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Generalized divergence criteria for model selection between random walk and AR(1) model

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Abstract

We investigate a general class of divergence measures among distributions for model selection. As alternative to the classical test of model choice, we introduce kernel type estimators of (ψ, ϕ) -divergence for continuous distributions based on model selection criteria in general non parametric case.

We introduce the Divergence Indicator \mathcal{DI} method by proposing a test for choosing between a random walk and a regression one, using a unified divergence measure. Under the assumptions of standard type about model densities, the asymptotic properties estimator of the expected divergence between the true unknown model and the candidate model are established. From the point of the resulting statistics divergence estimator, the performance of the discrepancy criteria is discussed and illustrated in various settings in model selection test.

Keywords: divergence measure, Kernel Estimator, Hypothesis testing
2010 MSC: 94A17, 62G07, 62G10.

1. Introduction

Statistical modeling technique using the functionals of information theory such as divergence measure, is not new. The divergence measures have provided several useful methods in statistical inference. For example, testing statistical hypotheses with type measures of information theory have been elaborated for models with continuous and discrete data. A comprehensive surveys on divergence measure in statistical testing have been proposed. In particular, among others, to Cressie and Read [6], Nayak [22], Read and Cressie [27], Zografos et al. [33], Salicru et al. [31], Menendez et al. [18, 19, 20], Pardo et al. [24], Morales et al. [21], Zografos [34, 35, 36] and references therein. Model selection is one of standard tools for time series econometricians for selecting the best model among competitor models. One can consider the model selection criteria as an approximately unbiased estimator of the discrepancy, between the true unknown model and a goodness-of-fit approximating model.

Many others model selection criterion have been introduced so far. One can cite the classical model selection criteria based on least-squares estimation, which makes them sensitive to non normalities in the case of finite samples and outliers.

To solve this drawback, robust versions of classical models criterion, which are not affected by outliers, have been proposed, in first, by Ronchetti [28], Ronchetti and Staudte [30]. Other references on this topic can be found in Maronna et al. [16]. On the other hand, a major problem with these tests (Dickey and Fuller) is that the decision on the level of differencing is then based on the outcome of a test at a significance level. A well known difficulty is that when these tests are applied to the same series, the result is that neither null hypothesis-stationarity or a unit autoregressive root-can be rejected at the usual significance levels.

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21 More recently, among the proposals for model selection we recall the criteria presented by Karagrigoriou et al.
 22 [13], the divergence information criteria (DIC) introduced by Mattheou et al. [17]. The DIC criteria use the density
 23 power divergences introduced by Basu et al. [3].

24
 25 In traditional method, Pearson type chi-square statistics have been used to test whether a specified model is con-
 26 sistent with observed data.

27 Because divergence statistic provide naturel measures for dissimilarity between the observed data and a specific
 28 model, it has been used through an informational criteria for discriminating among competing models. The statistic
 29 resulting of divergence estimator is asymptotically distributed as a chi-squared with d degrees of freedom. In this
 30 context, the main problem is that each divergence statistic tends to became large without no increase in its degrees of
 31 freedom as the sample size increases.

32 Hence the goodness-of-fit in forming type chi-squared statistics will generally (over) reject the correct specification
 33 of evry competitor model.

34 The most commonly used approach to this issue is through a method for model selection of Akaike (1973) Infor-
 35 mation Criterion (AIC).

36 This popular method consists in considering Pearson type chi-square statistics that the lower the value of criterion,
 37 the better is the approximated competitor model. In other wods, the model associated to smaller value of chi-square
 38 statistic is generally chosen as the best.

39 It is not at all sur that this approach accurately is entirely satisfactory : these chi-square tests based on the sample
 40 are random, in the sense that their actual values are subject (to fluctuation sample). As a consequence in terms of
 41 adequation, a model with a smaller value of criteria is not necessarily better than one with the a larger chi-square
 42 statistic.

43 It seems natural to explore new approach to the comparison of stationary models by for taking into account the
 44 stochastic nature of these differences. The modest aim of this paper is to address fundamental issues arising from
 45 the practical application of that approach. Our concern is considering an inference from the perspective of model
 46 selection based on divergence type statistics, by proposing some asymptotically standard normal tests.

47 Methodology considered here are testing the null hypothesis that the Random Walk is equally close to the data
 48 generating process (*DGP*) versus the alternative hypothesis that the Stationary *AR*(1) model is closer to the *DGP*
 49 where closeness of a model is measured according to the discrepancy implicit in the divergence type statistic used.

50
 51 The plan of the paper is as follows. In Section 2 we present the divergence measures. Then in Section 3 we
 52 develop our main results. Section 4 provides the results on nonparametric estimation and specification testing. Finally,
 53 in Section 5 we present our conclusion.

54 2. Formal Problem: Definitions and Estimation

One important aspect of statistical modeling is evaluating the fit of the chosen model. Marriott and Newbold [15] discussed the Bayesian goodness of the unit root as follows:

$$\begin{cases} H_0 : \rho = 1, \\ H_1 : |\rho| < 1 \end{cases}$$

in the model *AR*(1) with intercept

$$X_t - \mu 1_d = \rho(X_{t-1} - \mu 1_d) + \varepsilon_t,$$

where $d \in \mathbb{N}^*$, the d -dimensional vector $1_d = (1, \dots, 1)'$, $X_t \in \mathbb{R}^d$, $\forall t$ and ε_t are *i.i.d* Gaussian vector i.e $N(0_{\mathbb{R}^d}, \sigma^2 \Sigma_d)$, Σ_d is the identity matrix and μ is an unknown parameter. Marriott and Newbold [15] proposed to eliminate the parameter μ considering the sample $(W_1, \dots, W_n) \in \mathbb{R}^{d \times n}$ with zero mean vector instead of the sample (X_1, \dots, X_n) and

$$W_t = X_t - X_{t-1}, \quad \forall t = 1, \dots, n.$$

These authors transforme this problem of test by a comparison one between the two models, following the Bayesian approach:

$$\begin{cases} W_t = \varepsilon_t, & (M1), \\ W_t = \rho W_{t-1} + \varepsilon_t - \varepsilon_{t-1} & (M2). \end{cases}$$

Under the model (M1), the distribution function W_t given by:

$$f_1(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x'\Sigma_d x}{2\sigma^2}\right), \quad x \in \mathbb{R}^d.$$

And under the model (M2), the distribution function W_t can be expressed by :

$$f_2(x) = \frac{1}{\sqrt{2\pi\Lambda}} \exp\left(-\frac{x'\Sigma_d x}{2\Lambda}\right), \quad x \in \mathbb{R}^d.$$

where

$$W_t = \rho W_{t-1} + \varepsilon_t - \varepsilon_{t-1}$$

and

$$\Lambda = \text{Var}(W_t) = \rho^2 \text{Var}(W_{t-1}) + \text{Var}(\varepsilon_t - \varepsilon_{t-1}) + 2\rho \text{cov}(W_{t-1}, \varepsilon_t - \varepsilon_{t-1})$$

With a little algebra, we have:

$$\Lambda = \frac{2\sigma^2}{1-\rho}$$

55 Based on their methods, we propose a new approach based on the (ψ, ϕ) -divergence in order to find a goodness of
56 fit of the model.

57 2.1. A Brief Review of (ψ, ϕ) -divergence

58 The ϕ -divergence measure between the probability distributions p and q is defined by

$$\mathcal{D}_\phi(p, q) = \int_{\mathbb{R}^d} \phi\left(\frac{p(x)}{q(x)}\right) q(x) dx, \quad \phi \in \Phi^* \quad (1)$$

where Φ^* is the class of all convex function $\phi(x)$, $x \geq 0$, such that, $\phi(1) = 0$, $\phi'(1) = 0$ and $\phi''(1) = 1$.
For example: $\phi(x) = x \log(x) - x + 1$, we have Kullback-Leibler divergence

$$\mathcal{D}^{KL}(p, q) = \int_{\mathbb{R}^d} p(x) \log \frac{p(x)}{q(x)} dx.$$

Rényi [28] presented the first parametric generalization of Kullback-Leibler

$$\mathcal{D}_\alpha^R(p, q) = \frac{1}{\alpha - 1} \log \int_{\mathbb{R}^d} \left(\frac{p(x)}{q(x)}\right)^\alpha q(x) dx.$$

It is easy to prove that

$$\lim_{\alpha \rightarrow 1} \mathcal{D}_\alpha^R(p, q) = \mathcal{D}^{KL}(p, q) = \int_{\mathbb{R}^d} p(x) \log \frac{p(x)}{q(x)} dx,$$

$$\lim_{\alpha \rightarrow 0} \mathcal{D}_\alpha^R(q, p) = \mathcal{D}^{KL}(q, p) = \int_{\mathbb{R}^d} q(x) \log \frac{q(x)}{p(x)} dx.$$

59 Rényi are not ϕ -divergences measures. However, such measures can be written in the following form:

$$\mathcal{D}_\phi^\psi(p, q) = \psi\left(\mathcal{D}_\phi(p, q)\right), \quad (2)$$

where ψ is a differentiable increasing real function mapping from

$$\left[0, \quad \phi(0) + \lim_{t \rightarrow \infty} \frac{\phi(t)}{t}\right]$$

60 onto $[0, \infty)$; this condition will be justified in (Proposition 1.1, [23]), with $\psi(0) = 0$, $\psi'(0) > 0$, and $\phi \in \Phi^*$. In the
 61 following formulæ we list the functions ψ and ϕ that yield to the Rényi divergence measures:

$$62 \text{ Renyi : } \quad \psi(x) = \frac{1}{\alpha(\alpha-1)} \log(\alpha(\alpha-1)x + 1) \quad \phi(x) = \frac{x^\alpha - \alpha(x-1) - 1}{\alpha(\alpha-1)} \quad \alpha \neq 0, 1$$

$$63 \text{ Sharma-Mittal } \quad \psi(x) = \frac{1}{(s-1)} ((1 + \alpha(\alpha-1)x)^{\frac{s-1}{\alpha-1}} - 1) \quad \phi(x) = \frac{x^\alpha - \alpha(x-1) - 1}{\alpha(\alpha-1)} \quad s \neq 1$$

$$\alpha \neq 0, 1$$

$$64 \text{ Bhattachayya } \quad \psi(x) = -\log(-x + 1) \quad \phi(x) = -x^{\frac{1}{2}} + \frac{1}{2}(x + 1)$$

64 Now, let f be the unknown true density function (with respect to Lebesgue measure on \mathbb{R}^d) of the sample
 65 (W_1, \dots, W_n) with cumulative distribution function F . The distance between true density and those of the models
 66 can be measured by the (ψ, ϕ) -divergence of f and f_j , $j = 1, 2$ as follows

$$\mathcal{D}_\phi^\psi(f, f_j) = \psi(\mathcal{D}_\phi(f, f_j)).$$

67 For a given density of probability g defined on \mathbb{R}^d , we start by giving some notation and conditions that are needed for
 68 the forthcoming sections. Below, we will work under the following assumptions on f and g to establish our results.

69 **(F.1)** The functional $\mathcal{D}_\phi^\psi(f, g)$ as well-defined as (2), in the sense that $\mathcal{D}_\phi^\psi(f, g)$ is finite.

70 2.2. Nonparametric estimation of (ψ, ϕ) -divergence

71 To define our divergence estimator we define, in a first step, a kernel density estimator. Towards this aim, we
 72 introduce a measurable function $K(\cdot)$ fulfilling the following conditions.

73 **(K.1)** $K(\cdot)$ is of bounded variation on \mathbb{R}^d

74 **(K.1)** $K(\cdot)$ is right continuous on \mathbb{R}^d ,

75 **(K.2)** $\|K\|_\infty = \sup_{x \in \mathbb{R}^d} |K(x)| < \infty$,

76 **(K.3)** $\int_{\mathbb{R}^d} K(t) dt = 1$.

77 The well known Akaike-Parzen-Rosenblatt (refer to [1], [25] and [32]) kernel estimator of $f(\cdot)$ is defined, for any
 78 $x \in \mathbb{R}^d$, by

$$\widehat{f}_{n, h_n}(x) = \frac{1}{nh_n^d} \sum_{i=1}^n K\left(\frac{x - W_i}{h_n}\right),$$

79 where $0 < h_n \leq 1$ is the smoothing parameter. Assuming that the density f is continuous, one can obtain the normality
 80 asymptotic of the estimator \widehat{f}_{n, h_n} under conditions below see [14]. For more details of kernel estimators \widehat{f}_{n, h_n} , one can
 81 refer to [9], [10], [5], [26], [7], [11], [8] and the references therein, and their limiting behavior.

82 In a second step, given $\widehat{f}_{n, h_n}(\cdot)$, we estimate divergences $\mathcal{D}_\phi(f, g)$ and $\mathcal{D}_\phi^\psi(f, g)$ by using the representation (1) and (2)
 83 with f and g , by setting

$$\widehat{\mathcal{D}}_\phi(\widehat{f}_{n, h_n}, g) = \int_{\mathbb{R}^d} \phi\left(\frac{\widehat{f}_{n, h_n}(x)}{g(x)}\right) g(x) dx \quad (3)$$

$$84 \widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n, h_n}, g) = \psi\left(\widehat{\mathcal{D}}_\phi(\widehat{f}_{n, h_n}, g)\right), \quad (4)$$

$$= \psi\left(\int_{\mathbb{R}^d} \phi\left(\frac{\widehat{f}_{n, h_n}(x)}{g(x)}\right) g(x) dx\right) \quad (5)$$

85 The approach use to define the plug-in estimators $\widehat{\mathcal{D}}_\phi(\widehat{f}_{n,h_n}, g)$ and $\widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, g)$ are respectively developed in [4]
 86 and [12] in order to introduce a kernel-type estimators of Shannon's entropy and divergences.

87 In the next section, we wish to establish the asymptotic behavior for the estimates $\widehat{\mathcal{D}}_\phi(\widehat{f}_{n,h_n}, g)$ $\widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, g)$ and to
 88 give in application for testing hypothesis.

89 3. Main Results

90 First step we study the consistency of the estimator. In a second step we show the asymptotic normality of the
 91 term given in the function ψ and to deduce those of the general estimator.

92 **Theorem 1.** *Suppose that f is uniformly continuous on $] -\infty, +\infty[$, and that the window width h_n satisfies $h_n \rightarrow 0$
 93 and $nh_n \rightarrow \infty$ as $n \rightarrow \infty$*

$$|\mathcal{D}_\phi(\widehat{f}_{n,h_n}, g) - \mathcal{D}_\phi(f, g)| \rightarrow 0 \quad \text{with probability one as } n \rightarrow \infty \quad (6)$$

PROOF.

$$\begin{aligned} |\widehat{\mathcal{D}}_\phi(\widehat{f}_{n,h_n}, g) - \mathcal{D}(f, g)| &= \left| \int_{\mathbb{R}^d} \phi\left(\frac{\widehat{f}_{n,h_n}(x)}{g(x)}\right) g(x) - \phi\left(\frac{f(x)}{g(x)}\right) g(x) dx \right| \\ &\leq \int_{\mathbb{R}^d} \left| \phi\left(\frac{\widehat{f}_{n,h_n}(x)}{g(x)}\right) - \phi\left(\frac{f(x)}{g(x)}\right) \right| g(x) dx \end{aligned}$$

94 ϕ is a convex function therefore it is locally Lipschitz, so there exists real as k : $|\phi(x) - \phi(y)| \leq k|x - y|$,

95 for $x = \frac{\widehat{f}_{n,h_n}(x)}{g(x)}$ and $y = \frac{f(x)}{g(x)}$

$$\begin{aligned} \left| \phi\left(\frac{\widehat{f}_{n,h_n}(x)}{g(x)}\right) - \phi\left(\frac{f(x)}{g(x)}\right) \right| g(x) &\leq k \left| \frac{\widehat{f}_{n,h_n}(x)}{g(x)} - \frac{f(x)}{g(x)} \right| g(x) \\ &\leq k |\widehat{f}_{n,h_n}(x) - f(x)| \end{aligned}$$

$$|\widehat{\mathcal{D}}_\phi(\widehat{f}_{n,h_n}, g) - \mathcal{D}(f, g)| \leq k \int_{\mathbb{R}} |\widehat{f}_{n,h_n}(x) - f(x)| dx \quad (7)$$

96 Devroye and Györfi [9] shows that

$$\int |\widehat{f}_{n,h_n}(x) - f(x)| \rightarrow 0 \quad \text{with probability one as } n \rightarrow \infty \quad (8)$$

97 therefore after Eq 7 and Eq 8 :

$$|\widehat{\mathcal{D}}_\phi(\widehat{f}_{n,h_n}, g) - \mathcal{D}(f, g)| \rightarrow 0 \quad \text{with probability one as } n \rightarrow \infty$$

98 **Lemma 1.** *Under the assumptions of Theorem 1*

$$|\mathcal{D}_\phi^\psi(\widehat{f}_{n,h_n}, g) - \mathcal{D}_\phi^\psi(f, g)| \rightarrow 0 \quad \text{with probability one as } n \rightarrow \infty \quad (9)$$

99 PROOF. of after Theorem 1, and the effect that ψ is a convex function thus locally Lipschitz.

100 **Lemma 2.** *Let $K(\cdot)$ satisfy (K.1-2-3-4) and let $f(\cdot)$ be a bounded density fulfill (F.1). Suppose that $\phi \in C^1([0, \infty))$ and
 101 there exist a measurable and Lebesgue-integrable function $F(x)$ such that $|\phi'\left(\frac{f(x)}{g(x)}\right)| < F(x)$,*

102 *Then*

(i) *if $f \neq g$ we have*

$$\sqrt{nh_n^d} \left(\mathcal{D}_\phi(\widehat{f}_{n,h_n}, g) - \mathcal{D}_\phi(f, g) \right) \rightarrow \mathcal{N} \left(0, \left(\int_{\mathbb{R}^d} \sigma(x) \phi'\left(\frac{f(x)}{g(x)}\right) dx \right)^2 \right),$$

103 where $\sigma^2(x) := f(x) \int K^2(z) dz$
(ii) if $f = g$ we have

$$\frac{2nh_n^d}{\phi''(1) \int K^2(z) dz} \widehat{\mathcal{D}}_\phi(\widehat{f}_{n,h_n}, g) \rightarrow \chi^2(d)$$

104 **PROOF.** • if $f \neq g$

105 The first order Taylor expansion of $\phi\left(\frac{\widehat{f}_{n,h_n}(x)}{g(x)}\right)$ around $\frac{f(x)}{g(x)}$ gives

$$\int_{\mathbb{R}^d} \phi\left(\frac{\widehat{f}_{n,h_n}(x)}{g(x)}\right) g(x) dx = \int_{\mathbb{R}^d} \phi\left(\frac{f(x)}{g(x)}\right) g(x) dx + \int_{\mathbb{R}^d} \left(\frac{\widehat{f}_{n,h_n}(x)}{g(x)} - \frac{f(x)}{g(x)}\right) \phi'\left(\frac{f(x)}{g(x)}\right) g(x) dx + \int_{\mathbb{R}^d} o\left(\left\|\frac{\widehat{f}_{n,h_n}}{g} - \frac{f}{g}\right\|\right) g(x) dx$$

$$\begin{aligned} \widehat{\mathcal{D}}_\phi(\widehat{f}_{n,h_n}, g) &= \mathcal{D}_\phi(f, g) + \int_{\mathbb{R}^d} \left(\frac{\widehat{f}_{n,h_n}(x)}{g(x)} - \frac{f(x)}{g(x)}\right) \phi'\left(\frac{f(x)}{g(x)}\right) g(x) dx + \int_{\mathbb{R}^d} o\left(\left\|\frac{\widehat{f}_{n,h_n}}{g} - \frac{f}{g}\right\|\right) g(x) dx \\ \widehat{\mathcal{D}}_\phi(\widehat{f}_{n,h_n}, g) - \mathcal{D}_\phi(f, g) &= \int_{\mathbb{R}^d} \left(\widehat{f}_{n,h_n}(x) - f(x)\right) \phi'\left(\frac{f(x)}{g(x)}\right) dx + \int_{\mathbb{R}^d} o\left(\left\|\frac{\widehat{f}_{n,h_n}}{g} - \frac{f}{g}\right\|\right) g(x) dx \end{aligned} \quad (10)$$

107 note that we have from Theorem 2.2. p. 339 of Bulinski. A and Shashkin. A [2]

$$\sqrt{nh_n^d}(\widehat{f}_{n,h_n}(x) - f(x)) \rightarrow \mathcal{N}(0, \sigma^2(x)). \quad (11)$$

Then $\sqrt{nh_n^d} \int_{\mathbb{R}^d} o\left(\left\|\frac{\widehat{f}_{n,h_n}}{g} - \frac{f}{g}\right\|\right) g(x) dx = \sqrt{nh_n^d} o(O_p((nh_n^d)^{-\frac{1}{2}})) \int_{\mathbb{R}^d} g(x) dx = o_p(1)$

Therefore, the random variables

$$\sqrt{nh_n^d} \left(\widehat{\mathcal{D}}_\phi(\widehat{f}_{n,h_n}, g) - \mathcal{D}_\phi(f, g)\right)$$

and

$$\sqrt{nh_n^d} \int_{\mathbb{R}^d} \phi'\left(\frac{f(x)}{g(x)}\right) (\widehat{f}_{n,h_n}(x) - f(x)) dx$$

have the same asymptotic distribution. By 11 we have

$$\sqrt{nh_n^d} \int_{\mathbb{R}^d} \phi'\left(\frac{f(x)}{g(x)}\right) (\widehat{f}_{n,h_n}(x) - f(x)) dx \rightarrow \mathcal{N}\left(0, \left(\int_{\mathbb{R}^d} \sigma(x) \phi'\left(\frac{f(x)}{g(x)}\right) dx\right)^2\right)$$

108 • if $f = g$

109 The second order Taylor expansion of $\phi\left(\frac{\widehat{f}_{n,h_n}(x)}{g(x)}\right)$ around $\frac{f(x)}{g(x)}$ gives

$$\begin{aligned} \int_{\mathbb{R}^d} \phi\left(\frac{\widehat{f}_{n,h_n}(x)}{g(x)}\right) g(x) dx &= \int_{\mathbb{R}^d} \phi\left(\frac{f(x)}{g(x)}\right) g(x) dx + \int_{\mathbb{R}^d} \left(\frac{\widehat{f}_{n,h_n}(x)}{g(x)} - \frac{f(x)}{g(x)}\right) \phi'\left(\frac{f(x)}{g(x)}\right) g(x) dx \\ &+ \frac{1}{2} \int_{\mathbb{R}^d} \left(\frac{\widehat{f}_{n,h_n}(x)}{g(x)} - \frac{f(x)}{g(x)}\right)^2 \phi''\left(\frac{f(x)}{g(x)}\right) g(x) dx + \int_{\mathbb{R}^d} o\left(\left\|\frac{\widehat{f}_{n,h_n}}{g} - \frac{f}{g}\right\|^2\right) g(x) dx \\ &= \frac{1}{2} \int_{\mathbb{R}^d} \left(\widehat{f}_{n,h_n}(x) - f(x)\right)^2 \frac{\phi''(1)}{g(x)} dx + \int_{\mathbb{R}^d} o\left(\left\|\frac{\widehat{f}_{n,h_n}}{g} - \frac{f}{g}\right\|^2\right) g(x) dx \end{aligned}$$

$$\begin{aligned} \widehat{\mathcal{D}}_\phi(\widehat{f}_{n,h_n}, g) &= \frac{1}{2} \int_{\mathbb{R}^d} \left(\widehat{f}_{n,h_n}(x) - f(x)\right)^2 \frac{\phi''(1)}{g(x)} dx + \int_{\mathbb{R}^d} o\left(\left\|\frac{\widehat{f}_{n,h_n}}{g} - \frac{f}{g}\right\|^2\right) g(x) dx \\ &= \frac{1}{2} \int_{\mathbb{R}^d} \left(\widehat{f}_{n,h_n}(x) - f(x)\right)^2 \frac{\phi''(1)}{f(x)} dx + \int_{\mathbb{R}^d} o\left(\left\|\frac{\widehat{f}_{n,h_n}}{g} - \frac{f}{g}\right\|^2\right) g(x) dx \\ &= \frac{1}{2} \int_{\mathbb{R}^d} \left(\frac{\widehat{f}_{n,h_n}(x) - f(x)}{(f(x))^{1/2}}\right)^2 \phi''(1) dx + \int_{\mathbb{R}^d} o\left(\left\|\frac{\widehat{f}_{n,h_n}}{g} - \frac{f}{g}\right\|^2\right) g(x) dx \end{aligned}$$

$$\frac{2nh_n^d}{\phi''(1) \int K^2(z)dz} \widehat{\mathcal{D}}_\phi(\widehat{f}_{n,h_n}, g) = \int_{\mathbb{R}^d} \left(\frac{\sqrt{nh_n^d}(\widehat{f}_{n,h_n}(x) - f(x))}{\sigma(x)} \right)^2 dx + \int_{\mathbb{R}^d} o\left(\left\| \frac{\widehat{f}_{n,h_n}}{g} - \frac{f}{g} \right\|^2\right) g(x) dx$$

110 from Eq.11

$$\frac{2nh_n^d}{\phi''(1) \int K^2(z)dz} \widehat{\mathcal{D}}_\phi(\widehat{f}_{n,h_n}, g) \longrightarrow \chi^2(d)$$

Theorem 2. We consider the $\mathcal{D}_\phi^\psi(f, g)$ defined in Eq.2, then we have

$$\sqrt{nh_n^d} (\widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, g) - \mathcal{D}_\phi^\psi(f, g)) \rightarrow \mathcal{N}\left(0, \left\{ \psi' \left(\int_{\mathbb{R}^d} \phi\left(\frac{f(x)}{g(x)}\right) g(x) dx \right) \int_{\mathbb{R}^d} \sigma(x) \phi'\left(\frac{f(x)}{g(x)}\right) dx \right\}^2\right)$$

111 PROOF. A direct application of the Delta Method.

112 4. Applications for Testing Hypothesis

113 In this section, we use the estimators $\widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_j)$ $j = 1, 2$ to find the perform statistical tests on the model
114 defined in Section 2.

115 4.1. Goodness-of-Fit test

116 For completeness, we look at $\widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_j)$ in the usual way, i.e as a goodness-of-fit statistic. From the uniform-in-
117 bandwidth consistency of $\widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_j)$ for $\mathcal{D}_\phi^\psi(f, f_j)$, the null hypothesis when using the statistic $\widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_j)$ can be
118 given as $H_0 : \mathcal{D}_\phi^\psi(f, f_j) = 0$. Under the alternative hypothesis $H_1 : \mathcal{D}_\phi^\psi(f, f_j) \neq 0$.

119 4.2. Test for Model Selection

Introduce the divergence Indicator $\mathcal{DI} = \mathcal{D}_\phi^\psi(f, f_1) - \mathcal{D}_\phi^\psi(f, f_2)$. An estimator of the divergence indicator is defined
as:

$$\widehat{\mathcal{DI}}_n := \widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_1) - \widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_2).$$

120 Using the divergence indicator, we develop the following test hypothesis on the model under study

- 121 • $H_0^{eq} : \mathcal{DI} = 0$ means that the two models are equivalent.
- 122 • $H_1^{M_1} : \mathcal{DI} < 0$ means that model M_1 is better than model M_2 .
- 123 • $H_1^{M_2} : \mathcal{DI} > 0$ means that model M_2 is better than model M_1 .

124 $\widehat{\mathcal{DI}}_n$ converges to zero under the null hypothesis H_0^{eq} , but it converges to a strictly negative or positive constant when
125 $H_1^{M_1}$ or $H_1^{M_2}$ hold. These properties actually justify the use of $\widehat{\mathcal{DI}}_n$ as a model selection indicator and common
126 procedure of selecting the model with highest goodness-of-fit.

Theorem 3. Under the assumptions of Lemma 2.

1) Under the null hypothesis H_0^{eq} , $\sqrt{nh_n^d} \widehat{\mathcal{DI}}_n \longrightarrow \mathcal{N}(0, \Gamma^2)$

2) Under the $H_1^{M_1}$ hypothesis $\sqrt{nh_n^d} \widehat{\mathcal{DI}}_n \longrightarrow -\infty$

3) Under the $H_1^{M_2}$ hypothesis $\sqrt{nh_n^d} \widehat{\mathcal{DI}}_n \longrightarrow +\infty$

with

$$\Gamma^2 = \left\{ \int_{\mathbb{R}^d} \left[\psi' \left(\int_{\mathbb{R}^d} \phi\left(\frac{f(x)}{f_1(x)}\right) f_1(x) dx \right) \phi'\left(\frac{f(x)}{f_1(x)}\right) - \psi' \left(\int_{\mathbb{R}^d} \phi\left(\frac{f(x)}{f_2(x)}\right) f_2(x) dx \right) \phi'\left(\frac{f(x)}{f_2(x)}\right) \right] \sigma(x) dx \right\}^2$$

PROOF.

$$\begin{aligned}
\sqrt{nh_n^d} \widehat{\mathcal{D}I}_n &:= \sqrt{nh_n^d} \left(\widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_1) - \widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_2) \right) \\
&= \sqrt{nh_n^d} \left\{ \left[\widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_1) - \mathcal{D}_\phi^\psi(f, f_1) \right] - \left[\widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_2) - \mathcal{D}_\phi^\psi(f, f_2) \right] \right\} + \sqrt{nh_n^d} \left[\mathcal{D}_\phi^\psi(f, f_1) - \mathcal{D}_\phi^\psi(f, f_2) \right] \\
&= \sqrt{nh_n^d} \left\{ \left[\widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_1) - \mathcal{D}_\phi^\psi(f, f_1) \right] - \left[\widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_2) - \mathcal{D}_\phi^\psi(f, f_2) \right] \right\} + \sqrt{nh_n^d} \mathcal{D}I
\end{aligned}$$

127 ○ Under the null hypothesis H_0^{eq} , we have: $\mathcal{D}I = 0$

$$\sqrt{nh_n^d} \widehat{\mathcal{D}I}_n = \sqrt{nh_n^d} \left[\widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_1) - \mathcal{D}_\phi^\psi(f, f_1) \right] - \sqrt{nh_n^d} \left[\widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_2) - \mathcal{D}_\phi^\psi(f, f_2) \right] \quad (12)$$

A first order Taylor expansion of $\psi(y)$ around $y = y_0$ at $y = \widehat{y}$ gives

$$\psi(\widehat{y}) = \psi(y_0) + \psi'(y_0)(\widehat{y} - y_0) + o(\|\widehat{y} - y_0\|).$$

128 Now for $y_0 = \int_{\mathbb{R}^d} \phi\left(\frac{f(x)}{f_j(x)}\right) f_j(x) dx$ and $\widehat{y} = \int_{\mathbb{R}^d} \phi\left(\frac{\widehat{f}_{n,h_n}(x)}{f_j(x)}\right) f_j(x) dx$, with $j = 1, 2$ we get

$$\begin{aligned}
\psi\left(\int_{\mathbb{R}^d} \phi\left(\frac{\widehat{f}_{n,h_n}(x)}{f_j(x)}\right) f_j(x) dx\right) &= \psi\left(\int_{\mathbb{R}^d} \phi\left(\frac{f(x)}{f_j(x)}\right) f_j(x) dx\right) + \psi'\left(\int_{\mathbb{R}^d} \phi\left(\frac{f(x)}{f_j(x)}\right) f_j(x) dx\right) \left[\int_{\mathbb{R}^d} \phi\left(\frac{\widehat{f}_{n,h_n}(x)}{f_j(x)}\right) f_j(x) dx - \int_{\mathbb{R}^d} \phi\left(\frac{f(x)}{f_j(x)}\right) f_j(x) dx \right] \\
&\quad + o\left(\left\| \int_{\mathbb{R}^d} \phi\left(\frac{\widehat{f}_{n,h_n}(x)}{f_j(x)}\right) f_j(x) dx - \int_{\mathbb{R}^d} \phi\left(\frac{f(x)}{f_j(x)}\right) f_j(x) dx \right\|\right)
\end{aligned}$$

129

$$\begin{aligned}
\widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_j) - \mathcal{D}_\phi^\psi(f, f_j) &= \psi'\left(\int_{\mathbb{R}^d} \phi\left(\frac{f(x)}{f_j(x)}\right) f_j(x) dx\right) \left[\int_{\mathbb{R}^d} \phi\left(\frac{\widehat{f}_{n,h_n}(x)}{f_j(x)}\right) f_j(x) dx - \int_{\mathbb{R}^d} \phi\left(\frac{f(x)}{f_j(x)}\right) f_j(x) dx \right] \\
&\quad + o\left(\left\| \int_{\mathbb{R}^d} \phi\left(\frac{\widehat{f}_{n,h_n}(x)}{f_j(x)}\right) f_j(x) dx - \int_{\mathbb{R}^d} \phi\left(\frac{f(x)}{f_j(x)}\right) f_j(x) dx \right\|\right) \\
&= \psi'\left(\mathcal{D}_\phi(f, f_j)\right) \left[\widehat{\mathcal{D}}_\phi(\widehat{f}_{n,h_n}, f_j) - \mathcal{D}_\phi(f, f_j) \right] + o\left(\left\| \widehat{\mathcal{D}}_\phi(\widehat{f}_{n,h_n}, f_j) - \mathcal{D}_\phi(f, f_j) \right\|\right)
\end{aligned}$$

130 from Eq.10, replacing g by f_j , we have

$$\begin{aligned}
\widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_j) - \mathcal{D}_\phi^\psi(f, f_j) &= \psi'\left(\mathcal{D}_\phi(f, f_j)\right) \left\{ \int_{\mathbb{R}^d} (\widehat{f}_{n,h_n}(x) - f(x)) \phi'\left(\frac{f(x)}{f_j(x)}\right) dx + \int_{\mathbb{R}^d} o\left(\left\| \frac{\widehat{f}_{n,h_n}}{f_j} - \frac{f}{f_j} \right\|\right) f_j(x) dx \right\} \\
&\quad + o\left(\left\| \int_{\mathbb{R}^d} \phi\left(\frac{\widehat{f}_{n,h_n}(x)}{f_j(x)}\right) f_j(x) dx - \int_{\mathbb{R}^d} \phi\left(\frac{f(x)}{f_j(x)}\right) f_j(x) dx \right\|\right)
\end{aligned}$$

$$\begin{aligned}
\sqrt{nh^d} \left\{ \widehat{\mathcal{D}}_\phi^\psi(\widehat{f}_{n,h_n}, f_j) - \mathcal{D}_\phi^\psi(f, f_j) \right\} &= \psi'\left(\int_{\mathbb{R}^d} \phi\left(\frac{f(x)}{f_j(x)}\right) f_j(x) dx\right) \int_{\mathbb{R}^d} \left[\sqrt{nh^d} (\widehat{f}_{n,h_n}(x) - f(x)) \phi'\left(\frac{f(x)}{f_j(x)}\right) dx + o(1) \right] + o(1) \\
&\approx \int_{\mathbb{R}^d} \left[\psi'\left(\mathcal{D}_\phi(f, f_j)\right) \phi'\left(\frac{f(x)}{f_j(x)}\right) \right] \sqrt{nh^d} (\widehat{f}_{n,h_n}(x) - f(x)) dx \quad (13)
\end{aligned}$$

131 replacing Eq.13 in Eq.12

$$\begin{aligned}
\sqrt{nh_n^d} \widehat{\mathcal{D}I}_n &= \int_{\mathbb{R}^d} \left[\psi'\left(\mathcal{D}_\phi(f, f_1)\right) \phi'\left(\frac{f(x)}{f_1(x)}\right) \right] \sqrt{nh^d} (\widehat{f}_{n,h_n}(x) - f(x)) dx \\
&\quad - \int_{\mathbb{R}^d} \left[\psi'\left(\mathcal{D}_\phi(f, f_2)\right) \phi'\left(\frac{f(x)}{f_2(x)}\right) \right] \sqrt{nh^d} (\widehat{f}_{n,h_n}(x) - f(x)) dx \\
&= \int_{\mathbb{R}^d} \left[\psi'\left(\mathcal{D}_\phi(f, f_1)\right) \phi'\left(\frac{f(x)}{f_1(x)}\right) - \psi'\left(\mathcal{D}_\phi(f, f_2)\right) \phi'\left(\frac{f(x)}{f_2(x)}\right) \right] \times \sqrt{nh^d} (\widehat{f}_{n,h_n}(x) - f(x)) dx
\end{aligned}$$

By Eq 11, we have

$$\sqrt{nh_n^d} \widehat{\mathcal{DI}}_n \rightarrow \mathcal{N}(0, \Gamma^2)$$

where

$$\Gamma^2 = \left\{ \int_{\mathbb{R}^d} \left[\psi' \left(\mathcal{D}_\phi(f, f_1) \right) \phi' \left(\frac{f(x)}{f_1(x)} \right) - \psi' \left(\mathcal{D}_\phi(f, f_2) \right) \phi' \left(\frac{f(x)}{f_2(x)} \right) \right] \sigma(x) dx \right\}^2.$$

132 Note that in the case of the α -divergence the asymptotic variance Γ^2 is

$$\Gamma^2 := \Gamma^2(\alpha) = \left\{ \int_{\mathbb{R}^d} \left[\psi' \left(\int_{\mathbb{R}^d} \phi \left(\frac{f(x)}{f_1(x)} \right) f_1(x) dx \right) \phi' \left(\frac{f(x)}{f_1(x)} \right) - \psi' \left(\int_{\mathbb{R}^d} \phi \left(\frac{f(x)}{f_2(x)} \right) f_2(x) dx \right) \phi' \left(\frac{f(x)}{f_2(x)} \right) \right] \sigma(x) dx \right\}^2.$$

133 with $\psi(x) = x$ and $\phi(x) = \frac{1}{\alpha(\alpha-1)}(x^\alpha - \alpha(x-1) - 1)$

$$\Gamma^2(\alpha) = \left\{ \frac{1}{\alpha-1} \int_{\mathbb{R}^d} \left[\left(\frac{f(x)}{f_1(x)} \right)^{\alpha-1} - \left(\frac{f(x)}{f_2(x)} \right)^{\alpha-1} \right] \sqrt{f(x)} dx \int_{\mathbb{R}^d} K^2(z) dz \right\}^2$$

In the special case where $\alpha = 1/2$, this asymptotic variance does not depend to the unknown density f and it is expressed by :

$$\Gamma^2(1/2) = \left\{ 2 \int_{\mathbb{R}^d} \left(\sqrt{f_1(x)} - \sqrt{f_2(x)} \right) dx \int_{\mathbb{R}^d} K^2(z) dz \right\}^2.$$

134 But the case $\alpha \neq 1/2$, $\Gamma^2(\alpha)$ is unknown because it depends on f which is also unknown. In practice, one way solve
135 this problem is to substitute f with its consistency kernel estimator \widehat{f}_{nh_n} and to plug it in $\Gamma^2(\alpha)$.

136 5. Computational Results

137 5.1. Example

138 To illustrate the model procedure discussed in the preceding section. I rely on a simple specification such that:

$$\begin{cases} W_t = \varepsilon_t, & (M1), \\ W_t = -0.2W_{t-1} + \varepsilon_t - \varepsilon_{t-1} & (M2). \end{cases}$$

139 with $\varepsilon_t \sim \mathcal{N}(0, 1)$, It was in this case the densities under $M1$ and $M2$ respectively:

$$f_1(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad f_2(x) = \frac{1}{\sqrt{2\pi} \times 2.5} \exp\left(-\frac{x^2}{2 \times 2.5}\right)$$

We consider various sets of experiments in which data are generated from the mixture of a Normal $\mathcal{N}(0, 1)$ and Normal $\mathcal{N}(0, 2.5)$ distributions. Hence the DGP (Data Generating Process) is generated from $m(\pi)$ with the density

$$m(\pi) = \pi \mathcal{N}(0, 1) + (1 - \pi) \mathcal{N}(0, 2.5)$$

140 where $\pi(\pi \in [0, 1])$ is specific value to each set of experiments. In each set of experiments several random sample
141 are drawn from this mixture of distributions. The sample size varies from 100 to 2000, and for each sample size
142 the number of replication is 1000. we choose value of the parameter $\alpha = 0.5$, that corresponds to the Hellinger
143 distance (this choice provided to the known asymptotic variance). The aim is to compare the distance between true
144 density and the density $\mathcal{N}(0, 1)$, and the distance between the true density and the density $\mathcal{N}(0, 2.5)$.

145 We choose different values of π which are 0.00, 0.25, 0.43, 0.75, 1.00.. Although our proposed model selection pro-
146 cedure does not require that the data generating process belong to either of the competing models, we consider the
147 two limiting cases $\pi = 1.00$ and $\pi = 0.00$ for they correspond to the correctly specified cases. To investigate the
148 case where both competing models are misspecified but not at equal distance from the DGP, we consider the case
149 $\pi = 0.25$, $\pi = 0.75$ and $\pi = 0.43$. Second case is interpreted similarly as a $\mathcal{N}(0, 2.5)$ slightly contaminated by a
150 $\mathcal{N}(0, 1)$ distribution. The former case correspond to a DGP which is $\mathcal{N}(0, 1)$ but slightly contaminated by a $\mathcal{N}(0, 2.5)$

151 distribution. In the last case, $\pi = 0.43$ is the value for which the $\widehat{\mathcal{D}}_\alpha(\widehat{f}_n, f_1)$ and the $\widehat{\mathcal{D}}_\alpha(\widehat{f}_n, f_2)$ family are approxi-
 152 mately at equal distance to the mixture $m(\pi)$ according to the α -divergence with the above cells. Thus, this series of
 153 experiments approximates the null hypothesis of our proposed model selection test $\widehat{\mathcal{DI}}_\alpha$. The results of our different
 154 sets of experiments are presented in **Tables 1-5** .
 155

156 **Table 1.** $DGP = N(0, 1)$

n	20	100	300	500	1000	1500	2000
$\widehat{\mathcal{D}}_1$	-0.05	0.007	-0.002	0.016	-0.004	0.012	0.006
$\widehat{\mathcal{D}}_2$	0.16	0.12	0.14	0.16	0.14	0.14	0.14
$\widehat{\mathcal{DI}}_\alpha$	-0.21	-0.11	-0.15	-0.14	-0.146	-0.12	-0.14
Correct	8.4%	8%	26.4%	57.8%	95.6%	100%	100%
Indecisive	91.6%	92%	73.6%	42.2%	4.4%	0%	0%
Incorrect	0%	0%	0%	0%	0%	0%	0%

158 **Table 2.** $DGP = N(0, 2.5)$

n	20	100	300	500	1000	1500	2000
$\widehat{\mathcal{D}}_1$	0.26	0.14	0.22	0.28	0.23	0.24	0.24
$\widehat{\mathcal{D}}_2$	-0.039	-0.016	-0.008	-0.006	-0.004	-0.002	-0.001
$\widehat{\mathcal{DI}}_\alpha$	0.30	0.16	0.23	0.29	0.23	0.24	0.24
Correct	30.8%	68.4%	94.2%	99%	100%	100%	100%
Indecisive	69%	31.6%	5.6%	1%	0%	0%	0%
Incorrect	0.2%	0%	0.2%	0%	0%	0%	0%

160 **Table 3.** $DGP = .75 * N(0, 1) + .25 * N(0, 2.5)$

n	20	100	300	500	1000	1500	2000
$\widehat{\mathcal{D}}_1$	-0.014	0.015	-0.001	0.01	-0.002	0.01	0.01
$\widehat{\mathcal{D}}_2$	0.19	0.19	0.16	0.13	0.13	0.11	0.12
$\widehat{\mathcal{DI}}_\alpha$	-0.21	-0.17	-0.16	-0.12	-0.13	-0.1	-0.11
$N(0, 1)$	1.6%	5.4%	34.4%	67.4%	99%	100%	100%
Indecisive	98.4%	94.6%	64.4%	32.6%	1%	0%	0%
$N(0, 2.5)$	0%	0%	0%	0%	0%	0%	0%

162 **Table 4.** $DGP = .43 * N(0, 1) + .57 * N(0, 2.5)$

n	20	100	300	500	1000	1500	2000
$\widehat{\mathcal{D}}_1$	0.1	0.05	0.04	0.05	0.04	0.053	0.057
$\widehat{\mathcal{D}}_2$	0.08	0.02	0.06	0.04	0.05	0.056	0.058
$\widehat{\mathcal{DI}}_\alpha$	0.02	0.03	-0.02	0.01	-0.01	-0.002	-0.01
$N(0, 1)$	1.4%	0.2%	0.2%	0%	0%	0%	0%
Indecisive	98.4%	99.8%	99.8%	100%	100%	100%	100%
$N(0, 2.5)$	0.2%	0%	0%	0%	0%	0%	0%

164 **Table 5.** $DGP = .25 * N(0, 1) + .75 * N(0, 2.5)$

n	20	100	300	500	1000	1500	2000
$\widehat{\mathcal{D}}_1$	0.69	0.83	1.006	0.86	1.08	1.04	0.99
$\widehat{\mathcal{D}}_2$	-0.024	0.039	0.02	0.06	0.05	0.046	0.06
$\widehat{\mathcal{DI}}_\alpha$	0.67	0.79	1.04	0.8	1.03	0.99	0.92
$N(0, 1)$	0.6%	0%	0%	0%	0%	0%	0.1%
Indecisive	21%	17%	0.4%	0.2%	0.2%	0.2%	0.1%
$N(0, 2.5)$	78.4%	83%	99.6%	99.8%	99.8%	99.8%	99.9%

166 Thus this set of experiments corresponds approximately to the null hypothesis of our proposed model selection test
 167 $\widehat{\mathcal{DI}}$. The results of our different sets of experiments are presented in **Tables 1-5**. The first half of each table gives
 168 the distance between the true density f and f_1 sample take density model 1 \mathcal{D}_1 , the distance between f and f_2 Model
 169 2 \mathcal{D}_2 and the difference between the two distance. The second half of each table gives in percentage the number of
 170 times our proposed model selection procedure based on $\widehat{\mathcal{DI}}$ favors the model 1, the model 2, and indecisive. The tests

171 are conducted at 5% nominal significance level. In the first two sets of experiments ($\pi = 0.00$ and $\pi = 1.00$) where
 172 one model is correctly specified, we use the labels "correct, incorrect" and "indecisive" when a choice is made. The
 173 first halves of **Tables 1-5** confirm our asymptotic results.

174 In **Tables 4**, we observed a high percentage of bad decisions. This is because both models are now specified incor-
 175 rectly. In contrast, turning to the second halves of the **Tables 1** and **2**, we first note that the percentage of correct
 176 choices using \mathcal{DI} statistic steadily increases and ultimately converges to 100%

177 The preceding comments for the second halves of **Tables 1** and **2** also apply to the second halves of **Tables 3** and **5**.

178
 179 In *Figures 1,3, 5, 7* and *9* we plot the histograms of data sets and overlay the curves for $\mathcal{N}(0, 1)$ and $\mathcal{N}(0, 2.5)$
 180 distributions. When the *DGP* is correctly specified *Figure 1*, the $\mathcal{N}(0, 1)$ distribution has reasonable chance of being
 181 distinguished from $\mathcal{N}(0, 1)$ distribution.

182 Similarly, in *Figure 3*, as can be seen, the $\mathcal{N}(0, 2.5)$ distribution closely approximates the data sets. In *Figures 5* and
 183 *7* two distributions are close but the $\mathcal{N}(0, 1)$ (*Figure 5*) and the $\mathcal{N}(0, 2.5)$ distributions (*Figure 7*) does appear to be
 184 much closer to the data sets. When $\pi = 0.43$, the distribution for both (*Figure 9*) $\mathcal{N}(0, 1)$ distribution and $\mathcal{N}(0, 2.5)$
 185 distribution are similar.

186 As expected, our statistic divergence $\sqrt{nh_n^4} \widehat{\mathcal{DI}}_\alpha$ diverges to $-\infty$ (*Figures 2* and *6*) and to $+\infty$ (*Figures 4* and *8*)
 187 more rapidly symmetrical about the axis that passes through the mode of data distribution. This follows from the fact
 188 that these two distributions are equidistant from the *DGP* and would be difficult to distinguish from data in practice.

189 *Figure 10* allows a comparison with the asymptotic $\mathcal{N}(0, \Gamma)$ approximation under our null hypothesis of equivalence.

190 *Figure 11*, Hence the density indicator $\widehat{\mathcal{DI}}_\alpha$ is very closer to the $\mathcal{N}(0, \Gamma)$.

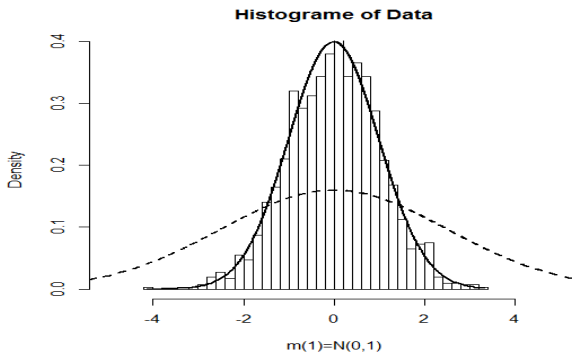


Figure 1: Histogram of ($DGP = \mathcal{N}(0, 1)$)

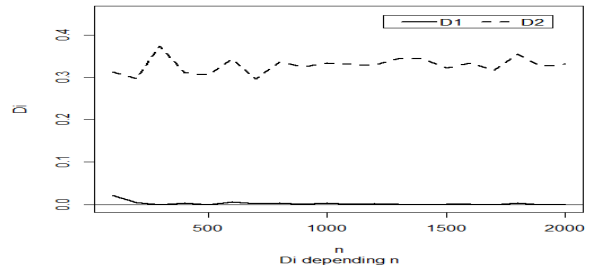


Figure 2: $\widehat{\mathcal{D}}_1$ and $\widehat{\mathcal{D}}_2$ depending n

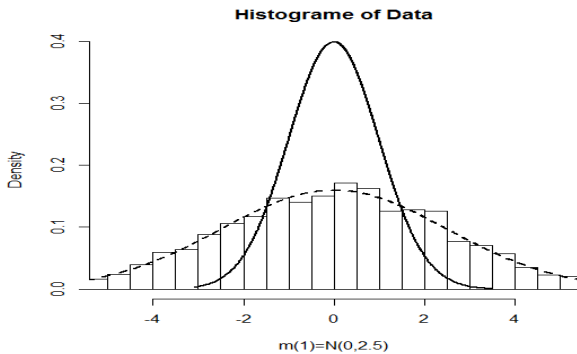


Figure 3: Histogram of ($DGP = \mathcal{N}(0, 2.5)$)

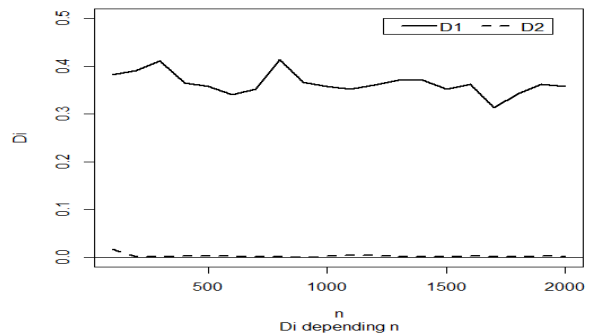


Figure 4: $\widehat{\mathcal{D}}_1$ and $\widehat{\mathcal{D}}_2$ depending n

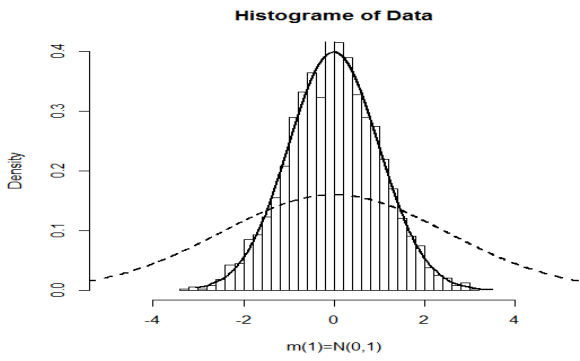


Figure 5: Histogram of $(DGP = .75 * N(0, 1) + .25 * N(0, 2.5))$

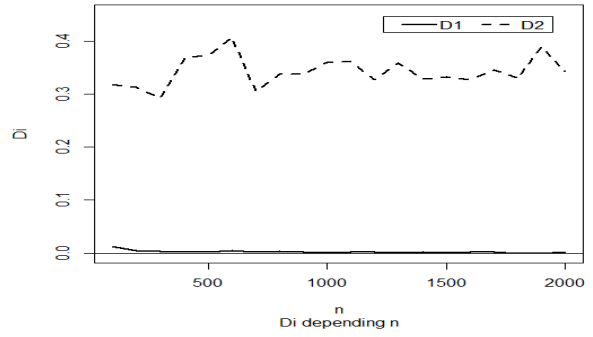


Figure 6: \widehat{D}_1 and \widehat{D}_2 depending n

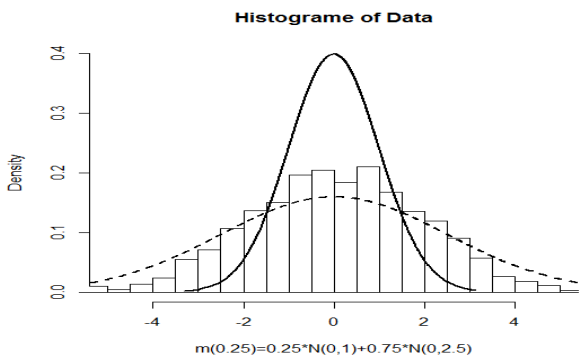


Figure 7: Coparaison barplot of D_i depending n ($DGP = .25 * N(0, 1) + .75 * N(0, 2.5)$)

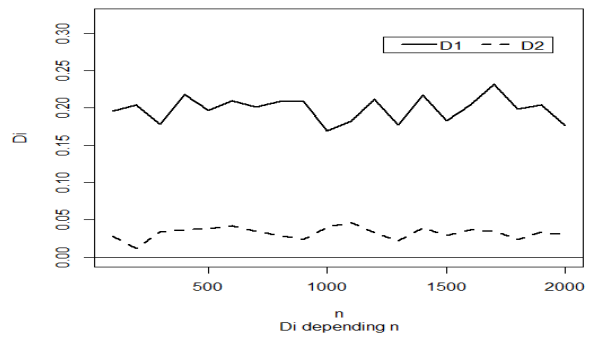


Figure 8: \widehat{D}_1 and \widehat{D}_2 depending n

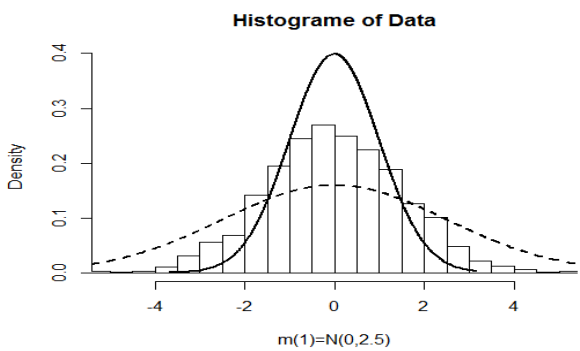


Figure 9: Coparaison barplot of D_i depending n ($DGP = .43 * N(0, 1) + .57 * N(0, 2.5)$)

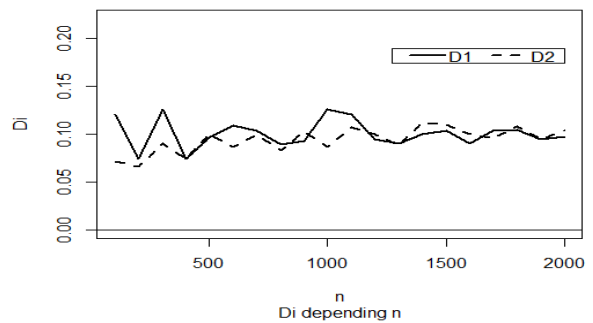


Figure 10: \widehat{D}_1 and \widehat{D}_2 depending n

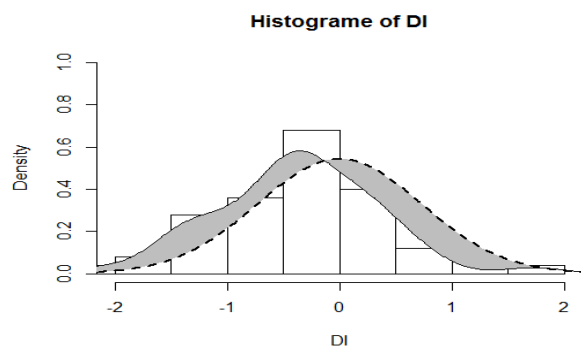


Figure 11: Coparaison of density depending n density data (continuous curve), density $\mathcal{N}(0, \Gamma)$ (dashed curve)

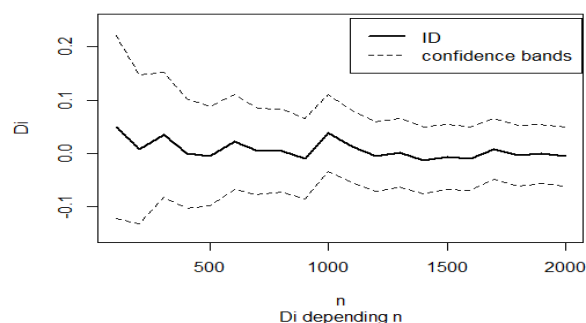


Figure 12: \widehat{DI} depending n aut its confidence bande at the levels 95%

6. Concluding remarks and future works

We have formulated the \mathcal{DI} method and applied it to the problem of choosing between a random walk and a stationary first order autoregressive model, using (ϕ, ψ) -divergence type statistics. In this context, we have considered some convenient asymptotically standard tests based on (ϕ, ψ) -divergence type statistics that use estimators in non parametric case. The results of the numerical experiments are most encouraging and show that \mathcal{DI} method performs very well and can be considered as a useful tool for addressing problems in model selection. these test allow to determine whether the competing model is as close to true distribution against the alternative hypothesis that one model is closer. Here closeness is evaluated according to the discrepancy implicit in the (ϕ, ψ) -divergence type statistic considered.

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