Nash Equilibrium for Femto-Cell Power Allocation in HetNets with channel uncertainty

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***Abstract*—We propose power allocation among femto-base stations (femto-BSs) in a heterogeneous network (HetNet) based on non cooperative games. A minimum level of quality of service has to be guaranteed at macro-user terminals (macro-UTs). Femto-BSs are unaware of the exact values of the channel parameters between them and macro-UTs because of the lack of cooperation and fading. First, we consider the design criterion where the outage probability has to be below a certain threshold at macro-UTs. The equilibrium concept is based on the Normalized Nash Equilibrium (NNE) since it caters to the distributed setting. NNE is unique only for a few strictly concave utility functions in this case. We introduce the concept of Weakly Normalized Nash Equilibrium (WNNE) which keeps the most of the appealing features of NNE but can be extended to a wide class of utility functions and can be incorporated with low complexity. Finally, we consider the design criterion where the expected SINR at a macro-UT has to be greater than a threshold. In this case, the NNE is always unique for any strictly concave utility functions.**

I. Introduction

Femto-cells are low-cost base stations which extend coverage and provide high bit rates in high critical environments like indoor. Thus, HetNets consisting of co-existing femto-cells and macro-cells, are recognized as key elements of next generation networks. Since femto-BSs and macro-base stations (macro-BSs) operate on the same frequency band, the interference management is a critical research problem in HetNets. Zhao et al. [1] identified the interference caused by femto-cell communications to downlink communications in macro-cells as the most detrimental kind of interference in standard HetNets. Since femto-BSs are not operated by the network providers which typically own and control the macro-BSs, the exact channel parameters between femto-BSs and macro-UTsre not known by the femto-BSs. The channels fade randomly making the interference management in the HetNets more challenging. We investigate the power allocation at the femto-BSs in presence of random channel gains under global constraints on a minimum level of quality-of-service to be guaranteed at the macro-UTs.

We consider a system of multiple femto-BSs and multiple macro-UTs. We formulate the power allocation problem at the femto-BSs as a coupled-constrained non cooperative game with femto-BSs as players. In order to minimize the cooperation and signaling, we assume that a femto-BS does not know the exact value of the channel parameter between itself and the macro-UTs, it only knows their probability distributions. We consider two design criteria in order to maintain an acceptable level of quality-of-service at the macro-UTs: i) The outage probability must be below a certain threshold, ii) The expectation of the signal-to-interference and noise ratio (SINR) at each macro-UT must be above a certain threshold.

We resort to the concept of NNE introduced in [6] since it caters to the distributed setting which we discuss in Section III-C. When we consider the first design criterion, the constraints reduce to posynomials [12] for Rayleigh fading channels and the coupled constrained game is not concave in general. Using the techniques of geometric programming we convert the set of constraints into a standard convex form through a variable transformation. The transformed game is not concave for well known functions e.g. Shannon Capacity function. Thus, the concept of NNE can not be applied for those utility functions. Nevertheless, we identify a class of utility functions of practical interests which are also strictly concave for the transformed game. We show that the NNE is unique for those games. We also propose a distributed algorithm which converges to the unique NNE where a femto-BS does not need to exchange information with other femto-BSs and a macro-UT only needs to keep track of the total interference. Since a large number of utility functions can not be cast into the concave game framework for the transformed problem, we resort to the concept of WNNE [10] which retains most of the useful properties of NNE and can be incorporated in a distributed setting.

We show that the game is always concave under the second design criterion. In this case the NNE is unique for strictly concave utility functions including the Shannon Capacity function unlike under the first design criterion. We also propose a distributed algorithm which converges to the unique NNE for any strictly concave utility function. Finally, we numerically investigate the characteristic of NNE strategy profiles under two design criteria in Section V.

In [2], [11], the power allocation problem in HetNets is formulated as a Stackelberg game with macro-BSs as leaders and femto-BSs as followers. However, this two-tier game does not match with the 5G driving concept where the downlink transmission in macro-cells is optimized via cooperation of the macro-UTs. In our previous work [10], we focus only on femto-BSs and we adopt the concept of NNE to design a distributed algorithm for power allocation among them. All the papers mentioned above assumed that the femto-BSs know the channel parameters perfectly.

Recently, [3] and [4] proposed power allocation algorithm in HetNets with uncertainty of channel gains. The former work adopts the framework of non cooperative game among femto-BSs constrained by second design criterion. In contrast to our work, the distributed algorithm proposed there requires the knowledge of the transmitted power from each femto-BS at each macro-UT and moreover, their approach can not be extended to the first design criterion. In [4] the authors con-

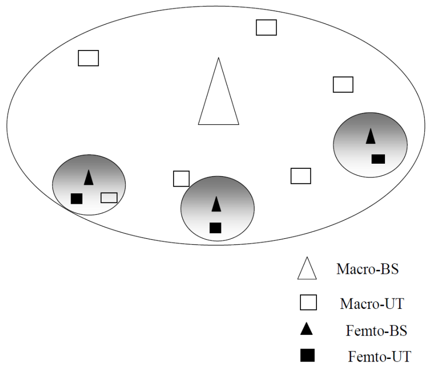

Fig. 1. Macro-BS, macro-UTs, femto-BSs and femto-UTs in a region. Circles represent the range of base stations. Macro-BS has higher coverage compared to femto-BS.

sider a distributed implementation of sum-rate maximization at high SNR regime with a single constraint on the maximum outage probability. Since femto-BSs are not managed by a single entity, thus each femto-BS only tries to maximize its own utility and thus, non cooperative game theoretic setting is more realistic approach for the analysis. The equilibrium selection concept does not arise in [4] and it is a salient feature of our game theoretic model. Moreover, the optimization framework considered in [4] does not offer the flexibility of our proposed approach in terms of utility functions and number of constraints. Additionally, [4] does not consider constraints on the expected SINR at each macro-UT.

Notation. Vectors are denoted by bold lower case letter; $\cdot^T$ denotes the transpose operator; the notation $\mathbf{x} \succeq \mathbf{0}$ stays for componentwise inequality. Given the real x, $(x)^+ = \max(x, 0)$. The vector $\mathbf{v}_{-i}$ is obtained from vector $\mathbf{v}$ by suppressing the ith component. $\mathbb{1}$ denotes the indicator function. $\exp(\mathbf{y}) = (\exp(y_1), \dots, \exp(y_F))$.

II. System Model

We consider a HetNet consisting of F femto-BSs equipped with single antenna and serving a single femto-UT per channel use and M macro-UTs (fig. 1). The macro-UTs can be served by different macro-BSs or the same macro-BS. We do not make any assumptions regarding the distribution of femto-BSs, femto-UTs and macro-UTs except the fact that femto-UT is located close to its serving femto-BS. The channel gain between femto-BS f and femto-UT f is denoted by h_f; $\widehat{h}_m^f$ is the channel gain between femto-BS f and macro-UT m. We denote the channel gain between the serving macro-BS and macro-UT m as $\overline{h}_m$. The femto-BS f transmits with power $p_f \geq 0$. For future use, it is convenient to define the following vectors $\mathbf{p} = (p_1, p_2, \dots, p_F)^T$, $\mathbf{h} = (h_1, h_2, \dots, h_F)^T$, $\widehat{\mathbf{h}}_m = (\widehat{h}_m^1, \widehat{h}_m^2, \dots, \widehat{h}_m^F)^T$, with $m = 1, \dots, M$,.

Due to the tight co-ordination between femto-BS f and its served femto-UT f, we assume that femto-BS f knows its channel gain h_f. We also assume that h_f is constant. Since the communications in the macro-cells and femto-cells are uncoordinated and the channel gain also fluctuates randomly due to the fading, thus, we assume that femto-BS f does not have the exact values of $\widehat{\mathbf{h}}_m, m = 1 \dots, M$ but it has the statistical Channel State information (CSI) i.e. it knows the probability distribution of $\widehat{\mathbf{h}}_m, m = 1 \dots, M$. In general, $\hat{h}_m^f$ is a product of two terms [5]:

$$\hat{h}_m^f = \widehat{\mu}_m^f r_m^f \tag{1}$$

where $\widehat{\mu}_m^f$ is the path loss component from femto-BS f to macro-UT m which corresponds to the signal attenuation because of the distance, the shadowing effect and the antenna gain between femto-BS f and macro-UT m . The above component is assumed to be constant throughout this work.

In (1), r_m^f models *Rayleigh Fading* [5]. r_m^f is a exponentially distributed random variable with mean 1. Hence, $\hat{h}_m^f$ is an exponentially distributed random variable with mean $\widehat{\mu}_m^f$.

The signal to noise ratio (SNR) at femto-UT f is given by

$$\gamma_f = \frac{p_f h_f}{\sigma^2} \tag{2}$$

where σ^2 is the variance of the additive white Gaussian noise that also accounts for interference from macro-BSs. We assume that the interference at a femto-UT from adjacent femto-BSs is negligible. SNR (γ_f) is only a function of p_f. When it is convenient, we explicitly point out this dependence by writing $\gamma_f(p_f)$, otherwise we use the short notation γ_f.

A. Constraint at macro-UT

The Signal to interference and noise ratio (SINR) at macro-UT m is given by

$$\mathrm{SINR}_m = \frac{\overline{p}\overline{h}_m}{\mathbf{p}^T \widehat{\mathbf{h}_m} + \overline{\sigma}^2} \tag{3}$$

where $\overline{p}$ is the signal power of macro-BS and $\overline{\sigma}^2$ is the variance of additive Gausian Noise at macro-UT m. The femto-BS does not know the exact values of $\overline{h}_m$. Thus same as for $\hat{h}_m^f$ we assume that $\overline{h}_m$ is an exponentially distributed random variable with mean $\overline{\nu}_m$. In order to guarantee the quality of service at a macro-UT, the SINR at a macro-UT has to be above a certain threshold γ_{th}. Thus, we enforce the constraint–

$$\mathrm{SINR}_m \geq \gamma_{th} \tag{4}$$

For notational simplicity we consider the threshold γ_{th} is the same for all macro-UTs. The generalization of the model to consider different thresholds is straightforward. From (3) we can also represent (4) as

$$\mathbf{p}^T \widehat{\mathbf{h}_m} + \overline{\sigma}^2 \leq \frac{\overline{p}\overline{h}_m}{\gamma_{th}} = I_m \tag{5}$$

The right hand side of the above inequality (i.e. I_m) is an exponentially distributed random variable with mean $\frac{\overline{\nu}_m \overline{p}}{\gamma_{th}} = \overline{\mu}_m$. We assume macro-UT m and all femto-BSs know $\overline{\mu}_m$.

We consider two different design criteria:

1) Assuming that macro-UT $m, m = 1, \dots, M$ suffers from the outage when the inequality in (5) is not satisfied, the femto-BSs need to guarantee that the outage probability is lower than a given value $\epsilon > 0$, or equivalently,

$$\Pr(\mathbf{p}^T \widehat{\mathbf{h}}_m + \overline{\sigma}^2 \leq I_m) \geq 1 - \epsilon \quad m = 1, \dots, M \tag{6}$$

2) The second design criterion is based in the fact that the expected value of SINR_m must be greater than γ_{th} i.e. for $m = 1, \dots, M$

$$\begin{aligned} &\mathrm{E}\left(\frac{\overline{p}\overline{h}_m}{\mathbf{p}^T \widehat{\mathbf{h}}_m + \overline{\sigma}^2}\right) \geq \gamma_{th} \\ &= \mathrm{E}(\overline{p}\overline{h}_m)\mathrm{E}\left(\frac{1}{\mathbf{p}^T \widehat{\mathbf{h}}_m + \overline{\sigma}^2}\right) \geq \gamma_{th} \end{aligned} \tag{7}$$

The last equality follows because of the independence between $\widehat{\mathbf{h}}_m$ and $\overline{h}_m$. Since $1/x$ is convex in x for positive x, thus, from Jensen's inequality we obtain

$$\mathrm{E}\left(\frac{1}{\mathbf{p}^T\widehat{\mathbf{h}}_m+\overline{\sigma}^2}\right)\geq\frac{1}{\mathrm{E}(\mathbf{p}^T\widehat{\mathbf{h}}_m+\overline{\sigma}^2)} \tag{8}$$

Hence, simplifying (8) the constraint (7) will be satisfied if the following holds:

$$\sum_{f=1}^{F}\mathrm{E}(\hat{h}_m^f)p_f\leq\overline{\mu}_m-\overline{\sigma}^2 \quad \text{for } m=1,\ldots,M \tag{9}$$

First, we consider the setting where femto-BSs must have to satisfy the constraint in (6) in Section III. Subsequently, we consider the setting where they have to satisfy the constraint in (9) in Section IV.

III. Power allocation for maximum outage probability constraint

A. Game formulation

Each femto-BS selects its transmission power with the objective of maximizing the quality of its communication in downlink. Its communication quality is characterized by $U_f(\gamma_f)$, where $U_f(\cdot)$ is a concave nondecreasing function.

We formulate the power allocation at femto-BSs as a non-cooperative game where each femto-BS aims to maximize its own utility $U_f(\gamma_f)$ under constraints (6).

More specifically, we define this non-cooperative game in a strategic form as

$$\mathcal{G}=\{\mathcal{F},\mathcal{P},\{u_f(\mathbf{p})\}_{f\in\mathcal{F}}\} \tag{10}$$

where the elements of the game are

- Player set: Set of the femto-BSs $\mathcal{F}=\{1,...,F\}$;
- Strategy set: $\mathcal{P}=\{\mathbf{p}|\mathbf{p}\in\mathbb{R}_+^F$ and constraint (6)$\}$ where $\mathbb{R}_+^F$ is the product space of F nonnegative real spaces $\mathbb{R}_+$.
- Utility set: The utility functions $u_f(\mathbf{p})$ are defined as $u_f(\mathbf{p})\equiv U_f(\gamma_f(p_f))=U_f\left(\frac{p_fh_f}{\sigma^2}\right)$, being $U_f(\cdot):\mathbb{R}_+\to\mathbb{R}_+$ a concave nondecreasing function in $\mathbb{R}_+$.

We adopt a NE of the non-cooperative game $\mathcal{G}$ as a power allocation policy for the femto-BSs of the heterogeneous network. More specifically, the power allocation vector $\mathbf{p}^*$ is a Nash Equilibrium (NE) if and only if for every $f\in\mathcal{F}$

$$U_f\left(p_f^*\frac{h_f}{\sigma^2}\right)\geq U_f\left(p_f\frac{h_f}{\sigma^2}\right) \tag{11}$$

for all p_f such that $(p_1^*,\ldots p_{f-1}^*,p_f,p_{f+1}^*,p_F^*)\in\mathcal{P}$. NE is in general not unique for coupled constrained game. Rosen has introduced the concept of NNE in [6] which provides a useful equilibrium selection criterion for concave games. Thus, we focus on the concave games.

B. Convexification–Concave Game

We must have convex $\mathcal{P}$ for concave games. However the constraint in (6) is not convex in general. But, we reduce it into a convex form for the Rayleigh fading channel. Since $\hat{h}_m^f p_f$ is an exponentially distributed random variable with mean $\widehat{\mu}_m^f p_f$, thus, similar to [13] we obtain from (6)

$$\Pr(\mathbf{p}^T\widehat{\mathbf{h}}_m+\overline{\sigma}^2\leq I_m)=e^{-\frac{\overline{\sigma}^2}{\overline{\mu}_m}}\prod_{f=1}^{F}\left(\frac{\overline{\mu}_m}{\overline{\mu}_m+\widehat{\mu}_m^f p_f}\right) \tag{12}$$

Note that the denominator of (12) is a posynomial [12] and we can proceed in similar manner of geometric programming ([12]) to represent the strategy set in the standard convex form. Since $p_f\geq 0$, we can set $y_f=\log(p_f)$ and thus, using (12) we can express the constraint in (6) for $m=1,\ldots,M$ as

$$e^{\frac{\overline{\sigma}^2}{\overline{\mu}_m}}\prod_{f=1}^{F}\left(\frac{\overline{\mu}_m+\widehat{\mu}_m^f\exp(y_f)}{\overline{\mu}_m}\right)\leq\frac{1}{1-\epsilon} \tag{13}$$

Taking logarithm of both sides we can equivalently express the above inequality as follows for $m=1,\ldots,M$:

$$\sum_{f=1}^{F}\log\left(1+\frac{\widehat{\mu}_m^f\exp(y_f)}{\overline{\mu}_m}\right)\leq\log\left(\frac{1}{1-\epsilon}\right)-\frac{\overline{\sigma}^2}{\overline{\mu}_m} \tag{14}$$

which is a convex function in $\mathbf{y}$. We assume that the r.h.s of (14) is always positive, otherwise the solution will not be feasible. Also note that the constraint in (6) has been reduced to (13) and then we take the logarithm to obtain (14). Thus, (14) is equivalent to the following

$$\log(\Pr(\mathbf{p}^T\widehat{\mathbf{h}}_m+\overline{\sigma}^2\leq I_m))\geq\log(1-\epsilon)$$

$$\text{Equivalently, }\log(\Pr(\mathbf{p}^T\widehat{\mathbf{h}}_m+\overline{\sigma}^2> I_m))\leq\log(\epsilon) \tag{15}$$

We use the above constraint to propose a distributed algorithm in Section III-F.

Change of decision variable: In consistence with the change of decision variables $y_f=\log(p_f)$, we consider an equivalent game $\widetilde{\mathcal{G}}=\{\mathcal{F},\widetilde{\mathcal{P}},\widetilde{u}_f(\mathbf{y})\}$ where $\widetilde{\mathcal{P}}$ is the set of constraints (14) (or constraints (15)) and $\mathbf{y}$ is the decision variable. Since $\gamma_f=\dfrac{p_fh_f}{\sigma^2}$, thus, $\widetilde{u}_f(\mathbf{y})=\widetilde{U}_f(\gamma_f(\exp(y_f)))$. Again we use $\gamma_f(\exp(y_f))$ to show the dependence of γ_f on $\exp(y_f)$ otherwise we use γ_f.

C. Normalized Nash Equilibrium

Since $p_f=\exp(y_f)$, femto-BS f can uniquely obtain p_f once it knows y_f due to the monotonicity of exponential function. We henceforth, consider the equilibrium strategy for the game $\widetilde{\mathcal{G}}$.

Since constraints in (14) is convex and closed, thus the modified $\widetilde{\mathcal{P}}$ is convex and closed. Moreover, if $\widetilde{U}_f(\cdot)$ is concave in y_f we can use the necessary and sufficient KKT conditions for constrained maxima (see, e.g. [7]) to obtain conditions satisfied by a NE $\mathbf{y}^*$. If $\mathbf{y}^*$ is a NE in $\widetilde{\mathcal{P}}$, then, there exist F vectors $\boldsymbol{\lambda}^f=(\lambda_1^f,\lambda_2^f\ldots,\lambda_M^f)$ with $\boldsymbol{\lambda}^f\geq\mathbf{0}$ such that $\mathbf{y}^*$ satisfies the following system of equations

$$\lambda_m^f(\sum_{f=1}^{F}\log\left(1+\exp(y_f)\frac{\widehat{\mu}_m^f}{\overline{\mu}_m}\right)-\log\left(\frac{1}{1-\epsilon}\right)+\frac{\sigma^2}{\overline{\mu}_m})=0,$$
$$m=1,\ldots M \text{ and } f=1,\ldots F \tag{16}$$

$$\frac{\partial\widetilde{U}_f(\gamma_f)}{\partial y_f}-\sum_{m=1}^{M}\lambda_m^f\frac{\partial}{\partial y_f}(\log(1+\exp(y_f)\frac{\widehat{\mu}_m^f}{\overline{\mu}_m}))=0,$$
$$f=1,\ldots F \tag{17}$$

We can also represent the Lagrangian for femto-BS f using the constraint (15) as

$$\widetilde{U}_f(\gamma_f)- \sum_{m=1}^{M} \lambda_m^f (\log(\Pr(\sum_{f=1}^{F} \exp(y_f)\hat{h}_m^f + \overline{\sigma}^2 > I_m)) - \log(\epsilon)) \quad (18)$$

$\mathbf{y}^*$ is a normalized Nash equilibrium (NNE) if KKT conditions in (16)-(17) are satisfied with $\boldsymbol{\lambda}^f = \boldsymbol{\lambda}$ for all $f \in \mathcal{F}$ i.e. the Lagrangians are identical for each player. It has several advantages: First, the Lagrangian multiplier λ_m^f can be viewed as the price for interference caused by player f at macro-UT m. Thus, a macro-UT does not have to select different prices for different players in order to implement a NNE. Additionally, it will be clear from the distributed algorithm proposed in Section III-F, the above property considerably reduces the cost and the complexity of the signaling among macro-UTs and femto-BSs. Second, if $\lambda_m^f = \lambda_m$ for all f, then using (18) we show that in the distributed algorithm, each Macro-UT only needs to track the sum of the interference in order to calculate the price for interference and does not need to track the interference from each player. Thus, the communication and signaling cost is significantly reduced.

Since NNE has favorable properties to be implemented in a decentralized fashion, we henceforth examine the computing and the uniqueness of NNE.

D. Uniqueness of Normalized Nash Equilibrium

The uniqueness of a NNE for concave games with coupled constraints has been studied in [6]. We summarize the relevant proposition in the following–

Proposition 1. *[6] Let*

$$\mathbf{G}(\mathbf{y}) = \begin{pmatrix} \frac{\partial^2 \widetilde{u}_1}{\partial y_1^2} & \frac{\partial^2 \widetilde{u}_1}{\partial y_2 \partial y_1} & \cdots & \frac{\partial^2 \widetilde{u}_1}{\partial y_F \partial y_1} \\ \frac{\partial^2 \widetilde{u}_2}{\partial y_1 \partial y_2} & \frac{\partial^2 \widetilde{u}_2}{\partial p_2^2} & \cdots & \frac{\partial^2 \widetilde{u}_2}{\partial y_F \partial y_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 \widetilde{u}_F}{\partial y_1 \partial y_F} & \frac{\partial^2 \widetilde{u}_F}{\partial y_2 \partial y_F} & \cdots & \frac{\partial^2 \widetilde{u}_F}{\partial y_F^2} \end{pmatrix}. \quad (19)$$

If the symmetric matrix $\mathbf{G}(\mathbf{y}) + \mathbf{G}^T(\mathbf{y})$ is negative definite for all $\mathbf{y} \in \widetilde{\mathcal{P}}$, then there exists a unique vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M)$ and the unique NE $\mathbf{y}^*$ which satisfy all the KKT conditions in (16)-(17) with $\lambda_m^f = \lambda_m$ for all $f \in \mathcal{F}$ and $m = 1, \ldots, M$, i.e. the NNE is unique [1].

Note that criterion in Proposition 1 is valid only when $\widetilde{u}_f(\cdot)$ is strictly concave in y_f. Even though $u_f(\cdot)$ is strictly concave in p_f, $\widetilde{u}_f(\cdot)$ may not be concave in y_f because of the relationship of $p_f = \exp(y_f)$. For example if the utility function is Shannon Capacity function i.e. $\widetilde{U}_f(\gamma_f) = \log(1 + \gamma_f)$. Then, $\widetilde{u}_f(\exp(y_f), \exp(\mathbf{y}_{-f})) = \log(1 + \frac{\exp(y_f)h_f}{\sigma^2})$, it is not concave in y_f, in fact it is convex in y_f, but $u_f(\mathbf{p}) = \log(1+\gamma_f(p_f))$ is concave in p_f. Hence, Proposition 1 can not be applied to the proposed example and NNE is not defined for such functions since it does not fall in the classical framework of concave games with coupled constraints. $\widetilde{u}_f(\cdot)$ does not depend on $\mathbf{y}_{-f}$, thus, non-diagonal elements of $\mathbf{G}(\mathbf{y})$ are always zero and the diagonal elements of $\mathbf{G}(\mathbf{y})$ are

$$\begin{aligned} \frac{\partial^2 \widetilde{u}_f}{\partial y_f^2} &= \widetilde{U}_f''(\frac{\partial \gamma_f}{\partial y_f})^2 + \widetilde{U}_f' \frac{\partial^2 \gamma_f}{\partial y_f^2} \\ &= \widetilde{U}_f'' \left(\frac{\exp(y_f)h_f}{\sigma^2}\right)^2 + \widetilde{U}_f' \left(\frac{\exp(y_f)h_f}{\sigma^2}\right) \end{aligned} \quad (20)$$

If the above is negative, then the condition in Proposition 1 is readily satisfied. However, it is not straightforward to determine whether some concave nondecreasing functions $\widetilde{U}_f(\exp(y_f))$ yield strictly concave utility functions in $\mathbf{y}$. In the following, we propose some utility functions of practical interest which yield the games $\widetilde{\mathcal{G}}$ with unique NNE.

As a first example, we consider the following utility function

$$\begin{aligned} \widetilde{U}_f(\gamma_f) &= \log(\log(1 + p_f h_f/\sigma^2)) \\ &= \log(\log(1 + \frac{\exp(y_f)h_f}{\sigma^2})) \end{aligned} \quad (21)$$

For certain applications the utility function of interest behaves as a logarithmic function of the throughput (Shannon Capacity). Hence, the above utility function is also of interest from a practical point of view.

The concavity of utility functions $\widetilde{u}_f$ in $\mathbf{y}$ is shown in the following lemma.

Lemma 1. $\widetilde{u}_f(\exp(y_f), \exp(\mathbf{y}_{-f})) = \log(\log(1 + \frac{\exp(y_f)h_f}{\sigma^2}))$ *is a strictly concave function in y_f.*

Proof. Note that

$$\begin{aligned} \frac{\partial^2 \widetilde{u}_f}{\partial^2 y_f} &= \frac{\exp(y_f)\frac{h_f}{\sigma^2}\{(\log(1 + \frac{\exp(y_f)h_f}{\sigma^2}) - \exp(y_f)h_f/\sigma^2\}}{(\frac{h_f}{\sigma^2}\exp(y_f) + 1)^2(\log(1 + \frac{\exp(y_f)h_f}{\sigma^2}))^2} \\ &= \frac{\exp(y_f)\frac{h_f}{\sigma^2}\{\log\left(\frac{1 + \frac{\exp(y_f)h_f}{\sigma^2}}{\exp(\exp(y_f)\frac{h_f}{\sigma^2})}\right)\}}{(\frac{h_f}{\sigma^2}\exp(y_f) + 1)^2(\log(1 + \frac{\exp(y_f)h_f}{\sigma^2}))^2} \\ &< 0 \quad (\text{since } \exp\left(\frac{\exp(y_f)h_f}{\sigma^2}\right) > 1 + \frac{\exp(y_f)h_f}{\sigma^2}) \end{aligned}$$

Hence, $\widetilde{u}_f(\cdot)$ is strictly concave in y_f. □

Another utility function of interest is $\widetilde{U}_f(\gamma_f) = \frac{(\gamma_f)^{1-\alpha}}{1-\alpha}$ for $\alpha \neq 1$.

- The above utility function corresponds to α proportional fair utility in SNR. When $\alpha = 2$, it corresponds to *harmonic fair* utility in SNR.[2]

[1] The condition is sufficient, but not necessary. In [6] a weaker condition is provided for the uniqueness of NNE. We do not consider that property since it is very difficult to verify that condition in practice.

[2] It is also known as *delay minimization utility*, because when $\alpha = 2$, $1-\alpha$ is negative, thus it corresponds to minimization the delay which is proportional to γ_f^{-1} in M/M/∞ queue.

When $p_f = \exp(y_f)$, then

$$\widetilde{U}_f(\gamma_f) = \frac{\exp(y_f(1-\alpha))}{1-\alpha}(\frac{h_f}{\sigma^2})^{1-\alpha} \quad (22)$$

By differentiating twice the above function, it is straightforward to discern that $\widetilde{U}_f(\cdot)$ (and thus, $\widetilde{u}_f$) is strictly concave in y_f *only when* $\alpha > 1$. The social utility is also maximized at the NNE strategy profile.

E. Computing NNE

By following the same lines as in our previous work [10] we can show that $\Phi(\mathbf{y}) = \sum_{f=1}^{F} \widetilde{u}_f(\mathbf{y})$ is a potential function [8] of the game $\widetilde{\mathcal{G}}$ and we can obtain the following result.

Proposition 2. *The NNE of the game $\widetilde{\mathcal{G}}$ with $\widetilde{u}_f(\cdot)$ as defined in (21) or (22) is equivalent to the solution of the following convex optimization problem CCPG-IN if $\widetilde{u}_f(\cdot)$ is strictly concave in $\mathbf{y}$:*

$$\textit{CCPG-IN :} \quad \underset{\mathbf{y}}{\textit{maximize}} \sum_{f=1}^{F} \widetilde{u}_f(\mathbf{y})$$
$$\textit{subject to } (14)(\textit{or } (15)) \quad \textit{for } m = 1, \ldots, M.$$

Thus, the computation of the NNE is equivalent to solving a convex optimization problem and the maximum of social utility is attained at the NNE.

F. Distributed Algorithm

Based on Proposition 2 we provide a distributed algorithm which converges to the unique NNE solution $\mathbf{y}^*$ and yields Lagrangian multipliers $\boldsymbol{\lambda}^*$ if $\widetilde{u}_f(\cdot)$ is strictly concave in y_f. For convenience, we assume that the femto-BSs only transmit in particular time slots.

Algorithm DIST-IN:

Initially macro-UT m selects $\boldsymbol{\lambda}^0 \in \mathbb{R}_+^M \setminus \{\mathbf{0}\}$ randomly.

Each power update iteration consists of N time slots. At power update iterations $k = 0, 1, \ldots,$ the following tasks are performed:

1) Each femto-BS f sets

$$y_f^k = \underset{y_f \geq 0}{\text{argmax}} \widetilde{U}_f(\gamma_f(\exp(y_f))) - \sum_{m=1}^{M} \lambda_m^k \log(1 + \exp(y_f)\frac{\widehat{\mu}_m^f}{\overline{\mu}_m})$$

All the femto-BSs update the power level $\mathbf{p}^k = (p_1^k, p_2^k, \ldots p_F^k)$ where $p_f^k = \exp(y_f^k)$. Femto-BS transmits with power $\mathbf{p}^k$ at each time slot in $[kN+1, \ldots, (k+1)N]$.

2) Macro-UT m does the following for the time slots $[kN + 1, \ldots, (k+1)N]$:
   - It keeps track of the total outage events N^k during the time slots $[kN+1, \ldots, (k+1)N]$. Specifically, macro-UT m keeps track of the total interference $\widehat{\mathbf{h}_m}^T \mathbf{p}^k$ at each time slot $i \in [kN+1, \ldots, (k+1)N]$ and compares with X_i an exponentially distributed random variable with mean μ_m same as I_m; calculate N^k as the following

$$N^k = \sum_{i=kN+1}^{(k+1)N} \mathbb{1}(\widehat{\mathbf{h}_m}^T \mathbf{p}^k + \overline{\sigma}^2 > X_i)$$

   - At the end of time slot $(k+1)N$, Macro-UT m calculates λ_m^{k+1} in the following manner

$$\lambda_m^{k+1} = (\lambda_m^k + \frac{1}{k+1}(\log(\max(\frac{N^k}{N}, \frac{1}{N}) - \log(\epsilon)))^+$$

   Macro-UT m reports the updated cost λ_m^{k+1} to all the femto-BSs.

The convergence of Algorithm DIST-IN follows immediately from known results in [9] and the law of large numbers. It is stated in the following proposition.

Proposition 3. *Algorithm DIST-IN converges* almost surely *to the unique optimal primal solution $\mathbf{y}^*$ and dual solution $\boldsymbol{\lambda}^*$ for large enough N* [3] *when $\widetilde{U}_f(\cdot)$ or $\widetilde{u}_f$, $f = 1, \ldots, F$ is strictly concave in y_f.*

Since $\widetilde{U}_f(\cdot)$ as defined in (21) and (22) are strictly concave in y_f (and thus, $\widetilde{u}_f$) thus, algorithm DIST-IN converges to the unique NNE for those functions.

Femto-BSs do not need to exchange information with each other to implement the algorithm DIST-IN. Macro-UT m keeps track of the total interference but it does need to know $\widehat{\mu}_m^f$, $\widehat{h}_m^f$ or p_f. In practice it is very difficult to obtain those values, thus our model is readily scalable and implementable in practice. Each macro-UT estimates the outage probability using the value of the total observed interference and generating a sequence of random variables before updating the price for causing interference. We show in Section V that the $N = 200$ is sufficient for the distributed algorithm to converge.

G. Weakly Normalized NE

We observe that NNE may not be unique or even may not be defined for some useful functions e.g. Shannon Capacity functions under the constraints in (14). However, NEs exist for those functions. We can still incorporate a NE which retains most of the properties of NNE for those functions by using the following proposition.

Proposition 4. *Let $\widetilde{\mathcal{G}} \equiv \left\{\mathcal{F}, \widetilde{\mathcal{P}}, \{\widetilde{U}_f(\gamma_f(\exp(y_f)))\}_{f\in\mathcal{F}}\right\}$ and $\bar{\mathcal{G}} \equiv \left\{\mathcal{F}, \widetilde{\mathcal{P}}, \{V_f(\gamma_f(\exp(y_f)))\}_{f\in\mathcal{F}}\right\}$ be two games of the kind defined in (10) with identical player and strategy sets but different utility sets. Let the functions V_f and $\widetilde{U}_f$, $f \in \mathcal{F}$, be strictly increasing functions in γ_f. Then, if $\mathbf{y}$ is a NE of game $\widetilde{\mathcal{G}}$, then it is also a NE for game $\bar{\mathcal{G}}$.*

Proof: Assume that $\mathbf{y}^*$ is a NE for $\widetilde{\mathcal{G}}$ but not for $\bar{\mathcal{G}}$. Then, there exists a y_f such that $(y_f, \mathbf{y}_{-f}^*) \in \widetilde{\mathcal{P}}$ and

$$V_f(\gamma_f(\exp(y_f))) > V_f(\gamma_f(\exp(y_f^*))).$$

V_f is increasing then $\gamma_f(\exp(y_f)) > \gamma_f(\exp(y_f^*))$. Since $\widetilde{U}_f$ is an increasing function in γ_f thus,

$$\widetilde{U}_f(\gamma_f(\exp(y_f))) > \widetilde{U}_f(\gamma_f(\exp(y_f^*))).$$

This contradicts the assumption that $\mathbf{y}^*$ is a NE for $\widetilde{\mathcal{G}}$. Thus, $\mathbf{y}^*$ is a NE for both $\widetilde{\mathcal{G}}$ and $\bar{\mathcal{G}}$. ■

We now define weakly normalized Nash equilibrium (WNNE) introduced in [10] using Proposition 4.

[3] As $N \to \infty$ $\Pr(N^k = 0) \to 0$ and thus, by the strong law of large numbers converge to the outage probability almost surely

Definition 1. *[10] Let game $\widetilde{\mathcal{G}}$ with utility set $\mathcal{U} \equiv \{\widetilde{U}_f(\gamma_f)|f \in \mathcal{F}\}$ and strictly increasing $\widetilde{U}_f(\cdot)$ have a NNE $\overline{\mathbf{y}}$. Then $\overline{\mathbf{y}}$ is also a NE of the game $\bar{\mathcal{G}}$ with utility set $\mathcal{V} \equiv \{V_f(\gamma_f)|f \in \mathcal{F}\}$ and strictly increasing $V_f(\cdot)$. This NE $\overline{\mathbf{y}}$ will be denoted as the* Weakly Normalized Nash equilibrium *(WNNE) of $\bar{\mathcal{G}}$ with respect to the utility set $\mathcal{U}$.*

Note that $\widetilde{U}_f(\gamma_f) = \log(\log(1+\gamma_f))$ is a strictly increasing function of γ_f. Thus, by Proposition 4 the NNE of the game $\widetilde{\mathcal{G}}$ is also an NE of the game $\bar{\mathcal{G}}$ with same set of constraints but utility function $V_f(\gamma_f) = \log(1+\gamma_f)$. Thus, the unique NNE $\mathbf{y}^*$ of the game $\mathcal{G}$ with utility function defined in (21) induces a WNNE $\mathbf{y}^*$ in the game $\bar{\mathcal{G}}$ with $V_f = \log(1+\gamma_f)$. Though the utility function $\log(1+\gamma_f)$ is not concave in $\mathbf{y}$, still we can find an NE which can be implemented using the DIST-IN algorithm. Also note that the unique NNE for the utility functions defined in (22) also induce a unique WNNE for Shannon Capacity functions for a given value of $\alpha > 1$.

IV. Power Allocation for Constrained SINRs in Expectations

In this section, we consider the setting where femto-BSs have to satisfy the constraint in (9). We denote the game as $\mathcal{G}' = \{\mathcal{F}, \mathcal{P}', \mathbf{u}\}$ where $\mathcal{P}'$ is the set of constraints in (9) and $\mathbf{p} \geq 0$. The only difference between the games $\mathcal{G}'$ and $\mathcal{G}$ (defined in (10)) is the strategy set. Note that the strategy set $\mathcal{P}'$ is always convex. Thus, the game $\mathcal{G}'$ always falls into the category of coupled constrained concave game studied in [6] and no transform of the decision variable p_f is required. Therefore, we study NNE in terms of $\mathbf{p}$ for the game $\mathcal{G}'$. In order to prove the uniqueness of NNE, the condition of Proposition 1 has to be satisfied in terms of $\mathbf{p}$ instead of $\mathbf{y}$. It is easy to discern that the condition in Proposition 1 is satisfied by any strictly concave function $U_f(\cdot)$ of p_f including the Shannon Capacity function unlike in Section III.

Moreover, similar to Proposition 2, the following proposition shows that $\sum_{f=1}^{F} u_f(\mathbf{p})$ is a concave potential game.

Proposition 5. *The NNE of the game $\mathcal{G}'$ is equivalent to the solution of the following convex optimization problem CCPG-EX when $u_f(\cdot)$ is strictly concave in p_f:*

$$\begin{aligned} &\textit{CCPG-EX}: \quad \underset{\mathbf{p}\geq 0}{\textit{maximize}} \sum_{f\in\mathcal{F}} u_f(\mathbf{p}) \\ &\textit{subject to} \quad (9) \quad \textit{for } m = 1,\ldots,M \end{aligned} \tag{23}$$

When $u_f(\mathbf{p})$ is strictly concave then the solution of a convex optimization problem is equivalent to compute a NNE and the maximum of the social utility is attained at the NNE.

A. Distributed Algorithm

By leveraging on the results of Proposition 5 we provide a distributed algorithm. We assume that the utilization of the channel is slotted. The distributed algorithm is similar to a stochastic gradient-descent approach:-

Algorithm DIST-EX

Initially macro-UT $m, m = 1,\ldots,M$ selects $\boldsymbol{\lambda}^0 \in \mathbb{R}_+^M \setminus \{\mathbf{0}\}$ randomly

Each femto-BS updates power after every N time slots. At power update iteration $k = 0, 1, \ldots,$ the following steps occur:

1) Each femto-BS $f, f \in \mathcal{F}$ sets
$$p_f^k = \underset{p_f\geq 0}{\operatorname{argmax}}\, U_f(\gamma_f) - \sum_{m=1}^{M} \lambda_m^k E(\hat{h}_m^f) p_f$$
Then, all the femto-BSs update power levels $\mathbf{p}^k = (p_1^k, p_2^k, \ldots p_F^k)$. All the femto-BSs transmits with power $\mathbf{p}^k$ at each time slots $[kN+1, \ldots, (k+1)N]$.
2) Each Macro-UT $m, m = 1,\ldots,M$ keeps track of the total interference at each time slot $[kN+1,\ldots,(k+1)N]$ and calculates the mean N_m^k. At the end of $(k+1)N$th time slot Macro-UT $m, m = 1,\ldots,M$ sets
$$\lambda_m^{k+1} = (\lambda_m^k + \frac{1}{k+1}(\sum_{f\in\mathcal{F}} N_m^k - \overline{\mu}_m + \overline{\sigma}^2))^+$$
Macro-UT m reports the updated cost λ_m^{k+1} to all the femto-BSs.

The convergence of Algorithm DIST-EX follows immediately from known results in [9] and the strong law of large number. This property is stated in the following proposition.

Proposition 6. *For large enough N* [4] *algorithm DIST-EX converges* almost surely *to the unique optimal primal solution $\mathbf{p}^*$ and dual solution $\boldsymbol{\lambda}^*$ when $U_f(\cdot)$ is strictly concave in $\mathbf{p}$.*

Each macro-UT only needs to track the total interference at each iteration. It does not need to know $\widehat{h}_m^f, \widehat{\mu}_m^f$ and p_f. Thus, communication and signaling cost is greatly reduced. Femto-BSs do not need to exchange information among themselves. In Section V we observe that $N = 100$ is enough for satisfactory performance of DIST-EX.

V. Numerical Results

We numerically evaluate the characteristics of an NNE strategy profile for several utility functions. To generate $\widehat{\mu}_m^f, h_f$, we first randomly place femto-BSs and macro-UTs in a disc of radius r_1. Then, we randomly place a femto-UT in a disc of radius r_2 around each femto-BS (fig. 1). We take $r_1 > r_2$ because in practice, a femto-UT is in a close vicinity of its femto-BS compared to the size of a macro-cell. We assume that $\widehat{\mu}_m^f$ includes only the path loss i.e. $\widehat{\mu}_m^f = (d_m^f)^{-\beta}$ where d_m^f denotes the distance between femto-BS f and macro-UT m and $\beta \in [2,4]$. We also assume $h_f = d_f^{-\beta}$ where d_f is the distance between femto-BS f and femto-UT f. For all simulations we take $\beta = 2, r_1 = 20, r_2 = 2$, $\overline{\sigma}^2 = 0.1$, $\overline{\mu}_m = 10$ for all $m = 1,\ldots,M$ and $\sigma^2 = 1$. We use the following metric:

$$U_{\text{NNE}} = \sum_{f\in\mathcal{F}} U_f(\gamma_f(p_{f,\text{NNE}})) \tag{24}$$

$\mathbf{p}_{\text{NNE}} = \{p_{1,NNE},\ldots,p_{F,NNE}\}$ is the NNE strategy profile.

A. Minimum Outage Probability as Constraint

We set the maximum outage probability as $\epsilon = 0.05$. Since $\mathrm{e}^{U_f(\gamma_f)}$ has a straightforward physical interpretation for utility function (21) , instead of plotting $U_f(\gamma_f)$, fig. 2 shows the variation of $\mathrm{e}^{U_f(\gamma_f)}$ at the NNE of the game for utility function (21) with the number of macro-UTs. Intuitively, as

[4] By the law of large number, the mean N_m^k converges to $\sum_{f=1}^{F} p_f^k \widehat{\mu}_m^f$

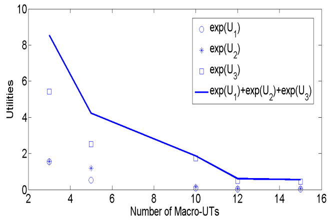

Fig. 2. $\exp(U_f)$, $f = 1, 2, 3$ versus number of macro-UTs when utility function is given by (21) and $F = 3$.

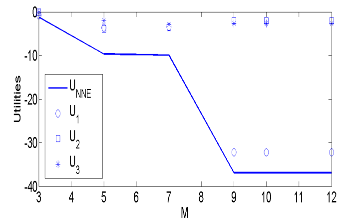

Fig. 3. U_{NNE} and U_f for utility function stated in (22) versus number of macro-UTs for $F = 3$.

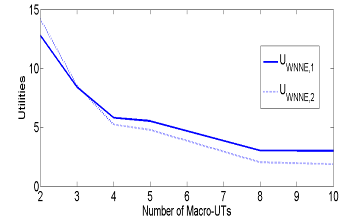

Fig. 4. Total utility at WNNE versus the number of macro-UTs for $F = 3$. $U_{\text{WNNE},1}$ corresponds to utility at the NNE for utility function (22) ($\alpha = 2$) and $U_{\text{WNNE},2}$ corresponds to the total utility at the NNE for the utility function (21).

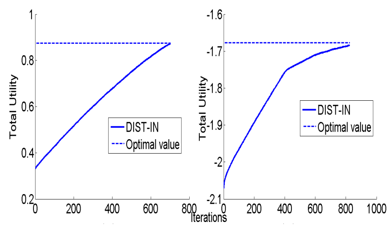

Fig. 5. Convergence analysis of Algorithm DIST-IN for $N = 200, F = 3$ and $M = 5$. The left hand side figure corresponds to the utility functions (21) and the right hand side corresponds to the utility functions (22) with $\alpha = 2$.

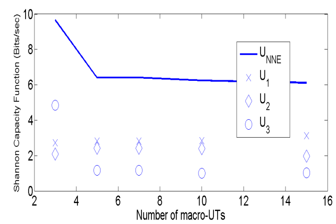

Fig. 6. Total and individual utility at NNE for Shannon Capacity function with $F = 3$ versus the number of macro-UTs.

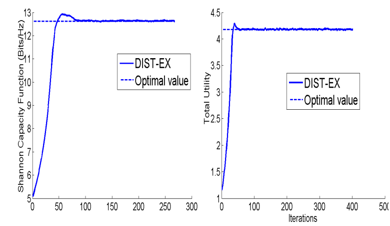

Fig. 7. Convergence analysis of Algorithm DIST-EX for $N = 100, F = 3$ and $M = 5$. The total utility is plotted against the iterations. The left and right hand side correspond to the utility function $U_f = \log(1 + \gamma_f)$ and $U_f = \log(\log(1 + \gamma_f))$ respectively.

the number of macro-UTs increase, the strategy set $\widetilde{\mathcal{P}}$ shrinks and the additional constraints imply that the power with which femto-BSs transmit decrease. Since the sum of utilities (21) corresponds to the proportional throughput, the utilities of femto-BSs are concentrated in a small range.

Fig. 3 shows that the individual utility and the total utility decreases when the number of macro-UTs increases for utility function (22) with $\alpha = 2$. Because of the harmonic fair allocation, only one of the femto-BSs reduces its power but others transmit with almost the same power even when the number of macro-UTs increases as shown in Fig. 3.

In fig. 4 we study the Shannon Capacity function i.e. $U_f(\gamma_f) = \log(1 + \gamma_f)$ as the utility function. There is no NNE for this utility function as shown in Section III-D. But, the NNE of the utility functions (21) and (22) induce two distinct WNNEs . Surprisingly, the WNNE corresponding to the game with utility function (22) with $\alpha = 2$ provides higher (lower, respectively) total utility compared to the WNNE corresponding to the game with utility function (21) when the number of macro-UTs is high (low, resp.).

Fig. 5 shows that the algorithm DIST-IN converges to the NNE value. The rate of convergence is slower for utility functions (22) compared to the utility functions (21).

B. Second Design Criterion

For the numerical analysis of the second design criterion we focus on the utility function $U_f(\gamma_f) = \log(1 + \gamma_f)$. Note that the NNE is unique for the above utility function in this case. Fig. 6 shows that the total utility and individual utility decreases as the number of macro-UTs increases since the strategy set shrinks. When the number of macro-UTs exceeds a certain threshold, the strategy set and thus, the utilities remain almost the same. The utility at the NNE is higher than the utility at the WNNEs shown in Fig. 4.

Fig. 7 shows that the distributed algorithm converges to the NNE solution for $N = 100$. Beside the Shannon Capacity function we also study the convergence of the DIST-EX algorithm for the utility function $U_f(\gamma_f) = \log(\log(1 + \gamma_f))$. The rate of convergence is faster for Shannon Capacity function. Convergence is faster for DIST-EX compared to DIST-IN.

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