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Abstract

This paper investigates the interval observer design for a class of nonlinear continuous systems, which can be represented as a superposition of a uniformly observable nominal subsystem with a Lipschitz nonlinear perturbation. It is shown in this case there exists an interval observer for the system that estimates the set of admissible values for the state consistent with the output measurements. An illustrative example of the observer application is given with simulation results.

1 Introduction

The state estimation problem of uncertain nonlinear systems is studied in this work. In particular we are interested in the case when the model is nonlinear parameterized by a vector of unknown parameters θ and the model equations do not belong to a canonical form. Usually in such a case it is necessary to apply a transformation of coordinates representing the system in a canonical form with posterior design of an observer Besançon (2007); Nijmeijer and Fossen (1999). The presence of unknown parameters may seriously complicate the design of a required transformation of coordinates, since the transformation has to be dependent on θ. In this case the initial problem of the state estimation can be replaced with a relaxed one dealing with approximation of the interval of admissible values of the state vector.

Suppose that the unknown (may be time-varying) parameters θ belong to a compact set $\Theta \subset \mathbb{R}^p$, then the plant dynamics under consideration is given by

$$\begin{cases} \dot{x} = f(x) + B(x,\theta)u + \delta f(x,\theta), \\ y = h(x) + \delta h(x,\theta), \end{cases} \tag{1}$$

where x belongs to an open subset Ω of $\mathbb{R}^n$ (it is assumed that $0 \in \Omega$) and the initial state value belongs to a compact set $I_0(x_0) = [\underline{x}_0, \bar{x}_0]$; $y \in \mathbb{R}$ and $u \in \mathbb{R}^m$ represent respectively the output and the input. The vector fields f and h are smooth, and δf, δh and B are assumed to be locally Lipschitz continuous.

Despite of the existence of many solutions for observer design Besançon (2007); Nijmeijer and Fossen (1999), a design of state estimators for (1) is rather complicated since the system is intrinsically nonlinear and it has uncertain terms in the state and in the output equations. Therefore, the whole system (1) may be even not observable, which means that an exact estimation is not possible. Under this situation, we can relax the estimation goal making an evaluation of the interval of admissible values for the state applying the theory of set-membership or interval estimation Gouzé et al. (2000); Mazenc and Bernard (2010); Walter et al. (1996). Contrarily to the conventional case, where a pointwise value of the state is the objective for estimation, in the interval estimation two bounds on the set of admissible values are calculated and the width of the estimated interval is dependent of the model uncertainty.

Recently the interval observers have been proposed for a special class of nonlinear systems Raïssi et al. (2012), the model (1) is a generalization of that case. Applying a coordinate transformation to a canonical form computed for the known nominal system, we are going to estimate the interval value of the state of the uncertain system (1) improving the result from Raïssi et al. (2012). Another solution has been presented in Meslem and Ramdani (2011), where a hybrid interval observer design is presented for a class of continuous-time nonlinear systems. In the present work we are going to avoid the complexity of the hybrid system framework developing a continuous-time interval observer. For upper-triangular systems, an iterative design procedure for robust interval observers is proposed in Mazenc and

Bernard (2012), which is started from the assumption that for each subsystem a robust interval observer has been designed. The result of this paper is an extension of our recent work in Zheng et al. (2013), and can be considered as a complementary method for such an observer syntheses for a nonlinear system. Comparing with the existing results in the literature, the present paper considers an interval observer design for more general uncertain nonlinear systems, which may be not observable. When an exact estimation for such systems becomes impossible, the main contribution of this paper is to present a method to obtain an interval estimation.

The outline of this paper is as follows. Some preliminary results and notations are given in Section 2. The precise problem formulation is presented in Section 3. The main results are described in Section 4. An example of computer simulation is given in Section 5.

2 Preliminaries

2.1 Notations

- $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.
- $L_f h(x) = \frac{\partial}{\partial x} h(x).f(x)$ denotes the Lie derivative of h along the vector field f, and $L_f^n h = L_f(L_f^{n-1} h)$ is the n-th Lie derivative of h along the vector field f.
- $a\mathscr{R}b$ represents the element-wise relation $\mathscr{R}$ (a and b are vectors or matrices): for example $a < b$ (vectors) means $\forall i : a_i < b_i$.
- for a matrix $P = P^T$, the relation $P \preceq 0$ means that the matrix is negative semidefinite.
- for a matrix $\mathscr{A} \in \mathbb{R}^{n\times n}$, define $\mathscr{A}^+ = \max\{0, \mathscr{A}\}$ [1] and $\mathscr{A}^- = \mathscr{A}^+ - \mathscr{A}$. For a vector $x \in \mathbb{R}^n$, define $x^+ = \max\{0, x\}$ and $x^- = x^+ - x$.
- for a matrix (function) $\mathscr{A}$ the symbol $\mathscr{A}_i$ denotes its i^{th} column, for a vector (function) b the symbol b_i denotes its corresponding element.
- a matrix $\mathscr{A} \in \mathbb{R}^{n\times n}$ is called Metzler if all its elements outside the main diagonal are nonnegative.
- a Lebesgue measurable function $u : \mathbb{R}_+ \to \mathbb{R}^m$ belongs to the space $\mathscr{L}_\infty$ if $ess \sup_{t\geq 0} \|u(t)\| < +\infty$.

2.2 Backgrounds on cooperative/comparison systems

The notions of Comparison systems and Cooperative systems have appeared separately, but they concern the same class of systems:

- *Comparison systems*: when dealing with a qualitative property involving solutions of a complex system, it is sometimes of interest to obtain a simpler system whose solutions overvalue the solutions of the initial system in some sense. For ODE (Ordinary Differential Equation), the contributions of Müller (1926); Kamke (1932); Wazewski (1950) are probably the most important in this field: they give necessary and sufficient hypotheses ensuring that the solution of $\dot{x} = f(t,x)$, with initial state x_0 at time t_0 and function f satisfying the inequality $f(t,x) \leq g(t,x)$ is overvalued by the solution of the so-called "comparison system" $\dot{z} = g(t,z)$, with initial state $z_0 \geq x_0$ at time t_0, or, in other words, conditions on function g that ensure $x(t) \leq z(t)$ for $t \geq t_0$. These results were extended to many different classes of dynamical systems (Bitsoris (1978); Dambrine (1994); Dambrine et al. (1995); Dambrine and Richard (1993, 1994); Grujič et al. (1987); Laksmikantham and Leela (1969); Matrosov (1971); Perruquetti et al. (1995a,b); Tokumaru et al. (1975)).
- *Cooperative systems*: this class of systems includes those involving in $\mathbb{R}^n$ preserving positive order relation on initial data and input signals Smith (1995), *i.e.* if the initial conditions and properly rescaled inputs are positive, then so is the corresponding solution.

From these results one can deduce the following corollary:

Corollary 1 *Smith (1995) Assume that:*

H1) A is a Metzler matrix,

H2) $b(t) \in \mathbb{R}_+^n, \forall t \geq t_0$, where t_0 represents the initial time,

H3) the system

$$\frac{dx(t)}{dt} = Ax + b(t), \tag{2}$$

possesses, for every $x(t_0) \in \mathbb{R}_+^n$, a unique solution $x(t)$ for all $t \geq t_0$.

Then, for any $x(t_0) \in \mathbb{R}_+^n$, the inequality

$$x(t) \geq 0$$

holds for every $t \geq t_0$.

In other word, under conditions of Corollary 1, $\mathbb{R}_+^n$ is positively invariant w.r.t (2).

2.3 Canonical representation of a nonlinear system

Based on the studied system (1), one obtains the *nominal drift-system* by setting $u = 0$, $\delta f = 0$, $\delta h = 0$ in (1):

$$\begin{cases} \dot{x} = f(x), \\ y = h(x). \end{cases} \tag{3}$$

For a nonlinear system, "observability" depends on the considered state (local property) and control: this is the main reason why many different concepts related to observability exists Besançon (2007); Nijmeijer and Fossen (1999);

[1] The $\max\{\cdot\}$ operation is applied element-wise.

Gauthier et al. (1992). This paper assumes that the nominal system (3) satisfies the observability rank condition, *i.e.* the following change of coordinates:

$$\Phi_{(3)} \; : \; \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto \left(h(x), L_f h(x), \ldots, L_f^{n-1} h(x)\right)^T, \tag{4}$$

defines a local diffeomorphism from Ω onto $\Phi_{(3)}(\Omega)$. With this diffeomorphism $\zeta = \Phi_{(3)}(x)$, it follows that, the system (3) can be rewritten as:

$$\begin{cases} \dot{\zeta} = \tilde{A}\zeta + \tilde{b}\varphi(\zeta), \\ y = \tilde{C}\zeta, \end{cases} \tag{5}$$

where

$$\tilde{A} = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}, \tag{6}$$
$$\tilde{b} = (0, \ldots, 0, 1)^T, \tag{7}$$
$$\tilde{C} = (1, 0, \ldots, 0,), \tag{8}$$
$$\varphi(\zeta) = L_f^n h(x). \tag{9}$$

The forthcoming analysis is based on this canonical form.

3 Problem formulation

The objective of this work is to design an interval observer for the system (1). We will not even assume that (1) is observable, but need only the observability for the nominal system (3).

Assumption A1): The nominal system (3) is observable and $f(0) = 0$, $h(0) = 0$ in (3).

Note that by a modification of δf and δh, the nominal system f, h can be freely assigned by the designer. Thus (1) may be not observable, however using a transformation of coordinates obtained for the nominal observable (under Assumption A1)) system (3), the system (1) can be transformed into the following one:

$$\begin{cases} \dot{\zeta} = \tilde{A}\zeta + \tilde{F}(\zeta, \theta) + \tilde{G}(\zeta, \theta)u, \\ y = \tilde{C}\zeta + \tilde{H}(\zeta, \theta), \end{cases} \tag{10}$$

where

$$\tilde{G}(\zeta, \theta) = \left(\tilde{G}_1(x, \theta), \ldots, \tilde{G}_n(x, \theta)\right)^T_{x = \Phi_{(3)}^{-1}(\zeta)},$$
$$\tilde{G}_i(x, \theta) = L_{B(x,\theta)} L_{f(x)}^{i-1} h(x), i = 1 \ldots n,$$
$$\tilde{F}(\zeta, \theta) = \left(\tilde{F}_1(x, \theta), \ldots, \tilde{F}_{n-1}(x, \theta)\right)^T_{x = \Phi_{(3)}^{-1}(\zeta)} + \tilde{b}\varphi(\zeta),$$
$$\tilde{F}_i(x, \theta) = L_{\delta f(x,\theta)} L_{f(x)}^{i-1} h(x), i = 1 \ldots n,$$
$$\tilde{H}(\zeta, \theta) = \delta h(x, \theta)|_{x = \Phi_{(3)}^{-1}(\zeta)}.$$

To proceed we need the following fact.

Claim 1 *Under Assumption A1), there exist a matrix $\tilde{L}$ and an invertible matrix P such that the matrix $\tilde{A} - \tilde{L}\tilde{C}$ is similar to a Metzler matrix $A - LC$, which means $A - LC = P(\tilde{A} - \tilde{L}\tilde{C})P^{-1}$.*

The conditions of the existence of such a transformation matrix P can be found in Raïssi et al. (2012), they are related with solution of a Sylvester equation. By Assumption A1) the pair $(\tilde{A}, \tilde{C})$ is observable, then there always exists a matrix $\tilde{L}$ such that the claim is satisfied Raïssi et al. (2012). Moreover, we can also impose the constraint to calculate the gain $\tilde{L}$ in the sense to minimize the influence of the uncertainties on the interval estimation accuracy Chebotarev et al. (2015).

Introducing the new coordinates $z = P\zeta$ we arrive at the desired representation of the system (1):

$$\begin{cases} \dot{z} = Az + F(z, \theta) + G(z, \theta)u, \\ y = Cz + H(z, \theta), \end{cases} \tag{11}$$

where the matrices A, C are given in Claim 1, and $H(z, \theta) = \tilde{H}(P^{-1}z, \theta)$, $F(z, \theta) = P\tilde{F}(P^{-1}z, \theta)$, $G(z, \theta) = P\tilde{G}(P^{-1}z, \theta)$.

Remark 2 *Since the origin of (3) is assumed to be an equilibrium and $\Phi_{(3)}$ is a diffeomorphism with $\Phi_{(3)}(0) = 0$, thus the origin is also an equilibrium for the both transformed systems in coordinates ζ and z for $F = 0$ and $u = 0$. By construction, F, H and G are locally Lipschitz continuous.*

Let us remind that, since the initial condition x_0 for (1) is only known within a certain interval $I(x_0) = [\underline{x}_0, \overline{x}_0]$, then using the diffeomorphism $\Phi_{(3)}(x)$, the initial condition $z_0 = P\Phi_{(3)}(x_0)$ is also known within a certain interval $I(z_0) = [\underline{z}_0, \overline{z}_0]$. Thus our original problem turns out to a dynamical system with the input (u, y) and the outputs $\underline{z}(t)$ and $\overline{z}(t)$ such that for all $t \geq 0$ we have:

$$\underline{z}(t) \leq z(t) \leq \overline{z}(t).$$

4 Main Results

4.1 Bounding functions

Since Θ is a compact set and by continuity of $F(z,\theta), H(z,\theta)$ and $G(z,\theta)$ (the functions $\delta f(x,\theta), B(x,\theta)$ and $\delta h(x,\theta)$ were assumed to be continuous and $\Phi_{(3)}$ given by (4) is a diffeomorphism), the element-wise minimum and maximum of $F(z,\theta), H(z,\theta)$ and $G(z,\theta)u$ (for a given u) in the domain $\theta \in \Theta$, $\underline{z} \leq z \leq \overline{z}$ exist. In order to built the observers, we need a more precise knowledge on these max and min functions. Let us firstly introduce the following lemma.

Lemma 3 Efimov et al. (2012) *Let $\mathscr{A} \in \mathbb{R}^{n\times n}$, by the definition $\mathscr{A} = \mathscr{A}^+ - \mathscr{A}^-$ and for any $[\underline{z}, \overline{z}] \subset \mathbb{R}^n$ and $z \in \mathbb{R}^n$, if $\underline{z} \leq z \leq \overline{z}$, then*

$$\mathscr{A}^+\underline{z} - \mathscr{A}^-\overline{z} \leq \mathscr{A}z \leq \mathscr{A}^+\overline{z} - \mathscr{A}^-\underline{z}$$

Lemma 4 *Let $\underline{x}, x, \overline{x} \in \mathbb{R}^n$ and $\underline{\mathscr{A}}, \mathscr{A}, \overline{\mathscr{A}} \in \mathbb{R}^{n\times m}$, then*

$$\underline{x} \leq x \leq \overline{x} \Longleftrightarrow \underline{x}^+ \leq x^+ \leq \overline{x}^+,\ \overline{x}^- \leq x^- \leq \underline{x}^-;$$
$$\underline{\mathscr{A}} \leq \mathscr{A} \leq \overline{\mathscr{A}} \Longleftrightarrow \underline{\mathscr{A}}^+ \leq \mathscr{A}^+ \leq \overline{\mathscr{A}}^+,\ \overline{\mathscr{A}}^- \leq \mathscr{A}^- \leq \underline{\mathscr{A}}^-.$$

PROOF. Let us prove the vector case only, the matrix case can be proven similarly. By definition for any $x \in \mathbb{R}^n$ we have $x = x^+ - x^-$, then using the inequalities in the right hand side

$$x^+ - x^- = x \leq \overline{x} = \overline{x}^+ - \overline{x}^-,$$
$$x^+ - x^- = x \geq \underline{x} = \underline{x}^+ - \underline{x}^-,$$

and the relations in the left hand side are satisfied. Moreover, the relation in the left hand side implies the inequalities in the right hand side by their definitions . ■

Lemma 5 Efimov et al. (2012) *Let $\underline{\mathscr{A}} \leq \mathscr{A} \leq \overline{\mathscr{A}}$ for some $\underline{\mathscr{A}}$, $\overline{\mathscr{A}}$, $\mathscr{A} \in \mathbb{R}^{n\times n}$ and $\underline{x} \leq x \leq \overline{x}$ for $\underline{x}$, $\overline{x}$, $x \in \mathbb{R}^n$, then*

$$\begin{aligned} \underline{\mathscr{A}}^+\underline{x}^+ - \overline{\mathscr{A}}^+\underline{x}^- - \underline{\mathscr{A}}^-\overline{x}^+ + \overline{\mathscr{A}}^-\overline{x}^- &\leq \mathscr{A}x \\ \leq \overline{\mathscr{A}}^+\overline{x}^+ - \underline{\mathscr{A}}^+\overline{x}^- - \overline{\mathscr{A}}^-\underline{x}^+ + \underline{\mathscr{A}}^-\underline{x}^-&. \end{aligned} \tag{12}$$

To apply these lemmas, we have to introduce the following standard (see Gauthier et al. (1992), for example) assumption in the estimation theory on the boundedness of the state x and the input u values for system (1).

<u>Assumption A2)</u>: For the system (1), it is assumed that $x(t) \in \mathscr{X}$ and $u(t) \in \mathscr{U}$ for all $t \geq 0$, where $\mathscr{X} \subset \Omega$ and $\mathscr{U} \subset \mathbb{R}^m$ are two given compacts.

Under this assumption, since $\zeta = \Phi_{(3)}(x)$ defined by (4) is a diffeomorphism and due to the fact that $z = P\zeta$, thus there exists a compact set $\mathscr{Z} \subset \mathbb{R}^n$ such that $z(t) \in \mathscr{Z}$ for all $t \geq 0$.

In Raïssi et al. (2012) it has been assumed that uncertain terms in the system equations admit known upper and lower bounding functions, in this work we are going to prove that these functional bounds exist and satisfy some useful properties.

Lemma 6 *There exist two functions $\overline{F}, \underline{F} : \mathbb{R}^{2n} \to \mathbb{R}^n$ such that, for all $\theta \in \Theta$ and $\underline{z} \leq z \leq \overline{z}$ with $z \in \mathscr{Z}$, the following inequalities hold:*

$$\underline{F}(\underline{z}, \overline{z}) \leq F(z,\theta) \leq \overline{F}(\underline{z}, \overline{z}), \tag{13}$$

and for a given submultiplicative norm $\|\cdot\|$ we have:

$$\|\overline{F}(\underline{z},\overline{z}) - F(z,\theta)\| \leq \overline{l}_{\overline{F}}\|\overline{z} - z\| + \underline{l}_{\overline{F}}\|\underline{z} - z\| + l_{\overline{F}},$$
$$\|\underline{F}(\underline{z},\overline{z}) - F(z,\theta)\| \leq \overline{l}_{\underline{F}}\|\overline{z} - z\| + \underline{l}_{\underline{F}}\|\underline{z} - z\| + l_{\underline{F}},$$

for some positive constants $\overline{l}_{\overline{F}}$, $\underline{l}_{\overline{F}}$, $l_{\overline{F}}$, $\overline{l}_{\underline{F}}$, $\underline{l}_{\underline{F}}$ and $l_{\underline{F}}$.

PROOF. Assume that $F(0,\theta) \neq 0$, then we can decompose $F(z,\theta) = \hat{F}(z,\theta) + F(0,\theta)$ with $\hat{F}(0,\theta) = 0$. Define

$$\overline{F}_0 = \max_{\theta\in\Theta} F(0,\theta),\ \underline{F}_0 = \min_{\theta\in\Theta} F(0,\theta),$$

then

$$\underline{F}_0 \leq F(0,\theta) \leq \overline{F}_0,$$
$$\|\underline{F}_0 - F(0,\theta)\| \leq l_{\underline{F}_0},\ \|\overline{F}_0 - F(0,\theta)\| \leq l_{\overline{F}_0},$$

for all $\theta \in \Theta$. Thus, for the further analysis, and without the loss of generality, we can assume that $F(0,\theta) = 0$.

Since $F(0,\theta) = 0$ and F is locally Lipschitz continuous on $\mathscr{Z} \times \Theta$, then this condition for $z \in \mathscr{Z}$ and $\theta \in \Theta$ can be written as follows

$$\|F(z,\theta)\| \leq l_F\|z\|$$

for some $l_F > 0$ (the Lipschitz constant of F on $\mathscr{Z} \times \Theta$). Assume that there is a matrix function $M : \mathscr{Z} \times \Theta \to \mathbb{R}^{n\times n}$ such that

$$F(z,\theta) = M(z,\theta)z.$$

Without loosing generality define $M(0,\theta) = 0$ and consider $z \in \mathscr{Z} \setminus \{0\}$:

$$\|F(z,\theta)\| = \|M(z,\theta)z\| \leq l_F\|z\|,$$

then for $\|z\| \neq 0$

$$\|M(z,\theta)\frac{z}{\|z\|}\| = \frac{\|M(z,\theta)z\|}{\|z\|} \leq l_F.$$

By definition, $\|M(z,\theta)\| = \sup_{v\in\mathbb{R}^n, \|v\|=1}\|M(z,\theta)v\|$, thus $\|M(z,\theta)\| \leq l_F$ everywhere in $\mathscr{Z} \times \Theta$, due to the equivalence of different norms it implies that there exist $\underline{M},\overline{M} \in \mathbb{R}^{n\times n}$ such that

$$\begin{aligned}\overline{M} &= \max_{\theta\in\Theta, z\in\mathscr{Z}} M(z,\theta)\\ \underline{M} &= \min_{\theta\in\Theta, z\in\mathscr{Z}} M(z,\theta)\end{aligned},$$

then $\underline{M} \le M(z,\theta) \le \overline{M}$ for all $\theta \in \Theta$ and $z \in \mathscr{Z}$. If we assume that $\underline{z} \le z \le \overline{z}$, then from Lemma 5 we have:

$$\begin{aligned}&\underline{M}^+\underline{z}^+ - \overline{M}^+\underline{z}^- - \underline{M}^-\overline{z}^+ + \overline{M}^-\overline{z}^- \le M(z,\theta)z\\ &\le \overline{M}^+\overline{z}^+ - \underline{M}^+\overline{z}^- - \overline{M}^-\underline{z}^+ + \underline{M}^-\underline{z}^-\end{aligned}$$

and we can define

$$\begin{aligned}\overline{F}(\underline{z},\overline{z}) &= \overline{M}^+\overline{z}^+ - \underline{M}^+\overline{z}^- - \overline{M}^-\underline{z}^+ + \underline{M}^-\underline{z}^-\\ \underline{F}(\underline{z},\overline{z}) &= \underline{M}^+\underline{z}^+ - \overline{M}^+\underline{z}^- - \underline{M}^-\overline{z}^+ + \overline{M}^-\overline{z}^-\end{aligned}$$

Also from $M(z,\theta)z = M^+(z,\theta)z^+ - M^+(z,\theta)z^- - M^-(z,\theta)z^+ + M^-(z,\theta)z^-$, we obtain:

$$\begin{aligned}\|\overline{F}(\underline{z},\overline{z}) - F(z,\theta)\| &= \|\overline{M}^+\overline{z}^+ - \underline{M}^+\overline{z}^- - \overline{M}^-\underline{z}^+\\ &\quad + \underline{M}^-\underline{z}^- - M(z,\theta)z\|\\ &= \|p_1 + p_2 + p_3 + p_4\|,\end{aligned}$$

where $p_1 = \overline{M}^+\overline{z}^+ - M^+(z,\theta)z^+$, $p_2 = M^+(z,\theta)z^- - \underline{M}^+\overline{z}^-$, $p_3 = M^-(z,\theta)z^+ - \overline{M}^-\underline{z}^+$ and $p_4 = \underline{M}^-\underline{z}^- - M^-(z,\theta)z^-$. Using

$$\begin{aligned}\|p_1\| &= \|\overline{M}^+[\overline{z}^+ - z^+] + [\overline{M}^+ - M^+(z,\theta)]z^+\|\\ &\le \|\overline{M}^+\|\|\overline{z}^+ - z^+\| + \|[\overline{M}^+ - M^+(z,\theta)]z^+\|\\ &\le \|\overline{M}^+\|\|\overline{z} - z\| + \|[\overline{M}^+ - M^+(z,\theta)]z^+\|\end{aligned}$$

and similar inequalities, which we can compute for p_2, p_3 and p_4, we finally substantiate

$$\|\overline{F}(\underline{z},\overline{z}) - F(z,\theta)\| \le \bar{l}_{\overline{F}}\|\overline{z} - z\| + \underline{l}_{\overline{F}}\|\underline{z} - z\| + l_{\overline{F}}$$

with $\bar{l}_{\overline{F}}, \underline{l}_{\overline{F}}, l_{\overline{F}}$ given by

$$\begin{aligned}&\bar{l}_{\overline{F}} = \|\overline{M}^+\| + \|\underline{M}^+\|, \underline{l}_{\overline{F}} = \|\overline{M}^-\| + \|\underline{M}^-\|,\\ &l_{\overline{F}} = \max_{\theta\in\Theta, z\in\mathscr{Z}} \{\|[\overline{M}^+ - M^+(z,\theta)]z^+\| +\\ &\|[\underline{M}^+ - M^+(z,\theta)]z^-\| + \|[\overline{M}^- - M^-(z,\theta)]z^+\|\\ &+ \|[\underline{M}^- - M^-(z,\theta)]z^-\|\}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|\underline{F}(\underline{z},\overline{z}) - F(z,\theta)\| &= \|\underline{M}^+\underline{z}^+ - \overline{M}^+\underline{z}^- - \underline{M}^-\overline{z}^+\\ &\quad + \overline{M}^-\overline{z}^- - M(z,\theta)z\|\\ &= \|p_5 + p_6 + p_7 + p_8\|,\end{aligned}$$

where $p_5 = \underline{M}^+\underline{z}^+ - M^+(z,\theta)z^+$, $p_6 = M^+(z,\theta)z^- - \overline{M}^+\underline{z}^-$, $p_7 = M^-(z,\theta)z^+ - \underline{M}^-\overline{z}^+$ and $p_8 = \overline{M}^-\overline{z}^- - M^-(z,\theta)z^-$. And applying the same technique, the following inequality can be proven:

$$\|\underline{F}(\underline{z},\overline{z}) - F(z,\theta)\| \le \bar{l}_{\underline{F}}\|\overline{z} - z\| + \underline{l}_{\underline{F}}\|\underline{z} - z\| + l_{\underline{F}}$$

with $\bar{l}_{\underline{F}}$ and $\underline{l}_{\underline{F}}$ given by

$$\begin{aligned}&\bar{l}_{\underline{F}} = \underline{l}_{\overline{F}}, \underline{l}_{\underline{F}} = \bar{l}_{\overline{F}},\\ &l_{\underline{F}} = \max_{\theta\in\Theta, z\in\mathscr{Z}} \{\|[\overline{M}^+ - M^+(z,\theta)]z^-\|\\ &+ \|[\underline{M}^+ - M^+(z,\theta)]z^+\| + \|[\overline{M}^- - M^-(z,\theta)]z^-\|\\ &+ \|[\underline{M}^- - M^-(z,\theta)]z^+\|\}. \blacksquare\end{aligned}$$

Remark 7 *Lemma 6 shows that the difference of functions $\overline{F}, \underline{F}$ and F has a linear growth with respect to the interval width estimates $\underline{z} - z$ and $\overline{z} - z$.*

Remark 8 *It is clear that the positive constants $\bar{l}_{\overline{F}}$, $\underline{l}_{\overline{F}}$, $l_{\overline{F}}$, $\bar{l}_{\underline{F}}$, $\underline{l}_{\underline{F}}$ and $l_{\underline{F}}$ depend on the choice of P. Due to the fact that $F(z,\theta) = P\tilde{F}(P^{-1}z,\theta)$, then we have $\underline{F}(\underline{z},\overline{z}) \le F(z,\theta) \le \overline{F}(\underline{z},\overline{z})$ where $\underline{F} = P^+\underline{\tilde{F}} - P^-\overline{\tilde{F}}$ and $\overline{F} = P^+\overline{\tilde{F}} - P^-\underline{\tilde{F}}$. The above relations imply that the result of Lemma 6 is equivalent to the following one:*

$$\begin{aligned}\|\overline{F}(\underline{z},\overline{z}) - F(z,\theta)\| &\le \|P\|\|P^{-1}\| \left[\bar{l}_{\overline{\tilde{F}}}\|\overline{z} - z\| + \underline{l}_{\overline{\tilde{F}}}\|\underline{z} - z\| + l_{\overline{\tilde{F}}}\right],\\ \|\underline{F}(\underline{z},\overline{z}) - F(z,\theta)\| &\le \|P\|\|P^{-1}\| \left[\bar{l}_{\underline{\tilde{F}}}\|\overline{z} - z\| + \underline{l}_{\underline{\tilde{F}}}\|\underline{z} - z\| + l_{\underline{\tilde{F}}}\right],\end{aligned}$$

for some positive constants $\bar{l}_{\overline{\tilde{F}}}$, $\underline{l}_{\overline{\tilde{F}}}$, $l_{\overline{\tilde{F}}}$, $\bar{l}_{\underline{\tilde{F}}}$, $\underline{l}_{\underline{\tilde{F}}}$ and $l_{\underline{\tilde{F}}}$, which are independent of P.

Remark 9 *Note that the values of constants $\bar{l}_{\overline{F}}$, $\underline{l}_{\overline{F}}$, $l_{\overline{F}}$, $\bar{l}_{\underline{F}}$, $\underline{l}_{\underline{F}}$, $l_{\underline{F}}$ and functions $\underline{F}$, $\overline{F}$ are the theoretically maximal bounds. The goal of the lemma is just to show that the bounds exist and to provide some approximate outer estimates for them. For the concrete applications, more accurate values may be computed.*

Using similar arguments to those for the proof of Lemma 6, a similar result can be established for H, *i.e.* there exist two functions $\overline{H}, \underline{H} : \mathbb{R}^{2n} \to \mathbb{R}^n$ such that, for all $\theta \in \Theta$ and $\underline{z} \le z \le \overline{z}$ with $z \in \mathscr{Z}$, the following inequality holds:

$$\underline{H}(\underline{z},\overline{z}) \le H(z,\theta) \le \overline{H}(\underline{z},\overline{z}), \tag{14}$$

and for a given submultiplicative norm $\|\cdot\|$ we have

$$\begin{aligned}\|\overline{H}(\underline{z},\overline{z}) - H(z,\theta)\| &\le \bar{l}_{\overline{H}}\|\overline{z} - z\| + \underline{l}_{\overline{H}}\|\underline{z} - z\| + l_{\overline{H}},\\ \|\underline{H}(\underline{z},\overline{z}) - H(z,\theta)\| &\le \bar{l}_{\underline{H}}\|\overline{z} - z\| + \underline{l}_{\underline{H}}\|\underline{z} - z\| + l_{\underline{H}},\end{aligned}$$

for some positive constants $\bar{l}_{\overline{H}}$, $\underline{l}_{\overline{H}}$, $l_{\overline{H}}$, $\bar{l}_{\underline{H}}$, $\underline{l}_{\underline{H}}$ and $l_{\underline{H}}$.

Similar relations for the term G can be also derived using Lemma 6, *i.e.* there exist two functions $\overline{G}, \underline{G} : \mathbb{R}^{2n+m} \to \mathbb{R}^n$

such that the following inequality holds for all $u \in \mathscr{U}$, $\theta \in \Theta$ and $\underline{z} \leq z \leq \overline{z}$:

$$\underline{G}_i(\underline{z}, \overline{z}, u_i) \leq u_i G_i(z, \theta) \leq \overline{G}_i(\underline{z}, \overline{z}, u_i) \tag{15}$$

for all $0 \leq i \leq m$, and for a given submultiplicative norm $\|\cdot\|$ we have

$$\begin{aligned} &\|\overline{G}_i(\underline{z}, \overline{z}, u_i) - G_i(z, \theta)u_i\| \leq |u_i|(\bar{l}_{\overline{G}}\|\overline{z} - z\| + \underline{l}_{\overline{G}}\|\underline{z} - z\| + l_{\overline{G}}), \\ &\|\underline{G}_i(\underline{z}, \overline{z}, u_i) - G_i(z, \theta)u_i\| \leq |u_i|(\bar{l}_{\underline{G}}\|\overline{z} - z\| + \underline{l}_{\underline{G}}\|\underline{z} - z\| + l_{\underline{G}}), \end{aligned}$$

for some positive constants $\bar{l}_{\overline{G}}$, $\underline{l}_{\overline{G}}$, $l_{\overline{G}}$, $\bar{l}_{\underline{G}}$, $\underline{l}_{\underline{G}}$ and $l_{\underline{G}}$.

4.2 Interval observer construction

We are now ready to give the interval observer equations. Let $\overline{z}$, $\underline{z}$ be the estimates of the transformed state z, whose dynamics constitute the interval observer as follows:

$$\begin{aligned} \dot{\overline{z}} &= A\overline{z} + \overline{G}(\underline{z}, \overline{z}, u) + \overline{F}(\underline{z}, \overline{z}) \\ &\quad + L(y - C\overline{z}) + L^+\overline{H}(\underline{z}, \overline{z}) - L^-\underline{H}(\underline{z}, \overline{z}), \\ \dot{\underline{z}} &= A\underline{z} + \underline{G}(\underline{z}, \overline{z}, u) + \underline{F}(\underline{z}, \overline{z}) \\ &\quad + L(y - C\underline{z}) + L^+\underline{H}(\underline{z}, \overline{z}) - L^-\overline{H}(\underline{z}, \overline{z}), \end{aligned} \tag{16}$$

where the observer gain $L = (l_1, \ldots, l_n)^T$ has to be designed. Defining the upper error $\overline{e} = \overline{z} - z$ and the lower error $\underline{e} = z - \underline{z}$, their dynamics read as:

$$\begin{aligned} \frac{d\overline{e}}{dt} &= (A - LC)\overline{e} + \overline{\Delta}(\underline{z}, \overline{z}, z, \theta, u, L), \\ \frac{d\underline{e}}{dt} &= (A - LC)\underline{e} + \underline{\Delta}(\underline{z}, \overline{z}, z, \theta, u, L), \end{aligned} \tag{17}$$

where $\overline{\Delta}(\underline{z}, \overline{z}, z, \theta, u, L)$ is the sum of the following terms:

$$\begin{aligned} \overline{\Delta}_G(\underline{z}, \overline{z}, z, \theta, u) &= \overline{G}(\underline{z}, \overline{z}, u) - G(z, \theta)u, \\ \overline{\Delta}_F(\underline{z}, \overline{z}, z, \theta) &= \overline{F}(\underline{z}, \overline{z}) - F(z, \theta), \\ \overline{\Delta}_{LH}(\underline{z}, \overline{z}, z, \theta, L) &= L^+\overline{H}(\underline{z}, \overline{z}) - L^-\underline{H}(\underline{z}, \overline{z}) + LH(z, \theta), \end{aligned}$$

and $\underline{\Delta}(\underline{z}, \overline{z}, z, \theta, u, L)$ is the sum of

$$\begin{aligned} \underline{\Delta}_G(\underline{z}, \overline{z}, z, \theta, u) &= G(z, \theta)u - \underline{G}(\underline{z}, \overline{z}, u), \\ \underline{\Delta}_F(\underline{z}, \overline{z}, z, \theta) &= F(z, \theta) - \underline{F}(\underline{z}, \overline{z}), \\ \underline{\Delta}_{LH}(\underline{z}, \overline{z}, z, \theta, L) &= -LH(z, \theta) - L^+\underline{H}(\underline{z}, \overline{z}) + L^-\overline{H}(\underline{z}, \overline{z})). \end{aligned}$$

Corollary 10 *For all $z \in \mathscr{Z}$, $u \in \mathscr{U}$ and $\theta \in \Theta$ there exist positive constants $\underline{l}_{\underline{\Delta}}$, $\bar{l}_{\underline{\Delta}}$, $\underline{l}_{\overline{\Delta}}$, $\bar{l}_{\overline{\Delta}}$, $l_{\overline{\Delta}}$, $l_{\underline{\Delta}}$ such that for a chosen submultiplicative norm $\|\cdot\|$*

$$\begin{aligned} &\|\overline{\Delta}(\cdot, L)\| \leq [\bar{l}_{\overline{\Delta}}\|\overline{z} - z\| + \underline{l}_{\overline{\Delta}}\|z - \underline{z}\| + l_{\overline{\Delta}}](1 + \|L\|), \\ &\|\underline{\Delta}(\cdot, L)\| \leq [\bar{l}_{\underline{\Delta}}\|\overline{z} - z\| + \underline{l}_{\underline{\Delta}}\|z - \underline{z}\| + l_{\underline{\Delta}}](1 + \|L\|). \end{aligned}$$

PROOF. The proof follows directly from the definition of functions $\overline{\Delta}(\underline{z}, \overline{z}, z, \theta, u, L)$, $\underline{\Delta}(\underline{z}, \overline{z}, z, \theta, u, L)$ and Lemma 6. ■

Remark 11 *As it has been explained in Remark 8, the result of Corollary 10 can be stated as well for some positive constants independent of P,* i.e. *there exist $\underline{l}_{\underline{\tilde{\Delta}}}$, $\bar{l}_{\underline{\tilde{\Delta}}}$, $\underline{l}_{\overline{\tilde{\Delta}}}$, $\bar{l}_{\overline{\tilde{\Delta}}}$, $l_{\overline{\tilde{\Delta}}}$, $l_{\underline{\tilde{\Delta}}}$, independent of P, such that*

$$\begin{aligned} \|\overline{\Delta}(\cdot, L)\| &\leq [\bar{l}_{\overline{\tilde{\Delta}}}\|\overline{z} - z\| + \underline{l}_{\overline{\tilde{\Delta}}}\|z - \underline{z}\| + l_{\overline{\tilde{\Delta}}}](1 + \|L\|)\|P\|\|P^{-1}\|, \\ \|\underline{\Delta}(\cdot, L)\| &\leq [\bar{l}_{\underline{\tilde{\Delta}}}\|\overline{z} - z\| + \underline{l}_{\underline{\tilde{\Delta}}}\|z - \underline{z}\| + l_{\underline{\tilde{\Delta}}}](1 + \|L\|)\|P\|\|P^{-1}\|. \end{aligned}$$

Lemma 12 *Assume that assumptions A1)–A2) are satisfied, then for any $u \in \mathscr{L}_\infty$ in $\mathscr{U}$ and any $(\overline{e}(t_0), \underline{e}(t_0)) \geq 0$ (componentwise), the inequality*

$$(\overline{e}(t), \underline{e}(t)) \geq 0,$$

holds for every $t \geq t_0$.

PROOF. First let us note that $(\overline{e}(t), \underline{e}(t)) \geq 0$ implies

$$\underline{z}(t) \leq z(t) \leq \overline{z}(t). \tag{18}$$

Now, by construction, $\forall u \in \mathscr{U}, \forall \theta \in \Theta$ we have:

$$\begin{aligned} \overline{\Delta}_G(\underline{z}, \overline{z}, z, \theta, u) \geq 0, \underline{\Delta}_G(\underline{z}, \overline{z}, z, \theta, u) \geq 0 \\ \overline{\Delta}_F(\underline{z}, \overline{z}, z, \theta) \geq 0, \underline{\Delta}_F(\underline{z}, \overline{z}, z, \theta) \geq 0 \\ \overline{\Delta}_{LH}(\underline{z}, \overline{z}, z, \theta, L) \geq 0, \underline{\Delta}_{LH}(\underline{z}, \overline{z}, z, \theta, L) \geq 0 \end{aligned}$$

as soon as (18) holds. Since the gain L is such that $(A - LC)$ is Metzler, then $\mathrm{diag}((A - LC), (A - LC))$ is also Metzler, and $(\overline{\Delta}^T(\underline{z}, \overline{z}, u, L), \underline{\Delta}^T(\underline{z}, \overline{z}, u, L))^T \geq 0$. The result is a direct application of Corollary 1 provided that Assumption A1) is satisfied. Therefore, the relation (18) is satisfied for $t = t_0$ by construction, then they are satisfied for all $t \geq t_0$ according to Corollary 1. ■

Theorem 13 *Suppose that Assumptions A1)–A2) are satisfied. For the constants $\underline{l}_{\underline{\Delta}}$, $\bar{l}_{\underline{\Delta}}$, $\underline{l}_{\overline{\Delta}}$, $\bar{l}_{\overline{\Delta}}$ deduced from Corollary 10, if there exist positive definite and symmetric matrices S, Q, O such that the following inequality is satisfied:*

$$D^T S + SD + SO^{-1}S + \alpha\|O\|I + Q \preceq 0, \tag{19}$$

where $D = A - LC$ and $\alpha = 3(1 + \|L\|)^2 \max\{\bar{l}_{\overline{\Delta}}^2 + \bar{l}_{\underline{\Delta}}^2, \underline{l}_{\overline{\Delta}}^2 + \underline{l}_{\underline{\Delta}}^2\}$, then the variables $\underline{z}(t)$ and $\overline{z}(t)$ are bounded. Moreover, (18) is satisfied for all $t > 0$ if it is valid for $t = 0$.

PROOF. We denote $e = (\overline{e}(t), \underline{e}(t))$ and the argument $(\cdot) = (\underline{z}, \overline{z}, z, \theta, u, L)$, then the observation error system is given by (17). Let us consider the following Lyapunov function:

$$V(e) = \underline{e}^T S\underline{e} + \overline{e}^T S\overline{e}. \tag{20}$$

The derivative of V is given by:

$$\frac{dV(e)}{dt} = \overline{e}^T(D^TS+SD)\overline{e} + \underline{e}^T(D^TS+SD)\underline{e} \\ +2\overline{e}^TS\overline{\Delta}(\cdot) + 2\underline{e}^TS\underline{\Delta}(\cdot).$$

Completing the squares as

$$2\overline{e}^TS\overline{\Delta}(\cdot) = 2\overline{e}^TSO^{-0.5}O^{0.5}\overline{\Delta}(\cdot) \\ \le \overline{e}^TSO^{-1}S\overline{e} + \overline{\Delta}^T(\cdot)O\overline{\Delta}(\cdot), \\ 2\underline{e}^TS\underline{\Delta}(\cdot) = 2\underline{e}^TSO^{-0.5}O^{0.5}\underline{\Delta}(\cdot) \\ \le \underline{e}^TSO^{-1}S\underline{e} + \underline{\Delta}^T(\cdot)O\underline{\Delta}(\cdot)$$

we obtain:

$$\frac{dV(e)}{dt} \le \overline{e}^T(D^TS+SD+SO^{-1}S)\overline{e} \\ +\underline{e}^T(D^TS+SD+SO^{-1}S)\underline{e} \\ +\overline{\Delta}(\cdot)^TO\overline{\Delta}(\cdot) + \underline{\Delta}(\cdot)^TO\underline{\Delta}(\cdot).$$

According to Corollary 10

$$\overline{\Delta}(\cdot)^TO\overline{\Delta}(\cdot) \le 3\|O\|(1+\|L\|)^2[\overline{l}^2_{\overline{\Delta}}\|\overline{e}\|^2 + \underline{l}^2_{\overline{\Delta}}\|\underline{e}\|^2 + l^2_{\overline{\Delta}}],$$

$$\underline{\Delta}(\cdot)^TO\underline{\Delta}(\cdot) \le 3\|O\|(1+\|L\|)^2[\overline{l}^2_{\underline{\Delta}}\|\overline{e}\|^2 + \underline{l}^2_{\underline{\Delta}}\|\underline{e}\|^2 + l^2_{\underline{\Delta}}],$$

then

$$\overline{\Delta}(\cdot)^TO\overline{\Delta}(\cdot) + \underline{\Delta}(\cdot)^TO\underline{\Delta}(\cdot) \le \alpha\|O\|\overline{e}^T\overline{e} + \alpha\|O\|\underline{e}^T\underline{e} + \beta$$

with

$$\beta = 3\|O\|\|(1+\|L\|)^2(l^2_{\overline{\Delta}} + l^2_{\underline{\Delta}})$$

and by using (19) we get

$$\frac{dV(e)}{dt} \le -\overline{e}^TQ\overline{e} - \underline{e}^TQ\underline{e} + \beta,$$

which proves the boundedness of e. The positivity of the observation error e has already been proven in Lemma 12, thus we conclude that the observation error e is always positive and bounded. ■

Note that if the relation (18) is satisfied and the variables $\underline{z}$ and $\overline{z}$ are bounded, then by standard arguments Jaulin et al. (2001) we can compute $\underline{x}(t) = \underline{\Psi}(\underline{z}(t), \overline{z}(t))$ and $\overline{x}(t) = \overline{\Psi}(\underline{z}(t), \overline{z}(t))$ (where $\underline{\Psi}, \overline{\Psi}$ depend on $\Phi_{(3)}$ and P) such that

$$\underline{x}(t) \le x(t) \le \overline{x}(t),$$

for all $t \ge 0$, *i.e.* we obtain the interval estimation for the original nonlinear system (1).

Remark 14 *In this paper, since the nominal system (3) with $h(x)$ and $f(x)$ can be chosen under the condition that (3) is observable, then it may be possible to choose h and f providing a monotone diffeomorphism with respect to its state. In this situation, the set inversion problem becomes feasible. Generally speaking, this problem is a hard task and using interval contractors cannot always allow obtaining a tight enclosure. Sometimes, we can use constraint propagation techniques based on interval analysis to solve this problem in a reliable way (see Jaulin et al. (2001) for more details). In addition, for the control purpose an inversion is not necessary, the regulation problem can be solved in the coordinates z (see Efimov et al. (2013)).*

If the output y equals to $h(x)$, *i.e.* there is no uncertainty $\delta h(x, \theta)$, then clearly the above theorem has more simple conditions.

Corollary 15 *Let Assumptions A1)–A2) be satisfied, and $y = h(x)$ in (1). For the deduced matrices $\tilde{L}$ and P in Claim 1, if there exist the positive definite and symmetric matrices S and Q such that the following LMI be true*

$$\begin{bmatrix} -I & S \\ S & D^TS+SD+\alpha I+Q \end{bmatrix} \preceq 0 \qquad (21)$$

where $D = P\tilde{A}P^{-1} - L\tilde{C}P^{-1}$, $\alpha = 3\max\{\overline{l}^2_{\overline{\Delta}} + \overline{l}^2_{\underline{\Delta}}, \underline{l}^2_{\overline{\Delta}} + \underline{l}^2_{\underline{\Delta}}\}$ with the constants $\underline{l}_{\underline{\Delta}}$, $\overline{l}_{\underline{\Delta}}$, $\underline{l}_{\overline{\Delta}}$ and $\overline{l}_{\overline{\Delta}}$ deduced from Corollary 10, then the variables $\underline{z}(t), \overline{z}(t)$ are bounded and (18) is satisfied for all $t \ge 0$.

PROOF. The inequality (21) follows from Theorem 13 by choosing $O = I$, skipping all terms dependent on H and applying the Schur complement. ■

Based on the result stated in Corollary 15, the following algorithm is presented to summarize the design procedure for the proposed interval observer:

S1: Since the nominal system of (1) is observable, compute the diffeomorphism $\Phi_{(3)}$ to obtain $\tilde{A}$ and $\tilde{C}$;

S2: Due to the fact that the pair $(\tilde{A}, \tilde{C})$ is observable, seek a matrix $\tilde{L}$ and an invertible matrix P such that the matrix $A - LC$ is Hurwitz and Metzler, where $A = P\tilde{A}P^{-1}$, $C = \tilde{C}P^{-1}$ and $L = P\tilde{L}$;

S3: Transform system (1) by applying the change of coordinates $z = P\Phi_{(3)}(x)$ to (11) with F, H and G, and calculate the positive constants $\overline{l}_*$, $\underline{l}_*$, and l_* where $*$ represents $\overline{F}, \overline{H}, \overline{G}$ and $\underline{F}, \underline{H}, \underline{G}$, which enables us to compute $\underline{l}_{\Delta}$, $\overline{l}_{\Delta}$, $l_{\overline{\Delta}}$ (see Corollary 10);

S4: Set $D = A - LC$ and $\alpha = 3\max\{\overline{l}^2_{\overline{\Delta}} + \overline{l}^2_{\underline{\Delta}}, \underline{l}^2_{\overline{\Delta}} + \underline{l}^2_{\underline{\Delta}}\}$. If the LMI (21) can be solved, then go to S5. Otherwise, go back to S2 by changing the choices of $\tilde{L}$ and P.

S5: Design the interval observer (16), whose observation error is bounded since (21) is satisfied.

Obviously, Corollary 15 just gave a way to check whether the chosen matrices $\tilde{L}$ and P enable us to design the interval observer (16). Thus, it does not present a constructive way to calculate the gain L. It can be seen, due to the fact that α depends on P and D depends on P and $\tilde{L}$, even for the simple case (no uncertainty in the output) it is not trivial to have a constructive way to calculate the matrices $\tilde{L}$ and P satisfying at the same time the inequality (21).

One solution to calculate the gain L constructively is to fix in advance some matrices, and then solve a set of inequalities on a grid or iteratively. More precisely, we can firstly choose the scalar γ and a Hurwitz and Metzler diagonal matrix D (since the nominal system is observable, D can be chosen as a diagonal matrix with desired eigenvalues on the main diagonal), and then seek the matrices P and L with symmetric positive definite matrices S and Q simultaneously:

$$\begin{bmatrix} -I & S \\ S & D^T S + SD + \gamma^2 \tilde{\alpha} I + Q \end{bmatrix} \preceq 0 \qquad (22)$$

$$P\tilde{A} - L\tilde{C} - DP = 0 \qquad (23)$$

$$\begin{bmatrix} -I & P \\ P^T & -\gamma^2 I \end{bmatrix} \preceq 0, \quad \begin{bmatrix} -I & P \\ P^T & -\gamma^{-2} I \end{bmatrix} \preceq 0, \qquad (24)$$

where $\tilde{\alpha} = 3\max\{\bar{l}_{\bar{\Delta}}^2 + \bar{l}_{\underline{\Delta}}^2, \underline{l}_{\bar{\Delta}}^2 + \underline{l}_{\underline{\Delta}}^2\}$ with the associated constants defined in Remark 11. If no solution is found, we can then modify γ and D, and repeat the test.

Remark 16 *It is clear that (22) is similar to (21) except that $\tilde{\alpha}$ in (22) does not depend on P. The additional equality (23) guarantees that the sought matrices P and $\tilde{L}$ will transform the matrix $(\tilde{A} - P^{-1}L\tilde{C})$ into the chosen Hurwitz and Metzler diagonal matrix D. The last inequality (24) is in fact equivalent to $\|P\| \preceq \gamma$, $\|P^{-1}\| \preceq \gamma$ with which we add the constraints for the norms of P and P^{-1} in order to facilitate the search of solutions.*

As it has been shown, an interval observer for the uncertain nonlinear system (1) is proposed using the transformation of coordinates calculated for the nominal system (3). It is worth noting that the original system may be non-uniformly observable, but if it is possible to extract from (1) a nominal observable system (3), then the proposed approach establishes the interval observer and the corresponding transformation of coordinates providing the interval state estimation for (1). Moreover, if Assumption A2 is not satisfied for (1) for all $t \geq 0$, the presented interval method is still valid during a finite time T if $x(t) \in \mathscr{X}$ and $u(t) \in \mathscr{U}$ for $T \geq t \geq 0$. Let us demonstrate the advantages of this approach on an example of a nonlinear non-observable system.

5 Example

Consider the following example of the nonlinear system (1):

$$\begin{aligned} \dot{x}_1 &= x_2 + a_1 \sin(\theta_1 x_1 x_2), \\ \dot{x}_2 &= -a_4 x_2 - a_2 \sin(\theta_2^2 x_1) + a_3 \cos(y) u, \\ y &= x_1 + c x_2 + \theta_3 x_1 x_2, \end{aligned}$$

where $a_1 = 0.25$, $a_2 = 19$, $a_3 = 1$, $a_4 = 2$ and $c = 0.526$ are given known parameters, the unknown parameters admit the condition $|\theta_i| \leq \bar{\theta}$ for $i = 1,2,3$ with $\bar{\theta} = 0.1$. For simulation we will use $\theta_1 = 0.1$, $\theta_2 = -0.1$, $\theta_3 = -0.05[1 + 0.25\sin(3t) + 0.25\cos(5t)]$ (it is a time-varying signal representing additional disturbance/noise) and $u(t) = 0.1\sin(t) + 0.75\cos(0.25t)$. It is straightforward to check that the linearization of this system at the origin for all admissible values of parameters is stable. We assume that $|x_1(0)| \leq 0.1$, $|x_2(0)| \leq 0.1$ and that solutions stay bounded and $|x_1(t)| \leq \bar{x}_1 = 0.2$, $|x_2(t)| \leq \bar{x}_2 = 0.2$ Therefore, Assumption A2) is satisfied for $\mathscr{X} = [-0.2, 0.2]^2$ and $\mathscr{U} = [-1, 1]$. Moreover, since the observability matrix of this system depends on the unknown parameters, thus the system is not always observable on these compact sets.

For this example, the following nominal system has been chosen:

$$f_1(x) = x_2,\ f_2(x) = -a_4 x_2,\ h(x) = x_1 + c x_2$$

then

$$\delta f_1(x,\theta) = a_1 \sin(\theta_1 x_1 x_2),\ \delta f_2(x,\theta) = -a_2 \sin(\theta_2^2 x_1), \\ \delta h(x,\theta) = \theta_3 x_1 x_2.$$

It is straightforward to check that the nominal system as a linear system in the canonical form is observable. Thus Assumption A1) is verified. Claim 1 is satisfied for the matrix

$$L = [1.9, 0]^T.$$

Let us compute the bounding functions for δf and δh. To this end, define the following two functions:

$$\text{Product}(\underline{x}, \bar{x}) = \begin{bmatrix} \min\{\bar{x}_1\bar{x}_2, \underline{x}_1\bar{x}_2, \bar{x}_1\underline{x}_2, \underline{x}_1\underline{x}_2\} \\ \max\{\bar{x}_1\bar{x}_2, \underline{x}_1\bar{x}_2, \bar{x}_1\underline{x}_2, \underline{x}_1\underline{x}_2\} \end{bmatrix},$$

$$\begin{bmatrix} \underline{\sin}(\underline{x}, \bar{x}) \\ \overline{\sin}(\underline{x}, \bar{x}) \end{bmatrix} = \begin{bmatrix} \sin(\underline{x}) \\ \sin(\bar{x}) \end{bmatrix}$$

corresponding to the interval of the product $x_1 x_2$ for $x = [x_1\ x_2]^T$ with $\underline{x} \leq x \leq \bar{x}$ and the interval of the function $\sin(x)$

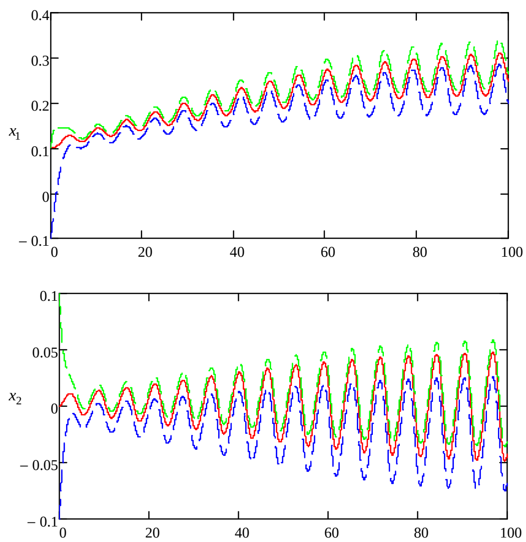

Fig. 1. The results of interval estimation for the coordinates x_1 and x_2

for a scalar x with $\underline{x} \le x \le \overline{x}$ (for $|x| \le \pi/2$). Then

$$
\begin{aligned}
\underline{\delta f}_1(\underline{x},\overline{x}) &= a_1 \underline{\sin}\left\{\rho(\bar{\theta},\underline{x},\overline{x})\right\},\\
\overline{\delta f}_1(\underline{x},\overline{x}) &= a_1 \overline{\sin}\left\{\rho(\bar{\theta},\underline{x},\overline{x})\right\},\\
\underline{\delta f}_2(\underline{x},\overline{x}) &= -a_2 \overline{\sin}\left\{\text{Product}\left(\begin{bmatrix}-\bar{\theta}\\ \bar{\theta}\end{bmatrix},\begin{bmatrix}\bar{x}_1\\ \bar{x}_1\end{bmatrix}\right)\right\},\\
\overline{\delta f}_2(\underline{x},\overline{x}) &= -a_2 \underline{\sin}\left\{\text{Product}\left(\begin{bmatrix}-\bar{\theta}\\ \bar{\theta}\end{bmatrix},\begin{bmatrix}\bar{x}_1\\ \bar{x}_1\end{bmatrix}\right)\right\},
\end{aligned}
$$

$$
\begin{bmatrix}\underline{\delta h}_1(\underline{x},\overline{x})\\ \overline{\delta h}_1(\underline{x},\overline{x})\end{bmatrix} = \rho(\bar{\theta},\underline{x},\overline{x}),
$$

where

$$
\rho(\bar{\theta},\underline{x},\overline{x}) = \text{Product}\left(\begin{bmatrix}-\bar{\theta}\\ \bar{\theta}\end{bmatrix},\text{Product}(\underline{x},\overline{x})\right).
$$

Take

$$
\bar{l}_{\underline{\Delta}} = \bar{l}_{\overline{\Delta}} = \underline{l}_{\overline{\Delta}} = \underline{l}_{\underline{\Delta}} = [a_2 + (a_1+1)\bar{x}_2]\bar{\theta},
$$

then $\alpha = 3(1+\|L\|)^2 \max\{\bar{l}_{\overline{\Delta}}^2 + \bar{l}_{\underline{\Delta}}^2, \underline{l}_{\overline{\Delta}}^2 + \underline{l}_{\underline{\Delta}}^2\} = 1.279$. For the chosen parameters, the matrix inequality from Theorem 13 is satisfied for:

$$
S = \begin{bmatrix}0.8 & -0.4\\ -0.4 & 1.8\end{bmatrix},\ O = I,\ Q = 0.8I,
$$

thus all conditions of Theorem 13 have been verified. The results of the interval estimation are given in Fig. 1.

6 Conclusion

The problem of state estimation is studied for an uncertain nonlinear system not in a canonical form. The uncertainty is presented by a vector of unknown time-varying parameters, the system equations depend on this vector in a nonlinear fashion. It is assumed that the values of this vector of unknown parameters belong to some known compact set. The idea of the proposed approach is to extract a known nominal observable subsystem from the plant equations, next a transformation of coordinates developed for the nominal system is applied to the original one. The interval observer is designed for the transformed system. It is shown that the residual nonlinear terms dependent on the vector of unknown parameters have linear upper and lower functional bounds, that simplifies the interval observer design and stability/cooperativity analysis. As a direction of future research, the problem of estimation accuracy optimization can be posed, *i.e.* how by a selection of the observer gain L to improve the asymptotic accuracy of estimation.

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