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A phase transition in the distribution of the length of integer partitions

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We assign a uniform probability to the set consisting of partitions of a positive integer n such that the multiplicity of each summand is less than a given number d and we study the limiting distribution of the number of summands in a random partition. It is known from a result by Erdős and Lehner published in 1941 that the distributions of the length in random restricted ($d = 2$) and random unrestricted ($d \geq n + 1$) partitions behave very differently. In this paper we show that as the bound d increases we observe a phase transition in which the distribution goes from the Gaussian distribution of the restricted case to the Gumbel distribution of the unrestricted case.

Keywords: Asymptotic expansions, integer partitions, multiplicities, limit distribution.

1 Introduction and statement of the results

The distribution of the number of summands in a random partition of an integer n was first studied by Erdős and Lehner [2] and then later by many other mathematicians. They showed that it follows a Gaussian distribution for restricted partitions (all parts distinct) and a Gumbel distribution for unrestricted partitions (arbitrary multiplicities). Their results were generalised and extended in many directions: for instance, analogous limit theorems were proved for general λ-partitions. See Haselgrove-Temperley [4], Richmond [8], and Lee [6] on unrestricted partitions, Hwang [5] on restricted partitions. We will closely follow the ideas of Hwang who proved that the distribution of the length of a random restricted λ-partition is asymptotically Gaussian.

In this paper, we consider partitions with no parts of multiplicity greater than d which has already been studied by Mutafchiev in [7], among others. Mutafchiev's result states that if $d \sim \alpha\sqrt{n}$ then among all partitions of n the set of partitions with no parts of multiplicity greater than d has a positive density asymptotically equal to

$$\prod_{\lambda}(1 - e^{-\alpha\lambda})^{-1}. \tag{1}$$

Here we are interested in the number of summands of such a partition, and we show that when d is asymptotically equal to $\sqrt{n}$, then we observe a phase transition in the distribution of the number of summands. More precisely we prove the following theorem:

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Theorem 1 *Let $S_{d,n}$ be the set of partitions of an integer n with no parts of multiplicity greater than d (d may be a function of n) and assume that all partitions in $S_{d,n}$ are equally likely. Then we have the following behaviour for the limit distribution of the number of summands in a random partition:*

- *if $d = o(\sqrt{n})$ then it is asymptotically Gaussian,*
- *if $dn^{-1/2}$ is unbounded then the distribution is asymptotically Gumbel,*
- *if $d \sim b\sqrt{n}$ where b is a positive constant, then when normalized, the distribution of the number of summands converges to a distribution with moment generating function given by*

$$M(x) = \prod_{\lambda} \frac{e^{-\frac{a}{\lambda}}}{1 - \frac{a}{\lambda}} \prod_{\lambda} \Big(\frac{1 - e^{-(\lambda - a)\vartheta}}{1 - e^{-\lambda\vartheta}} \Big) e^{a\vartheta/(e^{\lambda\vartheta}-1)}$$

where the product is taken over the set of positive integers, and

$$a = \frac{x}{\sqrt{\frac{\pi^2}{6} - \kappa}}, \ \vartheta = \frac{\pi}{\sqrt{6}} b, \quad \textit{and} \quad \kappa = \sum_{\lambda} \frac{\vartheta^2 e^{-\lambda\vartheta}}{(1 - e^{-\lambda\vartheta})^2}.$$

The mean and variance satisfy the following asymptotic formulae: for $d = o(\sqrt{n})$,

$$\mu_{d,n} \sim \log d \sqrt{\frac{6dn}{\pi^2(d-1)}}, \quad \sigma^2_{d,n} = \left(\frac{d-1}{2} - \frac{3d \log^2 d}{\pi^2(d-1)} \right) \sqrt{\frac{6dn}{\pi^2(d-1)}}$$

and for $d \gg \sqrt{n}$,

$$\mu_{d,n} \sim \frac{\sqrt{6n}}{2\pi} \log n, \quad \sigma^2_{d,n} \sim \left(\tfrac{\pi^2}{6} - \kappa \right) \frac{6n}{\pi^2}$$

as $n \to \infty$ ($\kappa = 0$ if $dn^{-1/2} \to \infty$).

These results are obtained by analysing the corresponding generating function. We are interested in the number of summands, and so the generating function for our problem is the following: for a positive integer d,

$$Q(d, u, z) = \prod_{\lambda} \sum_{j=0}^{d-1} u^j z^{j\lambda}, \tag{2}$$

where the product is taken over the set of positive integers, the second variable u counts the number of summands. Let $Q_{d,n}(u)$ be the coefficient of z^n in $Q(d, u, z)$ and let $\varpi_{d,n}$ be the random variable counting the number of summands in a random partition. Denote by $\mu_{d,n}$ and $\sigma_{d,n}$ its mean and its standard deviation, as in Theorem 1. Then we have the following immediate consequences:

$$\mathbb{E}(u^{\varpi_{d,n}}) = \frac{Q_{d,n}(u)}{Q_{d,n}(1)}, \tag{3}$$

$$\mu_{d,n} = \frac{\partial}{\partial u} \frac{Q_{d,n}(u)}{Q_{d,n}(1)} \Big|_{u=1}, \tag{4}$$

$$\sigma^2_{d,n} = \frac{\partial^2}{\partial^2 u} \frac{Q_{d,n}(u)}{Q_{d,n}(1)} \Big|_{u=1} + \mu_{d,n} - \mu^2_{d,n}. \tag{5}$$

Let us first define some useful functions that we will use throughout this paper:

$$
\begin{aligned}
F(u,\tau) &:= \log Q(d,u,e^{-\tau}),\\
f(u,\tau) &:= -\sum_{\lambda} \log(1-ue^{-\lambda\tau}),\\
g(\tau) &:= \sum_{\lambda} \frac{e^{-\lambda\tau}}{1-e^{-\lambda\tau}},\\
h(\tau) &:= \sum_{\lambda} \frac{e^{-\lambda\tau}}{(1-e^{-\lambda\tau})^2},\\
G(\tau) &:= g(\tau) - dg(d\tau),\\
H(\tau) &:= h(\tau) - d^2 h(d\tau).
\end{aligned}
$$

Note that we are omitting the parameter d in these definitions for simplicity. We denote by F_τ the partial derivative of F with respect to the second variable, $F_{\tau\tau}$, F_u,... are defined similarly. The same notations apply to the other functions.

It is possible to compute asymptotic formulae for the mean and variance by means of the saddle point method using equations (4) and (5), but we decided to not include these computations here explicitly since they do not differ much from those for the moment generating function that will be presented in more detail. Let us only state these asymptotic formulae, in which mean and variance are expressed in terms of the saddle point r_0, defined by the equation

$$n = -F_\tau(1, r_0). \tag{6}$$

For $d = o(\sqrt{n})$, we have

$$\mu_{d,n} = \frac{\log d}{r_0} + \mathcal{O}(d), \tag{7}$$

$$\sigma^2_{d,n} = \left(\frac{d-1}{2} - \frac{3d\log^2 d}{\pi^2(d-1)}\right) r_0^{-1} + \mathcal{O}\left(\frac{dr_0^c + r_0^{7c-3}\log^2\frac{1}{r_0}}{r_0}\right), \tag{8}$$

and for $d \gg \sqrt{n}$,we have

$$\mu_{d,n} = G(r_0) + \mathcal{O}(\log\tfrac{1}{r_0}) = (\log\tfrac{1}{r_0} + \gamma) r_0^{-1} + \mathcal{O}(\log\tfrac{1}{r_0}), \tag{9}$$

$$\sigma^2_{d,n} = \left(\tfrac{\pi^2}{6} - (dr_0)^2 h(dr_0)\right) r_0^{-2} + \mathcal{O}(r_0^{-1}\log^2\tfrac{1}{r_0}). \tag{10}$$

These asymptotic formulae for the mean and variance imply the formulae in Theorem 1 by using the Mellin transform method on Equation (6) (see the appendix for details and [3] for a nicely presented overview of the Mellin transform technique). We shall now prove the rest of Theorem 1 in a series of lemmas. Since the proofs of some of these lemmas are quite technical, they are mostly deferred to the appendix. The first ingredient is the following lemma, which plays an important role as we shall see in the next sections.

Lemma 2 *Let $2 \le d \le n$, and suppose that there are positive constants c_1 and c_2 such that $\frac{c_1}{\sqrt{n}} \le r \le \frac{c_2}{\sqrt{n}}$. If furthermore $\tau = r + iy$ with $\pi \ge |y| \ge r^{1+c}$, where c is any number within $(\frac{1}{3}, \frac{1}{2})$, and $\frac{1}{2} \le u \le 2$, then there are positive constants c_3 and δ depending only on c, c_1 and c_2 such that*

$$\frac{|Q(d,u,e^{-\tau})|}{Q(d,u,e^{-r})} \le e^{-c_3 n^{\delta}}$$

for sufficiently large n.

Proof: See appendix. □

2 The Case $d \gg \sqrt{n}$

Throughout this section, we assume that $d \gg \sqrt{n}$. To get the limit distribution we consider the normalized random variable

$$X_n = \frac{\varpi_{d,n} - \mu_{d,n}}{\sigma_{d,n}},$$

and we want to estimate the moment generating function

$$M_n(x) = \mathbb{E}(e^{xX_n}) = e^{-x\mu_{d,n}/\sigma_{d,n}} \frac{Q_{d,n}(e^{x/\sigma_{d,n}})}{Q_{d,n}(1)}. \tag{11}$$

It remains to determine an asymptotic formula for the coefficient $Q_{d,n}(u)$ for certain values of u. So we use the following integral representation:

$$Q_{d,n}(u) = \frac{e^{nr}}{2\pi} \int_{-\pi}^{\pi} \exp\Big(nit + F(u, r+it)\Big) dt. \tag{12}$$

From now on we set $u = e^{ar}$ where a is within some fixed interval around zero; a is always as such until the end of this section. We use the saddle point method and we choose $r = r(a,n)$ as the positive solution of the equation

$$n = -F_{\tau}(u,r). \tag{13}$$

It is not hard to check that the function on the right hand side is a monotone decreasing function of r for $r > 0$. So the solution exists, and it is unique. To obtain the asymptotic behaviour of the solution in terms of n we need the next result.

Lemma 3 *We have the estimates*

$$F_{\tau}(u,r) = -\frac{\pi^2}{6} r^{-2} + \mathcal{O}(r^{-1}\log\tfrac{1}{r}) \quad \textit{and} \quad F_{\tau\tau}(u,r) = \frac{\pi^2}{3} r^{-3} + \mathcal{O}(r^{-2}\log\tfrac{1}{r})$$

as $r \to 0^+$ uniformly in a.

Proof: See appendix. □

As a direct corollary of this lemma, we find that the solution r admits the asymptotic expansion

$$r = \frac{\pi}{\sqrt{6n}}(1 + \mathcal{O}(n^{-1/2} \log n))$$

as $n \to \infty$, uniformly in a and $d \gg \sqrt{n}$.

Then the next step is to split the integral (12) into three parts, namely the central part $|t| \le r^{1+c}$, from where the main term will come, and the tails. Here c is an arbitrary constant within the range $(1/3, 1/2)$. For $|t| \le r^{1+c}$, we have

$$nit + F(u, r + it) = F(u, r) - \frac{t^2}{2} F_{\tau\tau}(u, r) + O\Big(|t|^3 \max_{|\eta| \le r^{1+c}} \Big|F_{\tau\tau\tau}(u, r + i\eta)\Big|\Big).$$

One can use a similar approach as in the proof of Lemma 3 to obtain the following bound:

$$\max_{|\eta| \le r^{1+c}} \Big|F_{\tau\tau\tau}(u, r + i\eta)\Big| \ll r^{-4}.$$

Therefore, the central part of the integral can be estimated as:

$$e^{F(u,r)} \int_{r^{1+c}}^{r^{1+c}} e^{-F_{\tau\tau}(u,r)\frac{t^2}{2}} \, dt \left(1 + \mathcal{O}(r^{3c-1})\right). \tag{14}$$

Note that we can extend the range of integration in (14) to $(-\infty, \infty)$ at the cost of an exponentially small error:

$$2\Big|\int_{r^{1+c}}^{\infty} e^{-F_{\tau\tau}(u,r)\frac{t^2}{2}} \, dt\Big| \ll \int_{r^{1+c}}^{\infty} e^{-r^{c-2}t} dt = r^{2-c} e^{-r^{2c-1}}.$$

Thus the asymptotic formula for the central part of the integral follows:

$$e^{F(u,r)} \int_{-\infty}^{\infty} e^{-F_{\tau\tau}(u,r)\frac{t^2}{2}} \, dt \left(1 + \mathcal{O}(r^{3c-1})\right) = \frac{Q(d, u, e^{-r})}{\sqrt{2\pi F_{\tau\tau}(u, r)}}(1 + \mathcal{O}(n^{-\frac{(3c-1)}{2}}))$$

as $n \to \infty$. For the tails, we make use of Lemma 2. Indeed, we have

$$\frac{1}{Q(d, u, e^{-r})}\Big|\int_{r^{1+c} < |t| \le \pi} e^{nit + F(u, r+it)} dt\Big| \ll \int_{r^{1+c} < |t| \le \pi} \frac{Q(d, u, e^{-(r+it)})}{Q(d, u, e^{-r})} dt \ll e^{-c_3 n^{\delta}}.$$

Finally we obtain the following asymptotic formula:

$$Q_{d,n}(u) = \frac{e^{nr} Q(u, e^{-r})}{\sqrt{2\pi F_{\tau\tau}(u, r)}}(1 + \mathcal{O}(n^{-\frac{(3c-1)}{2}})) \tag{15}$$

as $n \to \infty$, uniformly in a, where $u = e^{ar}$. Now we use the latter asymptotic formula to derive an estimate for the moment generating function $M_n(x)$. For a fixed value of x, we define a and r such that r is the solution of

$$n = -F_\tau(e^{ar}, r) \text{ and } ar = \frac{x}{\sigma_{d,n}}.$$

This equation has a solution when x is within some appropriate fixed interval containing zero since $\sigma_{d,n}$ is of order $\sqrt{n}$, and so a is a bounded function of x, d and n. Before we continue our calculations, we call r_0 the value of r when $a = 0$ ($u = 1$). Then we deduce from (15) that

$$\frac{Q_{d,n}(e^{ar})}{Q_{d,n}(1)} = \exp(n(r - r_0) + F(e^{ar}, r) - F(1, r_0))(1 + o(1)) \tag{16}$$

as $n \to \infty$, uniformly in a.

The rest of the section is to estimate the exponent of (16) and to apply the result to determine the behaviour of (11). We first need to estimate the difference $|r - r_0|$.

Lemma 4 *We have*

$$|r - r_0| \ll \frac{\log n}{n}$$

as $n \to \infty$, uniformly in a.

Proof: See appendix. □

We can approximate $F(1, r)$ by means of the Taylor expansion around r_0. From Lemma 4, we get

$$F(1, r_0) = F(1, r) + n(r - r_0) + \mathcal{O}(n^{-1/2} \log^2 n). \tag{17}$$

Note here that $F_\tau(1, r_0) = -n$ by our choice of r_0. Hence the exponent of (16) is reduced to

$$F(e^{ar}, r) - F(1, r) + \mathcal{O}(n^{-1/2} \log^2 n),$$

and this estimate is uniform in a. We also have

$$\begin{aligned} F(e^{ar}, r) - F(1, r) =& a\, rG(r) + \sum_\lambda \Big(-\log\Big(1 - \frac{a}{\lambda}\Big) - \frac{a}{\lambda}\Big) \\ &+ \underbrace{f(1, dr) - f(e^{adr}, dr) + adr \cdot g(dr)}_{\text{(at most of constant order)}} + o(1). \end{aligned}$$

To see this, one only needs to take the Mellin transform of the left hand side, see the Appendix section for more details on this calculation.

Now we are going to use the latter equation to estimate (11). From the estimate (9) we have

$$\begin{aligned} \frac{x\mu_{d,n}}{\sigma_{d,n}} &= a\, rG(r_0) + \mathcal{O}(r \log \tfrac{1}{r}) \\ &= a\, rG(r) + \mathcal{O}(r \log^2 \tfrac{1}{r}), \end{aligned}$$

since

$$|G(r) - G(r_0)| \ll |G_\tau(r)||r - r_0| \ll \log^2 r.$$

Furthermore if we set $\vartheta := dr$, which is a function of x, d and n, then we finally have

$$M_n(x) \sim \prod_\lambda \frac{e^{-\frac{a}{\lambda}}}{1 - \frac{a}{\lambda}} \cdot \prod_\lambda \Big(\frac{1 - e^{-(\lambda - a)\vartheta}}{1 - e^{-\lambda\vartheta}} \Big) e^{a\vartheta/(e^{\lambda\vartheta} - 1)} \tag{18}$$

as $n \to \infty$ and $d \gg \sqrt{n}$.

Let us remark here that if $dn^{-1/2}$ goes to infinity then ϑ, which is a function of n, also goes to infinity, therefore

$$M_n(x) \to \prod_{\lambda} \frac{e^{-\frac{a}{\lambda}}}{1-\frac{a}{\lambda}} \text{ and } a \sim \frac{\sqrt{6}}{\pi} x$$

as $n \to \infty$, which is the moment generating function of the Gumbel distribution.

By Curtiss's theorem [1], the normalised random variable X_n converges in distribution to the Gumbel distribution as $n \to \infty$ just like in the case of unrestricted partitions ($d = n + 1$). This is not surprising since almost all partitions are covered in this case.

If now $dn^{-1/2}$ converges to some positive number b, then ϑ is asymptotically constant, more precisely $\vartheta \sim \frac{\pi}{\sqrt{6}} b$. These observations prove the second and the third part of our main theorem.

3 The case $d = o(n^{1/2})$

We will follow the lines in the previous section though there are several differences where we have to use other techniques. So the main goal is to comptute the moment generating function of the normalized random variable X_n. We need to have an estimate of $Q_{d,n}(u)$ for u within an interval containing 1 to understand the limit behaviour of X_n. Let $r = r(u, d, n)$ be the unique positive solution of the equation

$$n = -F_\tau(u, r). \tag{19}$$

The right hand side of (19) is a decreasing function of r if $r > 0$, and it tends to ∞ as $r \to 0^+$. This confirms the existence and the uniqueness of the solution r. Furthermore, the solution r goes to zero as n goes to infinity. We shall now find the asymptotic relation between r and n.

Lemma 5 *If $u = e^{x/\sigma_{d,n}}$, where x is a fixed real number, then*

$$F_\tau(u, y) = F_\tau(1, y)(1 + \mathcal{O}(\sqrt{d}\, n^{-1/4})) \tag{20}$$

as $n \to \infty$, uniformly for $y > 0$.

Proof: Since $\sigma_{d,n}$ is of order $\sqrt{d}n^{1/4}$, the result follows from the fact that

$$u^j = 1 + \mathcal{O}(\sqrt{d}n^{-1/4}),$$

uniformly for $0 \le j < d$ by replacing all powers of u in the expression of $F_\tau(u, y)$. □

This lemma implies that the solution of (19) is also of order $n^{-1/2}$ by estimating $F(1, y)$ (now that the parameter u is no longer present the asymptotic behaviour of this function can be determined easily). Therefore, dr is tending to zero as n tends to infinity. Furthermore, we have

$$n = \tfrac{\pi^2(d-1)}{6d} r^{-2} + \mathcal{O}(\sqrt{d}n^{3/4}). \tag{21}$$

We also need to estimate $F_{\tau\tau}(u, r)$ and $|F_{\tau\tau\tau}(u, r + it)|$ for $|t| \le r^{1+c}$.

Lemma 6 *If $u = e^{x/\sigma_{d,n}}$, where x is a fixed real number, then we have the estimates*

$$F_{\tau\tau}(u,r) \sim \frac{\pi^2(d-1)}{3d} r^{-3} \tag{22}$$

and

$$|F_{\tau\tau\tau}(u,r+it)| \ll r^{-4} \tag{23}$$

uniformly for $|t| \le r^{1+c}$.

Proof: See appendix. □

Now again the saddle point method applies and we get that if $u = e^{x/\sigma_{d,n}}$ for a fixed real number x, then

$$Q_{d,n}(u) \sim \frac{1}{\sqrt{2\pi F_{\tau\tau}(u,r)}} \exp\Big(nr + F(u,r)\Big) \tag{24}$$

as $n \to \infty$. This immediately implies that

$$\frac{Q_{d,n}(u)}{Q_{d,n}(1)} \sim \exp\Big(n(r-r_0) + F(u,r) - F(1,r_0)\Big) \tag{25}$$

as $n \to \infty$, since $F_{\tau\tau}(u,r)$ and $F_{\tau\tau}(1,r)$ are asymptotically equal, uniformly in u. It now remains to estimate the exponent of the right hand side of (25). As before at this stage we let r_0 be $r(1,d,n)$.

Lemma 7 *We have*

$$F(u,r) = F(1,r) + \frac{x}{r\sigma_{d,n}} \log d + \frac{(d-1)x^2}{4r\sigma_{d,n}^2} + o(1). \tag{26}$$

Proof: See appendix. □

On the other hand, we have

$$n(r-r_0) - F(1,r_0) = -F(1,r) + F_{\tau\tau}(1,r_0)\frac{(r-r_0)^2}{2} + \mathcal{O}(r_0^{-4}|r-r_0|^3)$$

since $F_\tau(1,r) = -n$ by definition, so we need an estimate of the difference $|r-r_0|$.

Lemma 8 *We have*

$$r - r_0 \sim (u-1)\frac{3d\log d}{\pi^2(d-1)} r_0 \tag{27}$$

if d is fixed, and

$$|r-r_0| = \mathcal{O}\Big(\frac{\log d}{\sqrt{d}} n^{-3/4}\Big). \tag{28}$$

if d goes to infinity with n.

Proof: See appendix. □

We deduce that

$$\begin{aligned} n(r-r_0)+F(1,r)-F(1,r_0) &= F_{\tau\tau}(1,r_0)\frac{(r-r_0)^2}{2}+\mathcal{O}(d^{-3/2}(\log d)^3\sqrt{r}) \\ &= \frac{3d(\log d)^2}{\pi^2(d-1)}\times\frac{x^2}{2r_0\sigma_{d,n}^2}+o(1) \end{aligned}$$

in the either case (if $d\to\infty$, then the first summand is also $o(1)$). By the latter equation combined with Lemma 7, we obtain the following formula for the exponent on the right hand side of (25):

$$n(r-r_0)+F(u,r)-F(1,r_0)=\frac{\log d}{r}\frac{x}{\sigma_{d,n}}+\Big(\frac{d-1}{2}+\frac{3d(\log d)^2}{\pi^2(d-1)}\Big)\frac{x^2}{2r_0\sigma_{d,n}^2}+o(1).$$

Hence, by using the estimates for $\mu_{d,n}$ and $\sigma_{d,n}^2$ in equations (7) and (8), respectively, we have, for a fixed real number x,

$$\begin{aligned} M_{d,n}(x) &= \mathbb{E}\Big(e^{\frac{x(\varpi_{d,n}-\mu_{d,n})}{\sigma_{d,n}}}\Big) \\ &= e^{-x\mu_{d,n}/\sigma_{d,n}}\frac{Q_{d,n}(e^{x/\sigma_{d,n}})}{Q_{d,n}(1)} \\ &= \exp\Big(\frac{-x}{\sigma_{d,n}}\Big(\frac{1}{r_0}-\frac{1}{r}\Big)\log d+\Big(\frac{d-1}{2}+\frac{3d(\log d)^2}{\pi^2(d-1)}\Big)\frac{x^2}{2r_0\sigma_{d,n}^2}+o(1)\Big) \\ &= \exp\Big(\frac{x^2}{2r_0\sigma_{d,n}^2}\Big(\frac{d-1}{2}-\frac{3d(\log d)^2}{\pi^2(d-1)}\Big)+o(1)\Big) \\ &= e^{\frac{x^2}{2}}(1+o(1)) \end{aligned}$$

as $n\to\infty$. This and Curtiss's theorem in [1] prove that if $d=o(n^{1/2})$ then we have convergence in law to the Gaussian distribution. That completes the proof of our main theorem.

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Appendix

Mellin transforms

Most of our functions are expressed in the form of harmonic sums, and we use the Mellin transform method to estimate them. More precisely, we are using the following result from [3]:

Theorem 9 *Let $\phi(x)$ be a continuous function on $(0,\infty)$ with Mellin transform $\phi^*(s)$ having a non empty fundamental strip $\langle\alpha,\beta\rangle$. Assume that $\phi^*(s)$ admits a meromorphic continuation to the strip $\langle\gamma,\beta\rangle$ for $\gamma<\alpha$ with a finite number of poles there, which is analytic on $\mathrm{Re}(s)=\gamma$. Assume also that there exists a real number $\eta\in(\alpha,\beta)$ such that*

$$\phi^*(s) = O(|s|^{-c}) \tag{29}$$

with $c>1$ as $|s|\to\infty$ in the strip $\gamma\le\mathrm{Re}(s)\le\eta$. If $\phi^*(s)$ admits the singular expansion for $s\in\langle\gamma,\alpha\rangle$

$$\phi^*(s) \asymp \sum_{(\xi,k)\in A} \frac{d_{\xi,k}}{(s-\xi)^k},$$

then an asymptotic expansion of $\phi(x)$ at 0, $x>0$, is

$$\phi(x) = \sum_{(\xi,k)\in A} \frac{(-1)^{k-1} d_{\xi,k}}{(k-1)!} x^{-\xi} (\log x)^{k-1} + O(x^{-\gamma}).$$

The advantage that we have is that most of our functions have a nicely behaved Mellin transform, for example:

$$\begin{aligned}
\mathcal{M}(f(1,r),s) &= \zeta(s+1)\Gamma(s)\zeta(s),\\
\mathcal{M}(g(r),s) &= \zeta^2(s)\Gamma(s),\\
\mathcal{M}(h(r),s) &= \zeta(s-1)\Gamma(s)\zeta(s).
\end{aligned}$$

The above functions are all expressed in terms of the Riemann zeta function $\zeta(s)$ and the gamma function $\Gamma(s)$. We know that $\zeta(s)$ admits a simple pole at $s=1$ with residue 1 and is analytic everywhere else in the complex plane, also $\Gamma(s)$ is analytic everywhere except for simple poles at $s = 0, -1, -2, \cdots$. Furthermore, all the above Mellin transforms satisfy the hypothesis of Theorem 9 therefore one has

$$f(1,r) = \frac{\pi^2}{6} r^{-1} - \frac{1}{2}\log\tfrac{1}{r} + \mathcal{O}(1),, \tag{30}$$

$$g(r) = (\log\tfrac{1}{r} + 2\gamma)r^{-1} + \mathcal{O}(1), \tag{31}$$

$$h(r) = \frac{\pi^2}{6} r^{-2} - \frac{1}{2} r^{-1} + \mathcal{O}(1), \tag{32}$$

where γ is the Euler-Mascheroni constant. In order to estimate $f(e^{ar}, r)$ for fixed a within the interval $(-1, 1)$, we also need the Hurwitz zeta function

$$\zeta(s, 1-a) = \sum_{\lambda} \frac{1}{(\lambda - a)^s}.$$

Note that the Mellin transform of the difference $f(e^{ar}, r) - f(1, r)$ is

$$\mathcal{M}(f(e^{ar}, r) - f(1, r), s) = \zeta(s+1)\Gamma(s)(\zeta(s, 1-a) - \zeta(s)).$$

The Hurwitz zeta function admits a simple pole at $s = 1$ with residue 1, therefore the pole $s = 1$ of the Mellin transform cancels out, and the pole at $s = 0$ becomes important. All we need to know for our purposes is that

$$\lim_{s\to 0} \frac{(\zeta(s, 1-a) - \zeta(s) - as\zeta(s+1))}{s} = \sum_{\lambda} \left(-\log\left(1 - \frac{a}{\lambda}\right) - \frac{a}{\lambda}\right),$$

and so we have the equation

$$f(e^{ar}, r) - f(1, r) = a\,rg(r) + \sum_{\lambda} \left(-\log\left(1 - \frac{a}{\lambda}\right) - \frac{a}{\lambda}\right) + o(1) \tag{33}$$

as $r \to 0^+$, the term $a\,rg(r)$ is the inverse Mellin transform of $a\zeta(s+1)\Gamma(s+1)\zeta(s+1)$.

Proofs of intermediate results

In the following, we give proofs of all the lemmas that are used in the proof of our main theorem.

Proof of Lemma 2: First we are going to estimate the quantity

$$\mathrm{Re}\Big(\sum_{\lambda}(e^{-\lambda r} - e^{-\lambda\tau})\Big),$$

which can be written in the following form:

$$\frac{1}{1-e^{-r}} - \mathrm{Re}\Big(\frac{1}{1-e^{-\tau}}\Big) = \frac{e^{-r}(1+e^{-r})(1-\cos y)}{(1-e^{-r})(1-2e^{-r}\cos y + e^{-2r})}$$
$$\gg \frac{|y|^2}{r(\max\{r, |y|\})^2} \gg r^{2c-1}.$$

as $r \to 0^+$. Now if $|z| \le 2$ then we claim that there are positive constants c_4 and c_5 such that

$$\frac{|1+z|}{1+|z|} \le e^{-c_4(|z|-\mathrm{Re}(z))}$$

and

$$\frac{|1+z+z^2|}{1+|z|+|z|^2} \le e^{-c_5(|z|-\mathrm{Re}(z))}.$$

Indeed for $|z| \le 2$ we have

$$\begin{aligned}\frac{|1+z|^2}{(1+|z|)^2} &= 1-2\frac{|z|-\mathrm{Re}(z)}{(1+|z|)^2}\\ &\le 1-\frac{2}{9}(|z|-\mathrm{Re}(z))\\ &\le e^{-\frac{2}{9}(|z|-\mathrm{Re}(z))}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{|1+z+z^2|^2}{(1+|z|+|z|^2)^2} &= 1-2(|z|-\mathrm{Re}(z))\frac{1+|z|^2+(2-\mathrm{Re}(z))(\mathrm{Re}(z)+|z|)}{(1+|z|+|z|^2)^2}\\ &\le 1-\frac{2}{49}(|z|-\mathrm{Re}(z))\\ &\le e^{-\frac{2}{49}(|z|-\mathrm{Re}(z))}.\end{aligned}$$

Hence for any $2 \le d \le n$

$$|1+z+z^2+z^3+\cdots+z^{d-1}| \le |1+z|+|z|^2|1+z|+\cdots,$$

where the last term is either $|z|^{d-2}|1+z|$ or $|z|^{d-3}|1+z+z^2|$ depending on the parity of d. Therefore by the claim we have

$$|1+z+z^2+\cdots+z^{d-1}| \le e^{-c_6(|z|-\mathrm{Re}(z))}(1+|z|+|z|^2+\cdots+|z|^{d-1}),$$

where $c_6 = \min\{c_4, c_5\}$. Now we set $z = ue^{-\lambda\tau}$ and take the product over all $\lambda \ge 1$ to obtain

$$\frac{|Q(u,e^{\tau})|}{Q(u,e^{r})} \le \exp\Big(-c_6\sum_{\lambda}(e^{-\lambda r}-\mathrm{Re}(e^{-\lambda\tau}))\Big).$$

This completes the proof. □

Proof of Lemma 3:

We start with $F_\tau(e^{ar}, r)$, which can be written as a difference of two sums:

$$\sum_{\lambda}\frac{d\lambda}{e^{(\lambda-a)dr}-1}-\sum_{\lambda}\frac{\lambda}{e^{(\lambda-a)r}-1}.$$

We estimate these sums separately. First we have

$$\sum_\lambda \frac{\lambda}{e^{(\lambda-a)r}-1} = \sum_\lambda \frac{\lambda-a}{e^{(\lambda-a)r}-1} + a\sum_\lambda \frac{1}{e^{(\lambda-a)r}-1},$$

and hence the Mellin transform can be computed as

$$\zeta(s)\Gamma(s)\left(\zeta(s-1,1-a)+a\zeta(s,1-a)\right).$$

The dominant singularity is at $s=2$ which is a simple pole, and the next singularity is at $s=1$ which is a double pole, therefore by Theorem 9 we have

$$\sum_\lambda \frac{\lambda}{e^{(\lambda-a)r}-1} = \frac{\pi^2}{6}r^{-2} + \mathcal{O}(r^{-1}\log\tfrac{1}{r})$$

as $r \to 0^+$. It also follows that

$$\begin{aligned} d\sum_\lambda \frac{\lambda}{e^{(\lambda-a)dr}-1} &= \mathcal{O}(d(dr)^{-2}) \\ &= \mathcal{O}(d^{-1}r^{-2}). \end{aligned}$$

Therefore, r is of order $n^{-1/2}$, and by the assumption that $d \gg \sqrt{n}$, dr is bounded below. Hence

$$d\sum_\lambda \frac{\lambda}{e^{(\lambda-a)dr}-1} = \mathcal{O}(r^{-1})$$

and the first part of the lemma follows. The second part is proved analogously. □

Proof of Lemma 4: Since $r = r(a) := r(a,d,n)$ is uniquely determined by a, d, and n, we can apply implicit differentiation on the equation

$$n = -F_\tau(e^{ar}, r).$$

We get

$$\frac{\partial}{\partial a}r(a)\Big|_{a=a_1} = -\frac{\frac{\partial}{\partial a}F_\tau(e^{ar(a_1)}, r(a_1))\Big|_{a=a_1}}{\frac{\partial}{\partial r}F_\tau(e^{a_1 r}, r)\Big|_{r=r(a_1)}}. \tag{34}$$

We can compute the numerator:

$$\frac{\partial}{\partial a}F_\tau(e^{ar}, r)\Big|_{a=a_1} = r\left(\sum_\lambda \frac{\lambda e^{-(\lambda-a)r}}{(1-e^{-(\lambda-a)r})^2} - d^2\sum_\lambda \frac{\lambda e^{-(\lambda-a)dr}}{(1-e^{-(\lambda-a)dr})^2}\right).$$

By using the Mellin transform we can show that

$$\sum_\lambda \frac{\lambda e^{-(\lambda-a)r}}{(1-e^{-(\lambda-a)r})^2} \ll r^{-2}\log\frac{1}{r}.$$

For the second term, we know that $dr \gg 1$, therefore we have

$$d^2 \sum_{\lambda} \frac{\lambda e^{-(\lambda-a)dr}}{(1-e^{-(\lambda-a)dr})^2} \ll r^{-2} \sum_{\lambda} (\lambda dr)^2 e^{-\lambda dr} \ll r^{-2}.$$

The denominator can also be estimated in the same way and we have

$$\left| \frac{\partial}{\partial r} F_\tau(e^{a_1 r}, r) \Big|_{r=r(a_1)} \right| \gg r^{-3}.$$

Therefore,

$$|r - r_0| \ll \sup_{a_1} \frac{\partial}{\partial a} r(a) \Big|_{a=a_1} \ll \mathcal{O}(r^2 \log \tfrac{1}{r})$$

which completes the proof. □

Proof of Lemma 6:
Let

$$A := [\tfrac{x}{r\sigma_{d,n}}] \quad \text{and} \quad a := \tfrac{x}{r\sigma_{d,n}} - A,$$

where $[.]$ denotes the nearest integer. For $\lambda \le A$ and for a fixed non-negative integer k, there are positive constants K_1 and K_2 depending only on k such that

$$K_1 d^{k+1} \le \sum_{j=0}^{d-1} j^k u^j e^{-\lambda jr} \le K_2 d^{k+1}, \tag{35}$$

since $u^j = 1 + \mathcal{O}(\sqrt{d} n^{-1/4})$ and $\lambda jr \ll \sqrt{d} n^{-1/4}$ as well. Now we split the series $F_{\tau\tau}(u, r)$ into two parts and we denote by S_1 the sum over $\lambda \le A$ and by S_2 the sum over $\lambda > A$. We are going to estimate them separately: we have

$$S_1 = \sum_{\lambda \le A} \lambda^2 \frac{\sum_{j=0}^{d-1} j^2 u^j e^{-\lambda jr} \sum_{j=0}^{d-1} u^j e^{-\lambda jr} - \left(\sum_{j=0}^{d-1} j u^j e^{-\lambda jr} \right)^2}{\left(\sum_{j=0}^{d-1} u^j e^{-\lambda jr} \right)^2}$$

and so

$$S_1 \le \sum_{\lambda \le A} \lambda^2 \frac{\sum_{j=0}^{d-1} j^2 u^j e^{-\lambda jr}}{\sum_{j=0}^{d-1} u^j e^{-\lambda jr}} \ll A^3 d^2 \ll n$$

by (35). For S_2 we shift the summation so that we can write the sum as

$$S_2 = \sum_{\lambda \ge 1} (\lambda + A)^2 \left(\frac{e^{-(\lambda-a)r}}{(1-e^{-(\lambda-a)r})^2} - \frac{d^2 e^{-(\lambda-a)dr}}{(1-e^{-(\lambda-a)dr})^2} \right).$$

Now we expand $(\lambda + A)^2$. Then the term with λ^2 is equal to $F_{\tau\tau}(e^{ar}, r)$, and the term with A^2 is almost the same as $H(r)$: the difference is that the sum is taken over a slightly shifted sequence, where the shift

a is at most $\frac{1}{2}$ in absolute value. Since the Dirichlet series of the shifted sequence is $\zeta(s, 1-a)$, the term with A^2 contributes only $\mathcal{O}(A^2 H(r))$. The term with $2A\lambda$ can be written as

$$2A \sum_{\lambda} (\lambda - a) \left(\frac{e^{-(\lambda-a)r}}{(1-e^{-(\lambda-a)r})^2} - \frac{d^2 e^{-(\lambda-a)dr}}{(1-e^{-(\lambda-a)dr})^2} \right) + \mathcal{O}(AH(r)).$$

The sum can be estimated by using the Mellin transform method, and we have

$$\sum_{\lambda} (\lambda - a) \frac{e^{-(\lambda-a)r}}{(1-e^{-(\lambda-a)r})^2} = \frac{\log \frac{1}{r}}{r^2} + \mathcal{O}(r^{-2})$$

and

$$d^2 \sum_{\lambda} (\lambda - a) \frac{e^{-(\lambda-a)dr}}{(1-e^{-(\lambda-a)dr})^2} = \frac{\log \frac{1}{dr}}{r^2} + \mathcal{O}(r^{-2}).$$

Putting everything together we get

$$F_{\tau\tau}(u, r) = F_{\tau\tau}(e^{ar}, r) + \mathcal{O}\left(\frac{n^{5/4} \log d}{\sqrt{d}} \right),$$

and we can estimate $F_{\tau\tau}(e^{ar}, r)$ again by Theorem 9 to get the estimate in (22).

The estimate in (23) is done in a similar manner. □

Proof of Lemma 7: Let v be $\frac{x}{\sigma_{d,n}}$, so that $u = e^v$, and set $A = [v/r]$. Moreover, we set

$$S_1' = \sum_{\lambda \le A} \log \Big(\sum_{j=0}^{d-1} e^{j(v-\lambda r)} \Big) \quad \text{and} \quad S_2' = \sum_{\lambda > A} \log \Big(\sum_{j=0}^{d-1} e^{j(v-\lambda r)} \Big),$$

so that $F(u, r) = S_1' + S_2'$. We estimate S_1' and S_2' separately. We know that v is of order $d^{-1/2} r^{1/2}$ and that $A = [\frac{v}{r}]$. Hence

$$\begin{aligned} S_1' &= \sum_{\lambda \le A} \log \Big(d + \frac{d(d-1)}{2} (v - \lambda r) + \mathcal{O}(d^2 r) \Big) \\ &= A \log d + \frac{(d-1)vA}{2} - \frac{r(d-1)A(A+1)}{4} + \mathcal{O}(\sqrt{dr}) \\ &= A \log d + \frac{(d-1)x^2}{2r\sigma_{d,n}^2} - \frac{(d-1)x^2}{4r\sigma_{d,n}^2} + \mathcal{O}(\sqrt{dr}) \\ &= A \log d + \frac{(d-1)x^2}{4r\sigma_{d,n}^2} + \mathcal{O}(\sqrt{dr}). \end{aligned}$$

To estimate S_2' we use the same trick as in the proof of Lemma 6 by shifting the sum and we get

$$\begin{aligned} S_2' - F(1, r) &= F(e^{ar}, r) - F(1, r) \\ &= f(e^{ar}, r) - f(1, r) - (f(e^{adr}, dr) - f(1, dr)) \\ &= a \log d + o(1), \end{aligned}$$

where $a = \frac{v}{r} - A$. Here we used Equation (33) to derive the last line from the second line. Combining the two, we get

$$\begin{aligned} F(u,r) = S_1' + S_2' &= F(1,r) + (A+a)\log d + \frac{(d-1)x^2}{4r\sigma_{d,n}^2} + o(1) \\ &= F(1,r) + \frac{v}{r}\log d + \frac{(d-1)x^2}{4r\sigma_{d,n}^2} + o(1) \end{aligned}$$

which completes the proof. □

Proof of Lemma 8: Let us assume first that d goes to infinity with n. As in the proof of Lemma 4 we use implicit differentiation and we get

$$\frac{\partial}{\partial u} r = -\frac{\frac{\partial}{\partial u} F_\tau(u,r)}{\frac{\partial}{\partial r} F_\tau(u,r)}. \tag{36}$$

Then we apply our routine calculation to estimate the numerator and the denominator. For the numerator we split the sum at $A = [\frac{\log u}{r}]$, and the first sum is

$$\sum_{\lambda \le A} \frac{\lambda}{u} \frac{\sum_{j=0}^{d-1} j^2 u^j e^{-\lambda j r} \sum_{j=0}^{d-1} u^j e^{-\lambda j r} - \sum_{j=0}^{d-1} j u^j e^{-\lambda j r} \sum_{j=0}^{d-1} j u^j e^{-\lambda j r}}{\left(\sum_{j=0}^{d-1} u^j e^{-\lambda j r}\right)^2}$$

which is of order $\mathcal{O}(A^2 d^2)$ by the same argument that we used in Lemma 6. After shifting the summation, the sum over $\lambda > A$ can be written as

$$\sum_{\lambda} \frac{\lambda + A}{u} \left(\frac{e^{-(\lambda - a)r}}{(1 - e^{-(\lambda - a)r})^2} - \frac{d^2 e^{-(\lambda - a)dr}}{(1 - e^{-(\lambda - a)dr})^2} \right).$$

Here we can see that this sum can be Mellin-transformed, and we can use Theorem 9 to prove that this sum is a $\mathcal{O}(r^{-2}\log d)$. We have already seen that the denominator admits the asymptotic estimate

$$F_{\tau\tau}(u,r) \gg r^{-3}.$$

These completes the case where d tends to infinity since

$$|r - r_0| \ll |u-1|(r\log d) \ll \frac{r\log d}{\sigma_{d,n}}.$$

If d is fixed then we have

$$-n = F_\tau(u,r) = F_\tau(1,r_0)$$

which implies that

$$F_\tau(u,r) - F_\tau(1,r) = -(F_\tau(1,r) - F_\tau(1,r_0)). \tag{37}$$

We estimate both sides of Equation (37). The right hand side is easier, and we get

$$-(F_\tau(1,r) - F_\tau(1,r_0)) = -F_{\tau\tau}(1,r_0)(r - r_0) + \mathcal{O}(r_0^{-4}|r - r_0|^2).$$

To estimate the left hand side, note that for any $0 \leq j < d$

$$u^j = 1 + j(u-1) + \mathcal{O}(dr),$$

and so for any positive integer λ we have

$$\begin{aligned}
&\frac{\sum_{j=0}^{d-1} ju^j e^{-\lambda jr}}{\sum_{j=0}^{d-1} u^j e^{-\lambda jr}} - \frac{\sum_{j=0}^{d-1} je^{-\lambda jr}}{\sum_{j=0}^{d-1} e^{-\lambda jr}} \\
&= \frac{\sum_{j=0}^{d-1} ju^j e^{-\lambda jr} \sum_{j=0}^{d-1} e^{-\lambda jr} - \sum_{j=0}^{d-1} je^{-\lambda jr} \sum_{j=0}^{d-1} u^j e^{-\lambda jr}}{\Big(\sum_{j=0}^{d-1} e^{-\lambda jr}\Big)^2 \Big(1 + \mathcal{O}(\sqrt{dr})\Big)} \\
&= (u-1) \frac{\sum_{j=0}^{d-1} j^2 e^{-\lambda jr} \sum_{j=0}^{d-1} e^{-\lambda jr} - \Big(\sum_{j=0}^{d-1} je^{-\lambda jr}\Big)^2}{\Big(\sum_{j=0}^{d-1} e^{-\lambda jr}\Big)^2} \Big(1 + \mathcal{O}(\sqrt{dr})\Big) \\
&+ \mathcal{O}\Big(dr \frac{\sum_{j=0}^{d-1} je^{-\lambda jr}}{\sum_{j=0}^{d-1} e^{-\lambda jr}}\Big).
\end{aligned}$$

Summing over all positive integers we have

$$\begin{aligned}
F_\tau(u,r) - F_\tau(1,r) &= (u-1)F_{u\tau}(1,r) \\
&\quad + \mathcal{O}\left(\sqrt{dr}|u-1||F_{u\tau}(1,r)| + dr|F_\tau(1,r)|\right).
\end{aligned}$$

Since r and r_0 are asymptotically equal, we have the asymptotic formulae

$$\begin{aligned}
&u - 1 \sim \frac{x}{\sigma_{d,n}}, \\
&F_\tau(1,r) \sim \frac{-\pi^2(d-1)}{6d} r^{-2}, \\
&F_{u\tau}(1,r) = G_\tau(r) \sim -(\log d) r^{-2}, \\
&F_{\tau\tau}(1,r_0) \sim \frac{\pi^2(d-1)}{3d} r^{-3}.
\end{aligned}$$

Finally we obtain

$$r - r_0 = \frac{-(u-1)F_{u\tau}(1,r)}{F_{\tau\tau}(1,r_0)} + \mathcal{O}(r^2 + r^{-1}|r-r_0|^2).$$

This gives the asymptotic formula in the statement of the lemma since r is asymptotically equal to r_0. $\square$