sqrt{t}+\frac{k}{2\sqrt{t}}\right),\quad k \geq 0, \tag{4.3}$$

and

$$\mathcal{L}^{-1}\left(\frac{e^{-k\sqrt{\lambda}}}{\sqrt{\lambda}(d+\sqrt{\lambda})}\right) = e^{dk}\,e^{d^2t}\,\mathrm{erfc}\left(d\sqrt{t}+\frac{k}{2\sqrt{t}}\right),\quad k \geq 0. \tag{4.4}$$

Thus, after using (4.1)-(4.2) and (4.3)-(4.4) as well as some recombination (see Figure 2),

$$\begin{aligned} p(t,x,y) = {} & \frac{1}{\sqrt{2\pi t}}e^{-\frac{(bt-(y-x))^2}{2t}} + \mathrm{sgn}(y)(2\beta-1)\frac{1}{\sqrt{2\pi t}}e^{-\frac{(|y|+|x|)^2}{2t}}\,e^{-b^2\frac{t}{2}}\,e^{b(y-x)} \\ & + \mathrm{sgn}(y)(H^-(y)-\beta)b(2\beta-1)e^{b(2\beta-1)(|y|+|x|)}e^{b(y-x)}e^{2b^2\beta(\beta-1)t} \\ & \times \mathrm{erfc}\left(b(2\beta-1)\sqrt{\frac{t}{2}}+\frac{|y|+|x|}{\sqrt{2t}}\right). \end{aligned}$$

We recover the density obtained in [3, 11] up to a conversion of erfc to its probabilistic counterpart.

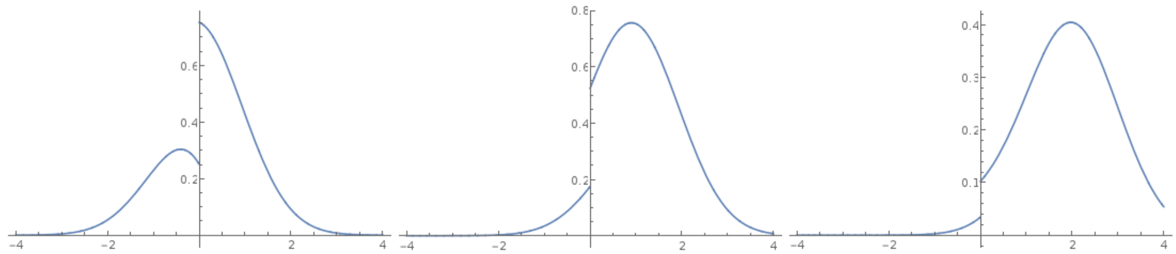

FIG 2. *The density $p(t,x,y)$ of the SBM for $\beta = \frac{3}{4}$ with a constant drift $b = 2$ at time $t = 1$ for the initial positions: $x = -1$, $x = 0$ and $x = 1$ (from left to right).*

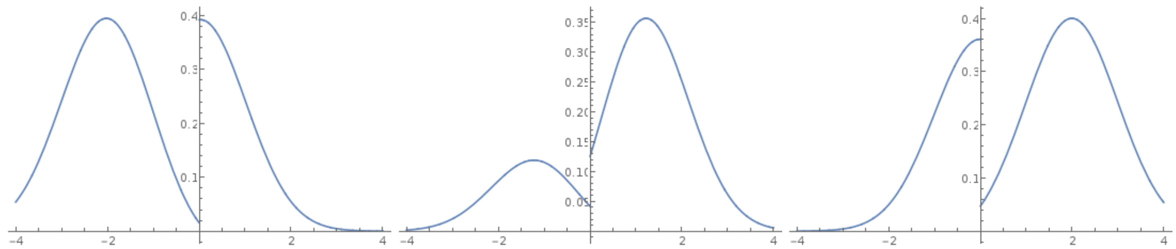

FIG 3. *The density $p(t,x,y)$ of the SBM for $\beta = \frac{3}{4}$ with a Bang bang drift $b = 2$ at time $t = 1$ for the initial positions $x = -1$, $x = 0$ and $x = 1$ (from left to right).*

4.3. The Bang-Bang Skew Brownian Motion

The Bang-Bang SBM is the diffusion corresponding to the choice of the coefficients $\widehat{\mathbf{a}}(\beta, b, -b)$ for some $b \neq 0$. This process was introduced in [14].

With this choice of coefficients, in Proposition 3.2,

$$A^{-+}(\lambda, y) = e^{2by}\frac{2\beta\sqrt{b^2+2\lambda}}{b+\sqrt{b^2+2\lambda}},\ A^{++}(\lambda, y) = e^{2by}\frac{(2\beta-1)\sqrt{b^2+2\lambda}-b}{b+\sqrt{b^2+2\lambda}},$$
$$A^{--}(\lambda, y) = e^{-2by}\frac{-(2\beta-1)\sqrt{b^2+2\lambda}-b}{b+\sqrt{b^2+2\lambda}} \text{ and } A^{+-}(\lambda, y) = e^{-2by}\frac{2(\beta-1)\sqrt{b^2+2\lambda}}{b+\sqrt{b^2+2\lambda}}.$$

Using (4.1)-(4.2) and (4.3)-(4.4), the density associated with the coefficients $\widehat{\mathbf{a}}(\beta, b, -b)$ is (see Figure 3)

$$\begin{aligned} p(t,x,y) = {} & \frac{1}{\sqrt{2\pi t}}e^{-\frac{(bt-\operatorname{sgn}(y)(y-x))^2}{2t}} \\ & + \operatorname{sgn}(y)(2\beta-1)\frac{1}{\sqrt{2\pi t}}e^{-\frac{(|y|+|x|)^2}{2t}}e^{-b^2\frac{t}{2}}e^{b(|y|-|x|)} \\ & + \operatorname{sgn}(y)(H^-(y)-\beta)be^{2b|y|}\operatorname{erfc}\left(b\sqrt{\frac{t}{2}}+\frac{|y|+|x|}{\sqrt{2t}}\right). \end{aligned}$$

This is the expression given in [14] up to a conversion of erfc into its probabilistic counterpart.

4.4. *The constant Péclet case*

Computing the transition density in all situations seems to be a very difficult problem. There are however other cases were simplifications occur. Among these situations, we consider the case where b_+ and b_- are linked by

$$b_- = \mu b_+ \text{ with } \mu = \frac{\beta}{1-\beta} \text{ or equivalently } (1-\beta)b_- = \beta b_+,\ \beta \neq \frac{1}{2}. \tag{4.5}$$

Remark 4.1. This assumption on b_+ and b_- is natural. If we consider the diffusion $X^{\mathbf{a}}$ with piecewise constant coefficients $\mathbf{a}$ as in Section 3.3, for $\phi(x) = x/\sqrt{a(x)\rho(x)}$, then $\phi(X)$ is a SBM with a piecewise constant drift. Is it easily checked that (4.5) is satisfied when

$$\frac{b_+}{\rho_+} = \frac{b_-}{\rho_-}.$$

When $a = 1$, the ratio b/ρ is called the *Péclet number*. It is a dimensionless quantity which plays a very important role in fluid mechanics by characterizing the effect of the convection against the diffusion and vice versa.

When (4.5) holds,

$$\Theta(\lambda, b_+, \mu b_-) = \beta\sqrt{b_+^2 + 2\lambda} + (1-\beta)\sqrt{b_-^2 + 2\lambda}.$$

Hence,

$$\frac{1}{\Theta(\lambda, b_+, \mu b_-)} = \frac{\beta\sqrt{b_+^2 + 2\lambda} - (1-\beta)\sqrt{b_-^2 + 2\lambda}}{\beta^2(b_+^2 + 2\lambda) - (1-\beta)^2(b_-^2 + 2\lambda)} = \frac{\beta\sqrt{b_+^2 + 2\lambda} - (1-\beta)\sqrt{b_-^2 + 2\lambda}}{2\lambda(2\beta - 1)}.$$

Therefore, for $x = 0$,

$$\begin{aligned} r(\lambda, y) = \frac{\beta\, e^{b_+ y}}{2\beta - 1} \frac{\beta\sqrt{b_+^2 + 2\lambda} - (1-\beta)\sqrt{b_-^2 + 2\lambda}}{\lambda} e^{-\sqrt{b_+^2 + 2\lambda} y} \mathbb{1}(y \geq 0) \\ - \frac{(1-\beta)\, e^{b_- y}}{2\beta - 1} \frac{\beta\sqrt{b_+^2 + 2\lambda} - (1-\beta)\sqrt{b_-^2 + 2\lambda}}{\lambda} e^{\sqrt{b_-^2 + 2\lambda} y} \mathbb{1}(y < 0). \end{aligned}$$

Lemma 4.1. *For $a \in \mathbb{R}$,*

$$\begin{aligned} \mathcal{L}^{-1}\left(\sqrt{a+s} e^{-\sqrt{s}\, y}\right)(t, y) = \frac{1}{\pi} \int_0^a \sin(y\sqrt{r})\sqrt{a-r}\, e^{-rt}\, \mathrm{d}r \\ - \frac{1}{\pi} \int_0^{+\infty} \cos(y\sqrt{r+a})\, \sqrt{r}\, e^{-(r+a)t}\, \mathrm{d}r. \end{aligned}$$

Proof. Inspired by [38], we use the Bromwich formula with the contour Γ illustrated in Figure 4.

Since the integrals on the outer and inner arcs as well as half-circles are null,

$$\begin{aligned} \mathcal{L}^{-1}\left(\sqrt{a+s}\, e^{-\sqrt{s}\, y}\right)(r, y) &= \frac{1}{2i\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} \sqrt{a+s}\, e^{-y\sqrt{s}} e^{st}\, \mathrm{d}s \\ &= \frac{1}{2i\pi} \left(\int_0^a (e^{iy\sqrt{r}} - e^{-iy\sqrt{r}})\sqrt{a-r}\, e^{-rt}\, \mathrm{d}r - \int_a^{+\infty} (e^{iy\sqrt{r}} + e^{-iy\sqrt{r}})\, i\sqrt{r-a}\, e^{-rt}\, \mathrm{d}r \right), \end{aligned}$$

hence the result. □

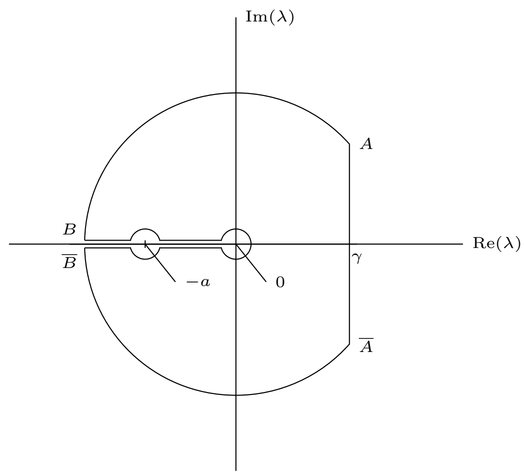

FIG 4. *The contour* Γ.

Proposition 4.1. *For any* $t \geq 0$ *and* $\widehat{\mathbf{a}}(\beta, b, \mu b)$*,*

$$
\begin{aligned}
p(t,0,y) = {} & \frac{2\,(H^{-}(y)-\beta)^2}{(2\beta-1)\sqrt{2\pi t}} e^{-\frac{(b(x)^2 t - y)^2}{2t}} \\
& + \frac{b(x)\,(H^{-}(y)-\beta)^2}{2(2\beta-1)} \left(\operatorname{erfc}\left(\frac{y}{\sqrt{2t}} - b(x)\sqrt{\frac{t}{2}} \right) - e^{2b(x)\,y} \operatorname{erfc}\left(\frac{y}{\sqrt{2t}} + b(x)\sqrt{\frac{t}{2}} \right) \right) \\
& - \frac{\beta\,(1-\beta)\,e^{b(x)\,y}}{2(2\beta-1)\pi} \int_0^t e^{\frac{-b(x)^2\,\tau}{2}} \int_0^{b^2(-x)-b^2(x)} \sin(y\sqrt{r})\sqrt{b^2(-x)-b^2(x)-r}\, e^{\frac{-r\,\tau}{2}}\,\mathrm{d}r\,\mathrm{d}\tau \\
& \frac{\beta\,(1-\beta)\,e^{b(x)\,y}}{2(2\beta-1)\pi} \int_0^t e^{-\frac{(b(-x)^2)\,\tau}{2}} \int_0^{+\infty} \cos(y\sqrt{r+b^2(-x)-b^2(x)})\sqrt{r}\, e^{\frac{-r\,\tau}{2}}\,\mathrm{d}r\,\mathrm{d}\tau.
\end{aligned}
$$

Proof. First,

$$
\begin{aligned}
\mathcal{L}^{-1}(r)(t,y) = {} & \frac{\beta^2\,e^{b_+\,y}}{2\beta-1} \mathcal{L}^{-1}\left(\frac{\sqrt{b_+^2+2\lambda}}{\lambda} e^{-\sqrt{b_+ + 2\lambda} y} \right)(t,y)\mathbb{1}(y \geq 0) \\
& - \frac{\beta\,(1-\beta)\,e^{b_+\,y}}{2\beta-1} \mathcal{L}^{-1}\left(\frac{\sqrt{b_-^2+2\lambda}}{\lambda} e^{-\sqrt{b_+ + 2\lambda} y} \right)(t,y)\mathbb{1}(y \geq 0) \\
& - \frac{\beta\,(1-\beta)\,e^{b_-\,y}}{2\beta-1} \mathcal{L}^{-1}\left(\frac{\sqrt{b_+^2+2\lambda}}{\lambda} e^{\sqrt{b_- + 2\lambda} y} \right)(t,y)\mathbb{1}(y < 0) \\
& + \frac{(1-\beta)^2\,e^{b_-\,y}}{2\beta-1} \mathcal{L}^{-1}\left(\frac{\sqrt{b_-^2+2\lambda}}{\lambda} e^{\sqrt{b_- + 2\lambda} y} \right)(t,y)\mathbb{1}(y < 0).
\end{aligned}
$$

By remarking that

$$\mathcal{L}^{-1}\left(\frac{\sqrt{b_+^2+2\lambda}}{\lambda}e^{-\sqrt{b_++2\lambda}y}\right)(t,y)=\mathcal{L}^{-1}\left(\frac{b_+^2}{\lambda\sqrt{b_+^2+2\lambda}}e^{-\sqrt{b_++2\lambda}y}\right)(t,y)$$
$$+2\,\mathcal{L}^{-1}\left(\frac{1}{\sqrt{b_+^2+2\lambda}}e^{-\sqrt{b_++2\lambda}y}\right)(t,y)\mathbb{1}(y\geq 0),$$

and using formulas (4.1), (4.2) and

$$\mathcal{L}^{-1}\left(\frac{e^{-ak\sqrt{s}}}{(s-d)\sqrt{s}}\right)=\frac{e^{dt}}{2d}\sqrt{d}e^{-ak\sqrt{d}}\,\mathrm{erfc}(\frac{ak}{2\sqrt{t}}-\sqrt{dt})$$
$$-\frac{e^{dt}}{2d}\sqrt{d}e^{ak\sqrt{d}}\,\mathrm{erfc}(\frac{ak}{2\sqrt{t}}+\sqrt{dt})$$

which can be found in [32],

$$\mathcal{L}^{-1}\left(\frac{\sqrt{b_+^2+2\lambda}}{\lambda}e^{-\sqrt{b_++2\lambda}y}\right)(t,y)=\frac{1}{2}b_+e^{-b_+\,y}\,\mathrm{erfc}\left(\frac{y}{\sqrt{2t}}-b_+\sqrt{\frac{t}{2}}\right)$$
$$-\frac{1}{2}b_+e^{b_+\,y}\,\mathrm{erfc}\left(\frac{y}{\sqrt{2t}}+b_+\sqrt{\frac{t}{2}}\right)+\frac{\sqrt{2}}{\sqrt{\pi t}}e^{-\frac{b_+^2\,t}{2}}e^{-\frac{y^2}{2t}}.$$

Then, using Lemma 4.1,

$$\mathcal{L}^{-1}\left(\frac{\sqrt{b_-^2+2\lambda}}{\lambda}e^{-\sqrt{b_++2\lambda}y}\right)(t,y)=$$
$$\frac{1}{2\pi}\int_0^t e^{\frac{-b_+^2\,\tau}{2}}\int_0^{b_-^2-b_+^2}\sin(y\sqrt{r})\sqrt{b_-^2-b_+^2-r}\,e^{\frac{-r\,\tau}{2}}\,\mathrm{d}r\,\mathrm{d}\tau$$
$$-\frac{1}{2\pi}\int_0^t e^{-\frac{b_+^2\,\tau}{2}}\int_0^{+\infty}\cos(y\sqrt{r+b_-^2-b_+^2})\sqrt{r}e^{\frac{-(r+b_-^2-b_+^2)\tau}{2}}\,\mathrm{d}r\,\mathrm{d}\tau.$$

Finally, when $y\geq 0$,

$$\mathcal{L}^{-1}(r)(t,y)=\frac{b_+\,\beta^2}{2(2\beta-1)}\,\mathrm{erfc}\left(\frac{y}{\sqrt{2t}}-b_+\sqrt{\frac{t}{2}}\right)\mathbb{1}(y\geq 0)$$
$$-\frac{b_+\,\beta^2\,e^{2b_+\,y}}{2(2\beta-1)}\,\mathrm{erfc}\left(\frac{y}{\sqrt{2t}}+b_+\sqrt{\frac{t}{2}}\right)\mathbb{1}(y\geq 0)+\frac{2\,\beta^2}{(2\beta-1)\sqrt{2\pi t}}e^{-\frac{(b_+t-y)^2}{2t}}$$
$$-\frac{\beta\,(1-\beta)\,e^{b_+\,y}}{2(2\beta-1)\pi}\int_0^t e^{\frac{-b_+^2\,\tau}{2}}\int_0^{b_-^2-b_+^2}\sin(y\sqrt{r})\sqrt{b_-^2-b_+^2-r}e^{\frac{-r\,\tau}{2}}\,\mathrm{d}r\,\mathrm{d}\tau\mathbb{1}(y\geq 0)$$
$$+\frac{\beta\,(1-\beta)\,e^{b_+\,y}}{2(2\beta-1)\pi}\int_0^t e^{-\frac{b_-^2\,\tau}{2}}\int_0^{+\infty}\cos(y\sqrt{r+b_-^2-b_+^2})\sqrt{r}e^{\frac{-r\,\tau}{2}}\,\mathrm{d}r\,\mathrm{d}\tau\mathbb{1}(y\geq 0).$$

The result proved using a symmetry argument when $y<0$. □

Conclusion

We have proposed a new method to compute the resolvent kernel of skew diffusions. In some cases, it leads to explicit expressions of their density.

Our approach is different from the one of B. Gaveau *et al.* [17]. Drift term can be taken in consideration, even when the drift is discontinuous. Not only known expressions are easily recovered but also new ones are computed. Hence, our approach overcomes the difficulties of a purely stochastic one.

At first glance, this approach seems to be rooted in the Feller theory of one-dimensional diffusions. Some generalizations are possible in particular cases of multi-dimensional diffusions. In addition, it leads to the development of new simulation techniques for the diffusions relying on exponential time steps instead of constant time step. This way, the expressions of the resolvent kernel are used as they are simpler than the ones of the density. These two directions of research are currently under investigation.

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