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L. Lenôtre
Project-Team Sage
Inria Rennes Bretagne-Atlantique
Campus de Beaulieu
35042 Rennes Cedex
France
e-mail: lionel.lenotre@inria.fr

Abstract: We provide two numerical schemes for the sample paths simulation of skew diffusions with piecewise constant coefficients. Both of these schemes use the resolvent kernel. Precisely, the first scheme use only the resolvent kernel while the second use it as less as possible. Moreover, in the second, some sampled positions are exact.

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Introduction

We consider the skew diffusions in dimension one, that is the continuous stochastic processes whose transition functions are the solutions to

$$\begin{cases} \frac{\partial}{\partial t}p(t,x,y) = \frac{\rho(y)}{2}\operatorname{div}\left(a(x)\nabla p(t,x,y)\right) + b(x)\nabla p(t,x,y),\ \forall x\in\mathbb{R},\ 0<t\leq T, \\ p(0,x,y) = \delta_y(x),\ y\in\mathbb{R} \end{cases} \tag{1}$$

when (a,ρ,b) are piecewise constant. We decide to call them constant skew diffusions.

Recently [3, 6, 7] new algorithms were developed to sample the paths of these processes. But, none of them actually give an answer for any combination of piecewise constant coefficients (a,ρ,b). Indeed, [6, 7] provide very interesting exact schemes but only suited for a single discontinuity while [3] is designed to handle multiple discontinuities but does not work for a discontinuous b.

In this note, we propose to give two new simulation schemes for one-dimensional constant skew diffusions with one discontinuity at 0 which can be extended in case of multiple discontinuities as they are of the Euler type [15]. Besides, both of our schemes can be mixed with a classical scheme as in [15] to ease the simulation of the skew diffusion when it is far from the discontinuities.

The outline of this note is as follows. In §1, using the idea of [11, 12], we build a simulation scheme for Feller processes that uses their resolvent density . Then, §2 we prove a Donsker invariance principle to argue for the consistency of this scheme. In §3 we quickly show that a constant skew diffusion is a Feller process and possesses a resolvent density. We also provide a change of variable allowing to simplify the coefficients (a,ρ,b) and to reduce the problem. After that, we introduce always in §3 a closed-form of the resolvent density for the constant skew diffusions with simplified coefficients that were computed in [14]. Finally in §4 and §5 we detail respectively the first and second schemes for the reduced problem.

1. A numerical scheme for Feller processes using their resolvent density

Let $X=(X_t)_{0\leq t\leq T}$ be a one-dimensional Feller process starting at x, $p(t,x,y)$ be the density of its transition kernel, $r(\lambda,x,y)$ be its resolvent density and τ be an exponentially distributed random variable with parameter λ which is independent of X.

Proposition 1.1. *The random variable X_τ possesses the density $\lambda\, r(\lambda,x,y)$.*

Proof. The result follows directly from the computation that follows.

$$\begin{aligned}\mathbb{P}_x[X_\tau \leq u] &= \int_0^{+\infty} \lambda\, e^{-\lambda t} \int_{-\infty}^{u} p(t,x,y)\, dy\, dt \\ &= \int_{-\infty}^{u} \lambda \int_0^{+\infty} e^{-\lambda t} p(t,x,y)\, dt\, dy \\ &= \int_{-\infty}^{u} \lambda\, r(\lambda,x,y)\, dy.\end{aligned}$$

□

Remark 1.1. Looking at this proof, we easily see that this result can be extended to higher dimensions.

From the definition of an exponential random variable, the larger λ is, the closest to its mean $m = 1/\lambda$ its samples are. Hence, using Proposition 1.1, we can consider for λ large enough that Algorithm 1 gives a discrete approximation of a path until the time T of X.

Data: An initial position x at time 0.
Result: An approximated path of X in n steps until the time T.
Set $X_0 = x$.
for $i = 1, \cdots, n$ **do**
  Sample $X_{\frac{\lfloor iT \rfloor}{n}}$ using the density $n\, r(n, X_{\frac{\lfloor (i-1)T \rfloor}{n}}, y)$.
end
return $(X_0, X_{\frac{\lfloor iT \rfloor}{n}}, \cdots, X_T)$.

Algorithm 1: A scheme using the resolvent.

2. An invariance principle for the numerical scheme

We denote here by a n-steps random walk on $[0,T]$ a sequence of n random variables indexed by n elements of $[0,T]$ and linearly interpolated on $[0,T]$. In order to show that Algorithm 1 produces an approximation of X, we give an invariance principle for the n-steps random walk produced by it.

Proposition 2.1. *Let $(X^n_t)_{0\leq t\leq T}$ be for each $n \geq 0$ the n-steps random walk defined by the random variables generated by Algorithm 1. Then, $(X^n_t)_{n\geq 0}$ converges in distribution to X.*

Proof. For the sake of clarity, the proof is given assuming that $T = 1$.

For each $n \geq 0$, we put $\tau_0 = 0$ and $\tau_k = \sum_{i=1}^{k} \theta_i$ for $k = 1, \ldots, n$ with $\theta_1, ..., \theta_n$ a finite sequence of independent exponential variables of parameter n. Using this sequence, we define for each $n \geq 0$ the n-steps random walk $(Y^n_t)_{0\leq t\leq 1}$ and $(Z^n_t)_{0\leq t\leq 1}$ from the linear interpolation of respectively $Y_{k/n} = X_{\tau_k}$ for each $k = 1, \ldots, n$ and $(X_{\tau_0}, \ldots, X_{\tau_n})$.

From their definition, $(Y^n_t)_{0\leq t\leq 1}$ and $(X^n_t)_{0\leq t\leq T}$ have the same law for each $n \geq 0$. Thus, if $(Y^n_t)_{0\leq t\leq 1}$ converges in distribution to X, then $(X^n_t)_{0\leq t\leq T}$ converges also to X.

By [2, Theorem 3.1], $(Y^n_t)_{0\leq t\leq 1}$ converges in distribution to X if we show

1. $d(Y^n, Z^n)$ converges almost surely 0 which reduced here from the interpolation properties to prove $\sup_{0\leq k\leq n} |\tau_k - k/n|$ converges a.s to 0.
2. $(Z^n)_{n\geq 0}$ converges in distribution to X.

By the construction above, $(\tau_k - k/n)_{0\leq k\leq n}$ is a martingale. Therefore, using the Doob's martingale inequality and the fact that the sums of exponential random variables are Gamma distributed,

$$\mathbb{P}\left[\sup_{0\leq k\leq n} (\tau_k - k/n) \geq K\right] \leq \frac{1}{K^p\, n^{\frac{p}{2}}}. \tag{2}$$

and we can choose p such that, for any $K > 0$,

$$\sum_{n=1}^{+\infty} \mathbb{P}\left[\sup_{0\le k\le n}(\tau_k - k/n) \ge K\right] < +\infty. \tag{3}$$

Now from the properties of Gamma variables, it comes that $\mathbb{P}\left[\sup_{0\le k\le n}(\tau_k - k/n) \ge K\right]$ converges in distribution to 0. Thus, using (3) and the Borel-Cantelli lemma, $\sup_{0\le k\le n}|\tau_k - k/n|$ converges a.s to 0 and 1. is proved.

Since Z^n is the linear interpolation of $(X_0, X_{\tau_1}, \cdots, X_{\tau_n})$,

$$|Z^n_t - X_t| \le |Z^n_t - X_{\tau_{\lfloor nt\rfloor}}| + |X_t - X_{\tau_{\lfloor nt\rfloor}}| \le |X_t - X_{\tau_{\lfloor nt\rfloor+1}}| + |X_t - X_{\tau_{\lfloor nt\rfloor}}| + |X_t - X_{\tau_{\lfloor nt\rfloor}}|.$$

Hence,

$$\mathbb{P}\left[\sup_{0\le t\le 1}|Z^n_t - X_t| \ge K\right] \le \mathbb{P}\left[\sup_{0\le t\le 1}|X_t - X_{\tau_{\lfloor nt\rfloor+1}}| \ge K\right] + 2\,\mathbb{P}\left[\sup_{0\le t\le 1}|X_t - X_{\tau_{\lfloor nt\rfloor}}| \ge K\right]. \tag{4}$$

and $(Z^n)_{n\ge 0}$ converges in probability to X if we show that both terms in the right hand side of (4) goes to 0 and 1.

The proof is similar for both terms of (4). Therefore, we only deals with the second one. As the paths of X are continuous, for any $K > 0$ and any $\varepsilon > 0$, it exists $\delta > 0$ such that

$$\mathbb{P}\left[\sup_{|t-s|<\delta}|X_t - X_t| \ge K\right] \le \varepsilon. \tag{5}$$

Subsequently, since

$$\mathbb{P}\left[\sup_{0\le t\le 1}|X_t - X_{\tau_{\lfloor nt\rfloor}}| \ge K\right] \le \mathbb{P}\left[\sup_{|t-s|<\delta}|X_t - X_t| \ge K\right] + \mathbb{P}\left[\sup_{0\le k\le n}(\tau_k - k/n) \ge \delta\right], \tag{6}$$

(2) and (5) immediately provide that (6) goes to 0. As a result, 2. is proved.

□

3. Existence of the resolvent density, problem reduction and formulae

We have the following fundamental result on skew diffusions.

Proposition 3.1. *Let the coefficients in* (1) *be such that*

$$(a, \rho, b) \text{ are measurable from } \mathbb{R}^3 \to \mathbb{R} \text{ with } \lambda \le a, \rho \le \Lambda \text{ and } |b| \le \Lambda.$$

Then, (1) *possesses a unique solution*

$$p(t, x, y) \in \mathrm{L}^2([0, T], \mathrm{H}^1(\mathbb{R})) \cap \mathcal{C}([0, T], \mathrm{L}^2(\mathbb{R})) \tag{7}$$

which is the transition function of a conservative Feller process $(X_t, \mathcal{F}_t, \mathbb{P}_x)_{t\ge 0, x\in\mathbb{R}}$ with continuous paths. Besides, X possesses the infinitesimal generator

$$A = \frac{\rho(x)}{2}\,\mathrm{div}(a(x)\nabla\cdot) + b(x)\nabla\cdot \text{ with } \mathrm{Dom}(A) = \left\{f \in \mathbb{C}_b(\overline{\mathbb{R}}, \mathbb{R}) \mid Af \in \mathbb{C}_b(\overline{\mathbb{R}}, \mathbb{R})\right\},$$

where

$$\mathbb{C}_b(\overline{\mathbb{R}}, \mathbb{R}) = \left\{f : \overline{\mathbb{R}} \mapsto \mathbb{R} \mid f \text{ is continuous and bounded}\right\}$$

and the resolvent of $(A, \mathrm{Dom}(A))$ defined for any $\lambda \in \mathbb{C}$ such that $\mathrm{Re}[\lambda] > 0$ admits a density with respect to Lebesgue measure

$$r(\lambda, x, y) = \int_0^{+\infty} p(t, x, y)e^{-\lambda t}\,dt, \quad \forall (x, y) \in \mathbb{R}^2,$$

that is the Laplace transform of $p(t, x, y)$.

Proof. First, the existence of the solution $p(t,x,y)$ and its regularity is proved in [1]. Secondly, the existence of the continuous stochastic process X is deducible through an adaptation of the results in [8, 9] as well as the infinitesimal generator (see [14] for more details). At last, the result on the density of the resolvent associated to the infinitesimal generator can be found in [10, 16]. $\square$

Remark 3.1. The above result is classical when the coefficients belongs to $\mathcal{C}^\infty(\mathbb{R})$ and the conservative Feller process obtained is a semi-martingale often called a diffusion process. This is in general not true when the coefficients are just measurable.

With this result, it is evident that Algorithm 1 can be used to simulate skew diffusions assuming a closed-form of the resolvent density is known.

Now let X be a constant skew diffusion having only one discontinuity at 0, that is with

$$\begin{gathered} a(x) = a_+\mathbb{1}(x \geq 0) + a_-\mathbb{1}(x < 0),\ \rho(x) = \rho_+\mathbb{1}(x \geq 0) + \rho_-\mathbb{1}(x < 0) \\ \text{and} \quad b(x) = b_+\mathbb{1}(x \geq 0) + b_-\mathbb{1}(x < 0), \\ \text{with } (a_+, a_-, \rho_+, \rho_-) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^* \text{ and } (b_+, b_-) \in \mathbb{R}^2. \end{gathered} \tag{8}$$

In [5, 14], it is shown that $X = \phi^{-1}(X')$ where $\phi(x) = x/\sqrt{a(x)\,\rho(x)}$ and X' is the constant skew diffusion whose coefficients are

$$\begin{gathered} \hat{a}(x) = \beta\,\mathbb{1}(x \geq 0) + (1-\beta)\,\mathbb{1}(x < 0),\ \hat{\rho}(x) = \beta^{-1}\,\mathbb{1}(x \geq 0) + (1-\beta)^{-1}\,\mathbb{1}(x < 0) \\ \text{with } \beta = \frac{\sqrt{a_+/\rho_+}}{\sqrt{a_+/\rho_+} + \sqrt{a_-/\rho_-}} \\ \text{and } \hat{b}(x) = \frac{b_+}{\sqrt{a_+\,\rho_+}}\mathbb{1}(x \geq 0) + \frac{b_-}{\sqrt{a_-\,\rho_-}}\mathbb{1}(x < 0) = \hat{b}_+\mathbb{1}(x \geq 0) + \hat{b}_-\mathbb{1}(x < 0). \end{gathered} \tag{9}$$

The constant skew diffusions with coefficients of type (9) are well-known. They are the unique strong solution to the stochastic differential equation involving local time

$$Y_t = Y_0 + B_t + \int_0^t \hat{b}(Y_s)\,ds + (2\beta - 1)L_t^0(Y), \qquad X_0 = x. \tag{10}$$

where $B = (B_t)_{t\geq 0}$ is a Brownian motion and $Y = (Y_t)_{t\geq 0}$ is the unknown process [13]. Besides, a closed-form of their resolvent density was recently computed in [14].

Proposition 3.2 (A. Lejay et al., [14])**.** *Let*

$$g(\gamma, \lambda, x, y) = \frac{1}{\sqrt{\gamma^2 + 2\lambda}} \begin{cases} e^{(-\gamma + \sqrt{\gamma^2+2\lambda})(x-y)} & \text{if } x < y, \\ e^{(-\gamma - \sqrt{\gamma^2+2\lambda})(x-y)} & \text{if } x \geq y. \end{cases} \tag{11}$$

The resolvent density of a skew diffusion X with coefficients satisfying (9) *is*

$$\begin{aligned} r(\lambda, x, y) = g(\hat{b}_+, \lambda, x, y)\mathbb{1}(x \geq 0) + A^{-+}(\lambda, y)g(\hat{b}_-, \lambda, x, y)\mathbb{1}(x < 0) \\ + A^{++}(\lambda, y)g(\hat{b}_+, \lambda, x, -y)\mathbb{1}(x \geq 0). \end{aligned} \tag{12}$$

and for $y < 0$,

$$\begin{aligned} r(\lambda, x, y) = g(\hat{b}_-, \lambda, x, y)\mathbb{1}(x < 0) + A^{--}(\lambda, y)g(\hat{b}_-, \lambda, x, -y)\mathbb{1}(x < 0) \\ + A^{+-}(\lambda, y)g(\hat{b}_+, \lambda, x, y)\mathbb{1}(x \geq 0) \end{aligned} \tag{13}$$

with

$$A^{-+}(\lambda,y)=\Theta(\lambda,\hat b_+,\hat b_-)^{-1}2\,\beta\sqrt{\hat b_-^2+2\lambda}\,e^{(\hat b_+-\hat b_-+\sqrt{\hat b_-^2+2\lambda}-\sqrt{\hat b_+^2+2\lambda})y},$$
$$A^{++}(\lambda,y)=\Theta(\lambda,\hat b_+,\hat b_-)^{-1}\big(\beta(\sqrt{\hat b_+^2+2\lambda}-\hat b_+)-(1-\beta)(\sqrt{\hat b_-^2+2\lambda}-\hat b_-)\big)e^{2\hat b_+y},$$
$$A^{--}(\lambda,y)=\Theta(\lambda,\hat b_+,\hat b_-)^{-1}\big(-\beta\,(\sqrt{\hat b_+^2+2\lambda}+\hat b_+)+(1-\beta)\,(\sqrt{\hat b_-^2+2\lambda}+\hat b_-)\big)e^{2\hat b_-y},$$
$$A^{+-}(\lambda,y)=\Theta(\lambda,\hat b_+,\hat b_-)^{-1}2(\beta-1)\sqrt{\hat b_+^2+2\lambda}e^{(\hat b_--\hat b_++\sqrt{\hat b_-^2+2\lambda}-\sqrt{\hat b_++2\lambda})y},$$

where the common denominator is

$$\Theta(\lambda,\hat b_+,\hat b_-)=\beta\big(\sqrt{\hat b_+^2+2\lambda}+\hat b_+\big)+(1-\beta)\big(\sqrt{\hat b_-^2+2\lambda}-\hat b_-\big). \tag{14}$$

With this last result and ϕ, we are finally able to simulate any constant skew diffusions with coefficients of type (8) using Algorithm 1. Furthermore, any scheme that sample X with (a,ρ,b) given by (9) simulate Y with $(\hat a,\hat\rho,\hat b)$ given by (8) after the application of ϕ^{-1}.

4. The first scheme

We treat the case where X has coefficients given by (9) through a series of algorithms which once nested one within the others give the scheme when the starting point of X is non negative. By the symmetry of (11), these algorithms can be easily adapted when the starting point is negative providing thus a complete scheme.

We start with a few notations. $\mathcal{E}(\alpha)$ denotes an exponential distribution with parameters α and $\mathcal{B}(p)$ a Bernoulli distribution with parameters p. The notations $\mathcal{Q}(\alpha,x)$ and $\mathcal{T}(\alpha,x)$ stand for the distribution whose densities are respectively

$$\alpha\,e^{-\alpha(y-x)}\,\mathbb{1}(y\ge x\ge 0)\text{ and }\frac{\alpha}{1-e^{\alpha x}}\,e^{\alpha y}\,\mathbb{1}(0\le y\le x)$$

From the expression in the Proposition 3.2,

$$\mathbb{P}[X_\tau X_0<0|X_0\ge 0]=\begin{cases}\frac{2\lambda\,(\beta-1)}{(\beta-1)(-\hat b_-+\sqrt{\hat b_-^2+2\lambda})+\beta\,(-\hat b_+-\sqrt{\hat b_+^2+2\lambda})}\,e^{(-\hat b_+-\sqrt{\hat b_+^2+2\lambda})x}\\ \times\int_{\mathbb{R}}e^{-(-\hat b_--\sqrt{\hat b_-^2+2\lambda})y}\,dy,\quad\text{ if } x\ge 0,\\ \frac{-2\lambda\,\beta}{(\beta-1)(-\hat b_-+\sqrt{\hat b_-^2+2\lambda})+\beta\,(-\hat b_+-\sqrt{\hat b_+^2+2\lambda})}\,e^{(-\hat b_-+\sqrt{\hat b_-^2+2\lambda})x}\\ \times\int_{\mathbb{R}}e^{-(-\hat b_++\sqrt{\hat b_+^2+2\lambda})y}\,dy,\text{ if } x<0.\end{cases}$$

Now using the densities of the exponential laws with parameters

$$\alpha_-=-\hat b_--\sqrt{\hat b_-^2+2\lambda}\text{ and }\alpha_+=-\hat b_++\sqrt{\hat b_+^2+2\lambda},$$

the probabilities simplifies to

$$\mathbb{P}[X_\tau X_0<0|X_0\ge 0]=\begin{cases}\frac{2\lambda\,(\beta-1)\,(-\hat b_--\sqrt{\hat b_-^2+2\lambda})^{-1}}{(\beta-1)(-\hat b_-+\sqrt{\hat b_-^2+2\lambda})+\beta\,(-\hat b_+-\sqrt{\hat b_+^2+2\lambda})}\,e^{(-\hat b_+-\sqrt{\hat b_+^2+2\lambda})x},\text{ if } x\ge 0,\\ \frac{-2\lambda\,\beta\,(-\hat b_++\sqrt{\hat b_+^2+2\lambda})^{-1}}{(\beta-1)(-\hat b_-+\sqrt{\hat b_-^2+2\lambda})+\beta\,(-\hat b_+-\sqrt{\hat b_+^2+2\lambda})}\,e^{(-\hat b_-+\sqrt{\hat b_-^2+2\lambda})x},\text{ if } x<0.\end{cases}$$

Hence,

$$\frac{\mathbb{P}[X_\tau\in dy|X_\tau<0,X_0\ge 0]}{\mathbb{P}[X_\tau X_0<0|X_0\ge 0]}=\alpha_+e^{-\alpha_+y}\,dy.$$

and the Algorithm 2 for sampling X_τ given $X_0\ge 0$ follows and give the basis of the scheme.

Data: An initial position $X_0 = x \geq 0$ at time 0.
Result: The position X_τ at time τ.
Sample $U \sim \mathcal{B}(\mathbb{P}[X_\tau X_0 < 0 | X_0 \geq 0])$.
if $U = 1$ **then**
  $X_\tau = E_+ \sim \mathcal{E}(\alpha_+)$.
else
  Sample X_τ using $(1 - \mathbb{P}[X_\tau X_0 < 0 | X_0 \geq 0])^{-1} \lambda \, r(\lambda, x, y) \, \mathbb{1}(y \geq 0) \, \mathbb{1}(x \geq 0)$.
end
return X_τ.

Algorithm 2: Sampling X_τ.

Using the densities of $\mathcal{Q}(\alpha_+, X_0)$,

$$\begin{aligned}\frac{(1 - \mathbb{P}[X_\tau X_0 < 0|X_0 \geq 0]) \sqrt{\hat{b}_-^2 + 2\lambda} \, (-\hat{b}_+ + \sqrt{\hat{b}_-^2 + 2\lambda})}{\lambda \, e^{-(\hat{b}_+ - \sqrt{\hat{b}_+^2 + 2\lambda}) X_0}} & \mathbb{P}[X_\tau \geq X_0 | X_0 \geq 0] \\ &= \frac{(1-\beta)(-\hat{b}_- + \sqrt{\hat{b}_-^2 + 2\lambda}) - \beta \, (-\hat{b}_+ - \sqrt{\hat{b}_+^2 + 2\lambda})}{(\beta - 1)(-\hat{b}_- + \sqrt{\hat{b}_-^2 + 2\lambda}) + \beta \, (-\hat{b}_+ - \sqrt{\hat{b}_+^2 + 2\lambda})} \, e^{(-\hat{b}_+ - \sqrt{\hat{b}_+^2 + 2\lambda}) X_0} \\ &\qquad + e^{(-\hat{b}_+ + \sqrt{\hat{b}_+^2 + 2\lambda}) X_0}.\end{aligned}$$

Henceforth, with a classical splitting of $(1 - \mathbb{P}[X_\tau X_0 < 0 | X_0 \geq 0])^{-1} \lambda \, r(\lambda, x, y) \, \mathbb{1}(y \geq 0) \, \mathbb{1}(x \geq 0)$, Algorithm 3 for sampling of X_τ given $X_\tau \geq 0$ comes naturally and complete Algorithm 2.

Data: An initial position $X_0 = x \geq 0$ at time 0.
Result: The position X_τ at time τ given $X_\tau \geq 0$.
Sample $U \sim \mathcal{B}(\mathbb{P}[X_\tau \geq X_0 | X_0 \geq 0])$.
if $U = 1$ **then**
  $X_\tau = F \sim \mathcal{Q}(\hat{b}_+ - \sqrt{\hat{b}_+^2 + 2\lambda}, X_0)$.
else
  Sample X_τ using $(1 - \mathbb{P}[X_\tau X_0 < 0 | X_0 \geq 0])^{-1} (1 - \mathbb{P}[X_\tau \geq X_0 | X_0 \geq 0])^{1} \lambda \, r(\lambda, x, y) \, \mathbb{1}(x > y \geq 0) \, \mathbb{1}(x \geq 0)$.
end
return X_τ.

Algorithm 3: Sampling of X_τ given $X_\tau \geq 0$.

From (12), $(1-P)^{-1}(1-Q)^{1} \lambda \, r(\lambda, x, y) \, \mathbb{1}(x > y \geq 0) \, \mathbb{1}(x \geq 0)$ is bounded density with its whole mass in the $(0, x)$. Thus, we can use a classical rejection sampling [4] to complete Algorithm 2 and obtain the scheme.

Remark 4.1. When

$$S = (1-\beta)(-\hat{b}_- + \sqrt{\hat{b}_-^2 + 2\lambda}) - \beta(-\hat{b}_+ + \sqrt{\hat{b}_+^2 + 2\lambda}) > 0,$$

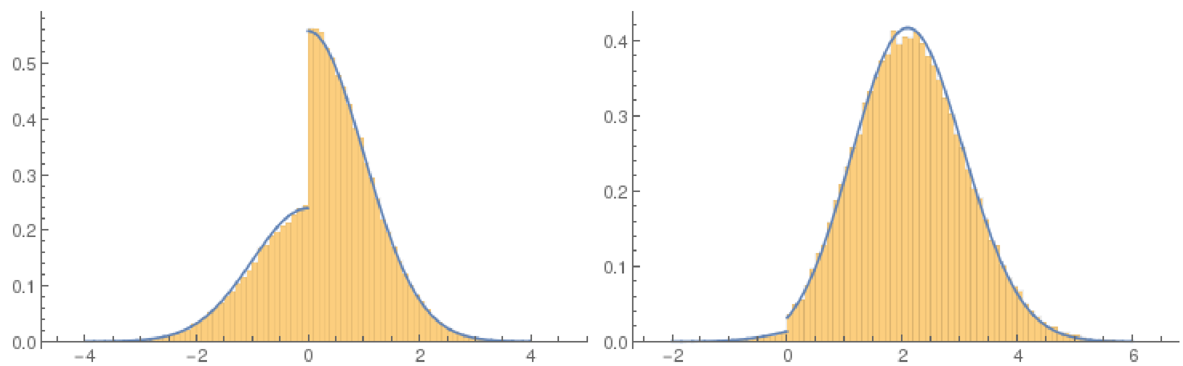
since

$$\begin{aligned}\frac{\sqrt{\hat{b}_+^2 + 2\lambda} \, \mathbb{P}[X_\tau \in dy | X_0 \geq 0, 0 \leq X_\tau < X_0]}{(1 - \mathbb{P}[X_\tau X_0 < 0 | X_0 \geq 0])^{-1} \, (1 - \mathbb{P}[X_\tau \geq X_0 | X_0 \geq 0])^{-1}} \\ = \frac{(1-\beta)(-\hat{b}_- + \sqrt{\hat{b}_-^2 + 2\lambda}) - \beta(-\hat{b}_+ + \sqrt{\hat{b}_+^2 + 2\lambda})}{(\beta - 1)(-\hat{b}_- + \sqrt{\hat{b}_-^2 + 2\lambda}) + \beta \, (-\hat{b}_+ - \sqrt{\hat{b}_+^2 + 2\lambda})} \, e^{-\hat{b}_+(x-y)} \, e^{-\sqrt{\hat{b}_+^2 + 2\lambda}(x+y)} \\ + e^{(-\hat{b}_+ - \sqrt{\hat{b}_+^2 + 2\lambda})(x-y)},\end{aligned}$$

a dichotomy immediately provides Algorithm 4 which gives an alternative to the rejection sampling.

Data: The position $X_0 = x \geq 0$ at time 0.
Result: The position X_τ at time τ knowing that $0 \leq X_\tau < X_0$ and $X_0 \geq 0$.
Sample $U \sim \mathcal{B}(P)$.
if $U = 1$ **then**
  $X_\tau = G_1$ with $E_+ \sim \mathcal{T}(-\hat{b}_+ - \sqrt{\hat{b}_+^2 + 2\lambda}, X_0)$.
else
  $X_\tau = G_2$ with $E_+ \sim \mathcal{T}(-\hat{b}_+ + \sqrt{\hat{b}_+^2 + 2\lambda}, X_0)$.
end
return X_τ.

Algorithm 4: Sampling of X_τ given $0 \leq X_\tau < X_0$ and $S > 0$.



(a) Skew Brownian motion (b) Skew Brownian motion with constant drift

FIG 1. *Histograms using the first algorithm and 100,000 final positions for a skew Brownian motion and a skew Brownian motion with constant drift* $b = 2$ *with the parameter* $\beta = 0.7$ *and* 0 *as starting point for both of them.*

The scheme we just present has shown relevant performances in a few numerical experiments. Basically, it competes with the exact scheme for the Skew Brownian Motion given in [15] (see Figure 2 and Table 1) and gives a reliable approximation of the Skew Brownian Motion with a constant drift (see Figure 1) which is the most complicated constant skew diffusion with transition function that can be drawn.

5. The second scheme

We consider a scheme for X with (a, ρ, b) satisfying (9) which is exact when the X stays on the positive or the negative half line and approximated when a crossing occurs. More precisely, we use an exact Euler scheme when the process starts from a point that is not 0 and check if the process has cross 0. If 0 is crossed, we estimate the time s the process takes to reach 0 and sample a position using the resolvent for the time length $\delta t - s$. This idea actually comes from the two steps scheme developed in [15] for the skew Brownian motion.

We start by proving the consistency of this approach. Since the local time in 0 of a semi-martingale grows only when the process cross x (see [17, Chapter VI, Proposition (1.3)]), (10) provides that X behaves exactly like a Brownian motion with drift when it stays on one side of 0.

	approximated	exact
max	4.71137	4.80811
75 % quantile	0.927567	0.922331
median	0.374352	0.365781
25 % quantile	-0.176939	-0.218652
min	-4.59835	-4.27381

TABLE 1
Statistical quantities collected from simulations of the first algorithm and the Exact SBM's algorithm of 100,000 final position of a skew Brownian motion with $\beta = 0.7$ *starting from* 0.

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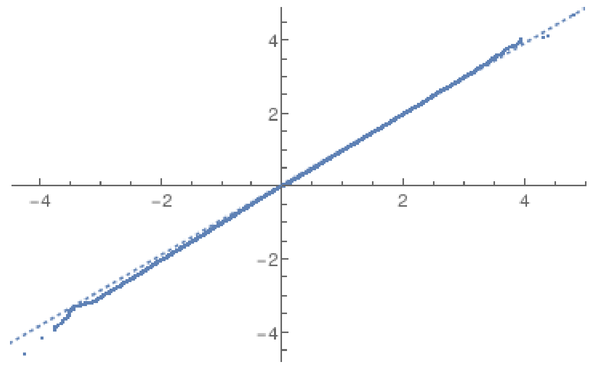

FIG 2. *A Q-Q plot between the first algorithm and the SBM's exact algorithm for a skew Brownian motion with* $\beta = 0.7$ *starting from* 0.

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As in §4, we gives a series of algorithms which must also be nested one within the others, cover only the case where X starts from non negative position and can be easily adapt in the other case.

The general structure of the scheme is given in Algorithm (5) where $B^{b,x}$ is Brownian Motion with drift and τ_x is its first hitting time of 0. At first glance, it appears that we must sample a stopped and pinned Brownian motion with drift. This is actually very easy to do since the exact scheme given in [15] for the case of a Brownian motion is completely suited since a bridge of a Brownian motion with drift is a simple Brownian bridge. As evidence of this statement, using a Doob's h-transform, the density of a bridge of a Brownian motion with drift is given by

$$\begin{aligned} q_T(t,x,y) &= \frac{p_b(t,x,y)p_b(T-t,y,z)}{p_b(T,x,z)} \\ &= \frac{\sqrt{T}}{\sqrt{2\pi t(T-t)}} \exp\left\{ \frac{(T(x-y)+t(z-x))^2}{2t(t-T)T} \right\} = \frac{p_0(t,x,y)p_0(T-t,y,z)}{p_0(T,x,z)}, \end{aligned}$$

where $p_b(t,x,y)$ is the transition function of $B^{b,x}$. For the sake of completeness, we recalled the scheme in Algorithm 7 where $\mathcal{IG}(\nu,\gamma)$ denote the inverse gaussian distribution.

Now to complete the skeleton given in Algorithm 5, we set $x = 0$ in (11). It comes that

$$\mathbb{P}[X_\tau \geq 0 | X_0 = 0] = \frac{2\lambda\,(\beta-1)\,(-\hat{b}_- - \sqrt{\hat{b}_-^2 + 2\lambda})^{-1}}{(\beta-1)(-\hat{b}_- + \sqrt{\hat{b}_-^2+2\lambda}) + \beta\,(-\hat{b}_+ - \sqrt{\hat{b}_+^2+2\lambda})},$$

$$\frac{\mathbb{P}[X_\tau \in dy | X_\tau \geq 0, X_0 = 0]}{\mathbb{P}[X_\tau \geq 0|X_0=0]} = \alpha_+ e^{-\alpha_+ y}\,dy \text{ and } \frac{\mathbb{P}[X_\tau \in dy | X_\tau < 0, X_0 = 0]}{\mathbb{P}[X_\tau < 0|X_0=0]} = \alpha_- e^{-\alpha_- y}\,dy.$$

and Algorithm 6 follows.

As for the first scheme, we perform some numerical experiments given in Table 2 and Figures 3 and 4. Actually, these experiments are exactly the same as those of §4.

Data: An initial position $X_0 = x > 0$ and a time $\delta t > 0$.
Result: An approximated position $X_{\delta t}$ at time δt.
Generate a realization (S, Y) of $(\delta_t \wedge \tau_x, B^{b,x}_{\delta t \wedge \tau_x})$.
if $S = \delta t$ **then**
| $X_{\delta_t} = Y$.
else
| Sample $X_{\delta t}$ using the resolvent for an exponential time step of mean $1/(\delta_t - s)$
end
return $X_{\delta t}$.

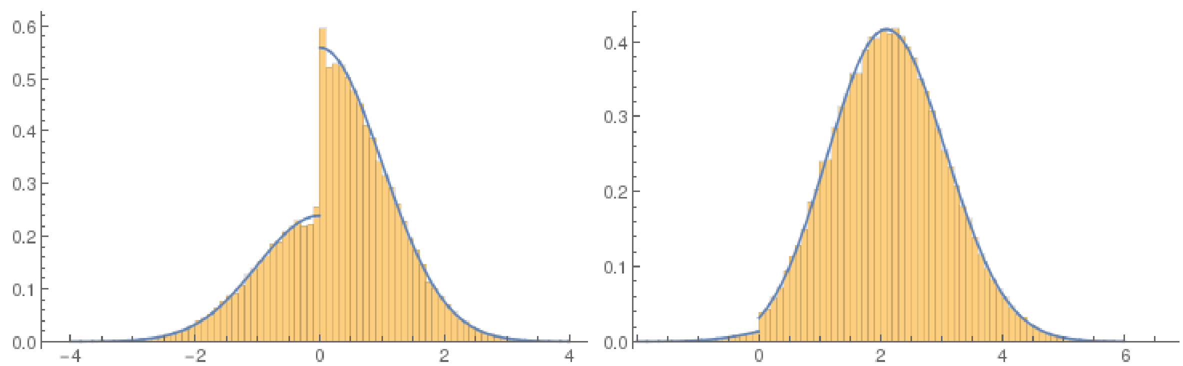
Algorithm 5: The skeleton of the second scheme

Data: An initial position $X_0 = 0$ and a time $\delta t > 0$.
Result: An approximated position $X_{\delta t}$ at time δt.
Sample $U \sim \mathcal{B}(\mathbb{P}[X_\tau \geq 0 | X_0 = 0])$.
if $U = 1$ **then**
  $X_\tau = E_+ \sim \mathcal{E}(\alpha_+)$.
else
  $X_\tau = E_+ \sim \mathcal{E}(\alpha_-)$.
end
return X_T.

Algorithm 6: Sampling from 0.

Data: An initial position x and a time $\delta t > 0$.
Result: A couple $(\tau_x, 0)$ or $(\delta t, B^{b,x}_{\delta t})$.
Sample a realization y of a $B^{b,x}$ with a Euler scheme.
if $xy < 0$ **then**
  Generate $\xi \sim \mathcal{IG}(\frac{|x|}{|y|}, \frac{x^2}{\delta t})$.
  Set $\tau_x = \frac{\delta t \, \xi}{1+\xi}$.
  return $(\tau, 0)$
else
  Generate $U \sim \mathcal{U}([0,1])$.
  if $U < e^{-\frac{2xy}{\delta t}}$ **then**
    Generate $\xi \sim \mathcal{IG}(\frac{|x|}{|y|}, \frac{x^2}{\delta t})$.
    Set $\tau_x = \frac{\delta t \, \xi}{1+\xi}$.
    return $(\tau, 0)$
  else
    return $(\delta t, y)$.
  end
end

Algorithm 7: Simulation of $(\delta t \wedge \tau_x, B^{b,x}_{\delta t \wedge \tau_x})$.



(a) Skew Brownian motion (b) Skew Brownian motion with constant drift

FIG 3. *The histograms using the second algorithm and 100,000 final positions for a skew Brownian motion and a skew Brownian motion with constant drift* $b = 2$ *with the parameter* $\beta = 0.7$ *and* 0 *as starting point for both of them.*

	approximated 1	approximated 2	exact
max	4.71137	4.07464	4.80811
75 % quantile	0.927567	0.923473	0.922331
median	0.374352	0.364082	0.365781
25 % quantile	-0.176939	-0.217547	-0.218652
min	-4.59835	-4.24103	-4.27381

TABLE 2
Statistical quantities collected from simulations of the first algorithm and the Exact SBM's algorithm of 100,000 final position of a skew Brownian motion with $\beta = 0.7$ *starting from* 0.

.

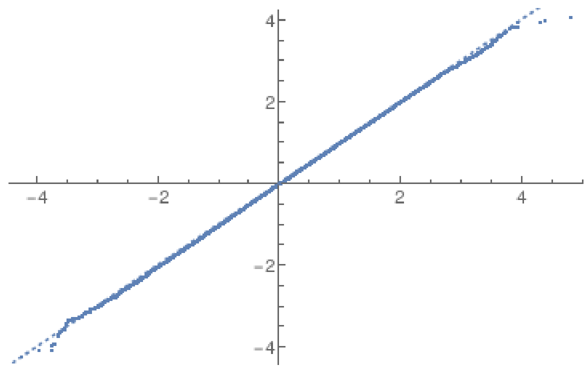


FIG 4. *A Q-Q plot between the second algorithm and the SBM's exact algorithm for a skew Brownian motion with* $\beta = 0.7$ *starting from* 0.

.

Conclusion

While both schemes clearly answer to the problem of simulating constant skew diffusions, they are not exact scheme and we think that it is possible to provide schemes which use the resolvent and are exact. In addition, it can be interesting to compare which schemes is the best particularly when they are extended to handle a large number of discontinuities. This is certainly a good topic for a future work.

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