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# *Rationally smooth Schubert varieties, inversion hyperplane arrangements, and Peterson translation*

William Slofstra\*

*Department of Mathematics, UC Davis, Davis, CA, USA*

**Abstract.** We show that an element  $w$  of a finite Weyl group  $W$  is rationally smooth if and only if the hyperplane arrangement  $\mathcal{I}(w)$  associated to the inversion set of  $w$  is inductively free, and the product  $(d_1 + 1) \cdots (d_l + 1)$  of the coexponents  $d_1, \dots, d_l$  is equal to the size of the Bruhat interval  $[e, w]$ . We also use Peterson translation of coconvex sets to give a Shapiro-Steinberg-Kostant rule for the exponents of  $w$ .

**Résumé.** Nous montrons qu'un élément  $w$  d'un groupe de Weyl fini est rationnellement lisse si et seulement si l'arrangement des hyperplans associé à l'ensemble d'inversion de  $w$  est libre, et le produit  $(d_1 + 1) \cdots (d_l + 1)$  des coexposants  $d_1, \dots, d_l$  est égal à la cardinalité de l'intervalle  $[e, w]$  pour l'ordre de Bruhat. Nous donnons une règle de Shapiro-Steinberg-Kostant pour calculer les exposants de  $w$  en utilisant traduction de Peterson sur des sous-ensembles coconvexes.

**Keywords:** rational smoothness, Schubert varieties, inversion sets, inversion arrangements, hyperplane arrangements, Peterson translation

## 1 Introduction

Let  $R$  be a crystallographic root system in a Euclidean space  $V$ , and let  $R^+$  be the subset of positive roots. If we identify  $V$  with  $V^*$  using the inner product, then the vectors of  $R^+$  cut out a hyperplane arrangement in  $V$ . It is well-known that the characteristic polynomial  $\chi(t)$  of this arrangement is equal to a product  $\prod_{i=1}^l (t - m_i)$ , where  $l$  is the rank of  $R$ . The integers  $m_1, \dots, m_l$  that appear in this factorization are called the exponents of  $R$ , and arise in many other contexts. In particular, if  $W$  is the Weyl group of  $R$ , and  $\ell$  is the length function on  $W$ , then the Poincaré polynomial  $P(q) = \sum_{w \in W} q^{\ell(w)} = \prod_{i=1}^l [m_i + 1]_q$ , where  $[m]_q$  is the  $q$ -integer  $(1 + q + \dots + q^{m-1})$ . If  $X$  is the generalized flag variety of  $R$ , then  $P(q^2) = \sum_i q^i \dim H^i(X)$ , so the exponents can be used to calculate the Betti numbers of  $X$ . The exponents can also be calculated directly from  $R$  via the Shapiro-Steinberg-Kostant rule: the multiplicity of  $m$  as an exponent of  $R$  is the number of positive roots of height  $m$  minus the number of positive roots of height  $m + 1$ .

\*Email: [wslofstra@math.ucdavis.edu](mailto:wslofstra@math.ucdavis.edu)

In this paper (an extended abstract of [Slo13b] and [Slo13a]), we show that the picture above extends to any rationally smooth Schubert variety of the flag variety  $X$ . Furthermore, by combining the condition that the inversion arrangement be free with a condition introduced by Hultman, Linusson, Shareshian, and Sjöstrand, we can characterize the Schubert varieties which are rationally smooth.

The Schubert varieties  $X(w)$  of  $X$  are indexed by the elements  $w$  of the Weyl group  $W$ . Let  $\leq$  denote the Bruhat order on  $W$ , and let  $[e, w]$  denote the interval in Bruhat order between the identity  $e$  and the element  $w$ . The Poincaré polynomial of  $w$  is the polynomial

$$P_w(q) = \sum_{x \in [e, w]} q^{\ell(x)}.$$

Note that  $P_w(q)$  has degree  $\ell(w)$ . As with the flag variety,  $P_w(q^2) = \sum_i q^i \dim H^i(X(w))$ . A theorem of Carrell and Peterson states that  $X(w)$  is rationally smooth if and only if  $P_w(q)$  is palindromic, meaning that  $q^{\ell(w)} P_w(q^{-1}) = P_w(q)$  [Car94]. We say that  $w \in W$  is rationally smooth if this latter condition is satisfied. By combining deep results of Gasharov, Billey, Billey-Postnikov, and Akyildiz-Carrell, we get the following result:

**Theorem 1.1** ([Gas98], [Bil98], [BP05], [AC12]). *Let  $W$  be a finite Weyl group. An element  $w \in W$  is rationally smooth if and only if*

$$P_w(q) = \prod_{i=1}^l [m_i + 1]_q$$

for some collection of non-negative integers  $m_1, \dots, m_l$ .

Theorem 1.1 allows us to make the following definition:

**Definition 1.2.** *Let  $W$  be a finite Weyl group. If  $w \in W$  is rationally smooth, then the exponents of  $w$  are the integers  $m_1, \dots, m_l$  appearing in Theorem 1.1.*

Given an element  $w \in W$ , the inversion set of  $w$  is the set

$$I(w) = \{\alpha \in R^+ : w^{-1}\alpha \in R^-\},$$

where  $R^-$  is the set of negative roots of  $R$ . The inversion hyperplane arrangement  $\mathcal{I}(w)$  of  $w$  is the hyperplane arrangement in  $V$  cut out by the elements of  $I(w)$ . If  $w_0$  is the longest element of  $W$ , then  $X(w_0) = X$ ,  $I(w_0) = R^+$ , and  $\mathcal{I}(w_0)$  is the arrangement cut out by  $R^+$  mentioned above.

Given an arrangement  $\mathcal{A} = \mathcal{A}(T)$  in  $V$  cut out by a set of vectors  $T$ , let  $V_{\mathbb{C}} = V \otimes \mathbb{C}$ ,  $R = S^*V_{\mathbb{C}}$ , and  $Q$  be the polynomial  $\prod_{\alpha \in T} \alpha$  in  $R$  cutting out  $\mathcal{A}$ . Let  $\text{Der}(\mathcal{A})$  be the set of derivations of  $R$  which preserve the ideal generated by  $Q$ . The set  $\text{Der}(\mathcal{A})$  is an  $R$ -module, called the module of derivations of  $\mathcal{A}$ . The arrangement  $\mathcal{A}$  is said to be free if  $\text{Der}(\mathcal{A})$  is a free  $R$ -module. In this case,  $\text{Der}(\mathcal{A})$  has a homogeneous basis, and the polynomial degrees  $d_1, \dots, d_l$  of the elements of this basis are called the coexponents of  $\mathcal{A}$ . When  $\mathcal{A}$  is free, a theorem of Terao [Ter81] states that the characteristic polynomial  $\chi(\mathcal{A}; t)$  of  $\mathcal{A}$  factors as

$$\chi(\mathcal{A}; t) = \prod_i (t - d_i).$$

It is well-known that the arrangement  $\mathcal{I}(w_0)$  cut out by  $R^+$  is free, with coexponents corresponding to the exponents of  $R$ . We can now state the main theorem:

**Theorem 1.3.** *Let  $W$  be a finite Weyl group. An element  $w \in W$  is rationally smooth if and only if the inversion hyperplane arrangement  $\mathcal{I}(w)$  is free, and the product  $\prod_i (1 + d_i)$  of the coexponents  $d_1, \dots, d_l$  is equal to the size of the Bruhat interval  $[e, w]$ . Furthermore, if  $w$  is rationally smooth then the coexponents  $d_1, \dots, d_l$  are equal to the exponents of  $w$ .*

When  $\mathcal{I}(w)$  is free, the product  $\prod_i (1 + d_i) = (-1)^l \chi(\mathcal{I}(w); -1)$  is equal to the number of chambers of  $\mathcal{I}(w)$ . The condition that the number of chambers of  $\mathcal{I}(w)$  be equal to the size of the Bruhat interval  $[e, w]$  has previously been studied by Hultman, Linusson, Shareshian, and Sjöstrand (type A, [HLSS09]) and Hultman (all finite Coxeter groups, [Hul11]). Accordingly, we call this the HLSS condition (see Section 5).

As an immediate corollary of Theorem 1.3, we have:

**Corollary 1.4.** *Let  $W$  be a finite Weyl group. If  $w \in W$  is rationally smooth, then*

$$\chi(\mathcal{I}(w); t) = \prod_{i=1}^l (t - m_i),$$

where  $m_1, \dots, m_l$  are the exponents of  $w$ .

In type A, Corollary 1.4 has previously been proved by Oh, Postnikov, and Yoo [OPY08]. Their proof implicitly shows that  $\mathcal{I}(w)$  is free. The general case of Corollary 1.4 answers a conjecture of Yoo [Yoo11, Conjecture 1.7.3]. Oh, Postnikov, and Yoo also show, in type A, that  $w$  is rationally smooth if and only if the Poincaré polynomial  $P_w(q)$  is equal to the wall-crossing polynomial of  $\mathcal{I}(w)$ . This result has been extended to all finite-type Weyl groups by Oh and Yoo [OY10], using what we will call chain Billey-Postnikov (BP) decompositions (this is a modest variation on the terminology in [OY10]).

Inspired by [OY10], we list the rationally smooth elements in finite type which do not have a chain BP decomposition. We also show that an element  $w$  of an arbitrary finite Coxeter group has a chain BP decomposition if and only if  $\mathcal{I}(w)$  has a modular coatom of a certain form. From these two results, we show that  $\mathcal{I}(w)$  is inductively free when  $w$  is rationally smooth, with coexponents equal to the exponents of  $w$ . To prove that  $w$  is rationally smooth when  $\mathcal{I}(w)$  is free and the HLSS condition holds, we use the root-system pattern avoidance criterion for rational smoothness due to Billey and Postnikov [BP05].

The roots in  $R^+$  can be ordered by dominance order  $\succeq$ , so  $\alpha \succeq \beta$  if and only if  $\beta - \alpha$  is a sum of simple positive roots. Given a lower order ideal  $T \subset R^+$  with respect to dominance order, let  $h_i$  be the number of roots in  $T$  of height  $i$ . Since  $T$  is a lower order ideal, we always have  $h_i \geq h_{i+1}$ . Let  $\text{Exp}(T)$  be the multiset which contains  $i$  with multiplicity  $h_i - h_{i+1}$ . One example of a lower order ideal is the set  $-\Omega T_e X(w)$ , where  $\Omega T_e X(w) \subset R^-$  is the set of torus weights of the Zariski tangent space of  $X(w)$  at the identity. A theorem of Akyildiz-Carrell states that when  $X(w)$  is smooth (a stronger condition than being rationally smooth), the set  $\text{Exp}(-\Omega T_e X(w))$  is precisely the set of exponents of  $w$  [AC12]. This gives an analogue of the Shapiro-Steinberg-Kostant rule for the exponents of a smooth Schubert variety  $w$ . A theorem of Summers-Tymoczko [ST06] and Abe-Barakat-Cuntz-Hoge-Terao [ABC<sup>+</sup>13] states that the arrangement  $\mathcal{A}(T)$  cut out by the elements of a lower order ideal  $T \subset R^+$  is always free, with coexponents equal to  $\text{Exp}(T)$ . Thus when  $T = -\Omega T_e X(w)$  and  $w$  is smooth, we get that  $\mathcal{I}(w)$  and  $\mathcal{A}(T)$  are free with the same coexponents. In Section 6, we show that any inversion set  $I(w)$  can be transformed into a lower order ideal  $T$  using a combinatorial version of Peterson translation. In the simply-laced types and type B, if  $\mathcal{I}(w)$  is free then the arrangement  $\mathcal{A}(T)$  is free, and  $\mathcal{I}(w)$  and  $\mathcal{A}(T)$  have the same coexponents. Thus for these types we get an analogue of the Shapiro-Steinberg-Kostant rule for calculating the exponents of a rationally smooth element  $w \in W$ .

## 2 Chain Billey-Postnikov decompositions

Let  $S$  be the set of simple generators of  $W$ . Given a subset  $J \subset S$ , we let  $W_J$  denote the parabolic subgroup generated by  $J$ ,  $W^J$  denote the set of minimal length left coset representatives, and  ${}^JW$  denote the set of minimal length right coset representatives. Every element  $w \in W$  can be written uniquely as  $w = vu$ , where  $v \in W^J$  and  $u \in W_J$ . This factorization is called the right parabolic decomposition of  $w$ . Left parabolic decompositions are defined similarly. If  $v \in W^J$ , then the Poincaré polynomial of  $v$  relative to  $J$  is the polynomial

$$P_v^J(q) = \sum_{x \in [e, v] \cap W^J} q^{\ell(v)}.$$

If  $w = vu$  is the parabolic decomposition of  $w$  with respect to  $J \subset S$ , then multiplication gives an injective map

$$([e, v] \cap W^J) \times [e, u] \rightarrow [e, w]. \quad (1)$$

If  $x = v_1u_1$  is the parabolic decomposition of an element  $x \in [e, w]$ , then  $v_1 \leq v$ . However, it is not necessarily true that  $u_1 \leq u$ , even though  $u_1 \leq w$  and  $u_1 \in W_J$ .

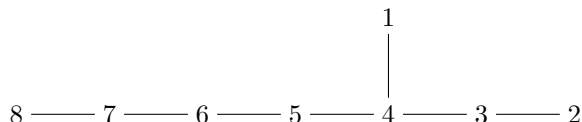
Let  $S(w) \subset S$  denote the support set of an element  $w \in W$  (i.e. the set of simple generators which appear in some reduced decomposition of  $w$ ), and let  $D_L(w)$  denote the left descent set.

**Lemma 2.1** ([BP05], [OY10], [RS13]). *Let  $w = vu$  be the parabolic decomposition of  $w$  with respect to  $J$ . Then the following are equivalent:*

- (a) *The map in equation (1) is surjective (hence bijective).*
- (b)  *$u$  is the maximal element of  $[e, w] \cap W_J$ .*
- (c)  *$S(v) \cap J \subseteq D_L(u)$ .*
- (d)  *$P_w(q) = P_v^J(q)P_u(q)$ .*

If any of the equivalent conditions of Lemma 2.1 are satisfied, then we say that  $w = vu$  is a (right) Billey-Postnikov (BP) decomposition with respect to  $J$ . If in addition  $[e, v] \cap W^J$  is a chain, or equivalently  $P_v^J(q) = [\ell(v) + 1]_q$ , then we say that  $w = vu$  is a chain BP decomposition. Left BP decompositions and left chain BP decompositions are defined similarly. Note that a left parabolic (resp. BP) decomposition of  $w$  is the same as a right parabolic (resp. BP) decomposition of  $w^{-1}$ .

Deep results of Gasharov [Gas98], Billey [Bil98], and Billey-Postnikov [BP05] imply that every rationally smooth element  $w \in W$  has either a left or right BP decomposition. In addition, most rationally smooth elements have a chain BP decomposition. In this section, we list the few exceptions. To do so, we use the following labelling for the Dynkin diagram of  $E_8$ :



We write elements of  $E_8$  as products of the simple generators  $s_i$ ,  $i = 1 \dots, 8$ , where  $s_i$  corresponds to the node in the Dynkin diagram labelled by  $i$ . For this section, we let  $S_k = \{s_1, \dots, s_k\}$ , and take the convention that  $E_6$  and  $E_7$  are embedded in  $E_8$  as  $W_{S_6}$  and  $W_{S_7}$  respectively. Finally, we let  $J_k = S_k \setminus \{s_2\}$ ,  $\tilde{u}_k$  be the maximal element of  $W_{J_k}$ , and  $\tilde{v}_k$  be the maximal element of  $W_{S_k}^{J_k}$ .

**Theorem 2.2.** *Suppose that  $w \in W$  is a rationally smooth element of a finite Weyl group  $W$ , that  $\ell(w) \geq 2$ , and that  $w$  has no (left or right) chain BP decomposition. Then  $w$  is one of the following elements:*

- *The maximal element of  $D_n$ ,  $n \geq 4$ .*
- *The maximal element of  $E_n$ ,  $n = 6, 7, 8$ .*
- *The element  $w_{kl} = \tilde{v}_l \tilde{u}_k$  in  $E_8$ , or its inverse  $w_{kl}^{-1}$ , where  $5 \leq l < k \leq 8$ .*
- *The maximal element of  $F_4$ .*

For the proof of Theorem 2.2 in the classical types, we refer to [Gas98] and [Bil98], as well as the summary of this work in [OY10]. For the exceptional types it is possible to check Theorem 2.2 by computer. We give a human-readable proof based on the existence theorems of [BP05] (which also use computer verification) in [Slo13b].

Theorem 2.2 allows us to give a direct proof of Theorem 1.1. If  $w$  is rationally smooth and has a chain BP decomposition  $w = vu$  or  $w = uv$ , then  $P_w(q) = [\ell(v) + 1]_q P_u(q)$ . The element  $u$  is also rationally smooth, and we can proceed by induction. As mentioned in the introduction, Theorem 1.1 for maximal elements is well-known. Furthermore,  $P_w(q) = P_{w^{-1}}(q)$ , so we only need to check Theorem 1.1 for the elements  $w_{kl}$ . If  $\tilde{w}_l$  is the maximal element of  $W_{S_l}$ , then  $P_{w_{kl}}(q) = P_{\tilde{v}_l}^{J_l}(q) P_{\tilde{u}_k}(q) = P_{\tilde{w}_l}(q) P_{\tilde{u}_k}(q) P_{\tilde{w}_l}^{-1}(q)$ , where the last equality uses the fact that  $\tilde{w}_l = \tilde{v}_l \tilde{u}_l$ . The Poincaré polynomials  $P_{\tilde{w}_l}(q)$  and  $P_{\tilde{u}_k}(q)$  are well-known, so it is easy to check that  $P_{w_{kl}}(q)$  is a product of  $q$ -integers as desired. For example, the exponents of  $w_{87}$  are 1, 6, 7, 9, 11, 11, 13, 17.

### 3 The HLSS condition and nbc-sets

In this section we give some background on the Hultman-Linusson-Shareshian-Sjöstrand (HLSS) condition which is necessary for the proof of Theorem 1.3. If  $s \in S$  is a simple generator, let  $\alpha_s$  denote the corresponding simple root. Conversely, let  $t_\alpha \in W$  denote the reflection corresponding to a root  $\alpha \in R$ . Given an order  $<$  on a set  $T$  of vectors in  $V$ , a broken circuit is defined to be an ordered subset  $\{v_1 < \dots < v_k\}$  of  $T$  such that there is  $v_{k+1} > v_k$  for which  $\{v_1, \dots, v_{k+1}\}$  is a minimal linearly dependent set in  $T$ . An nbc-set is an ordered subset of  $T$  which does not contain a broken circuit. The number of nbc-sets is equal to the number of chambers in the arrangement  $\mathcal{A}(T)$ . In particular, the number of nbc-sets does not depend on the chosen order.

If  $s_1 \cdots s_k$  is a reduced expression for an element  $w \in W$ , we can order the inversion set  $I(w)$  by  $\beta_1 < \dots < \beta_{\ell(w)}$ , where  $\beta_i = s_1 \cdots s_{i-1} \alpha_{s_i}$ . A total order on  $I(w)$  constructed in this way is called a convex order. Let  $2^{I(w)}$  denote the power set of  $I(w)$ . Given a convex order, we can define a surjective map

$$\phi : 2^{I(w)} \rightarrow [e, w] : \{\beta_1, \dots, \beta_k\} \mapsto t_{\beta_1} \cdots t_{\beta_k} w, \text{ where } \beta_1 < \dots < \beta_k.$$

**Theorem 3.1** (Hultman-Linusson-Shareshian-Sjöstrand [HLSS09]). *Choose a convex order for  $I(w)$ , and let  $\text{nbc}(I(w))$  denote the set of nbc-sets of  $I(w)$  with respect to the chosen order. Then the restriction of  $\phi$  to  $\text{nbc}(I(w))$  is injective.*

In particular, the number of nbc-sets of  $I(w)$  is less than the size of the Bruhat interval. The restriction of  $\phi$  to  $\text{nbc}(I(w))$  will be surjective if and only if the number of nbc-sets is equal to the size of the Bruhat interval. As mentioned in the introduction, when the restriction of  $\phi$  is surjective we say that  $w$

satisfies the HLSS condition. A theorem of Hultman-Linusson-Shareshian-Sjöstrand (type  $A$  [HLSS09]) and Hultman (all finite Coxeter groups [Hul11]) characterizes when this condition holds in terms of the directed Bruhat graph of  $[e, w]$ . In particular, the HLSS condition is weaker than being rationally smooth:

**Theorem 3.2** ([Hul11]). *If  $w$  is rationally smooth, then  $w$  satisfies the HLSS condition.*

## 4 Chain BP decompositions and modular flats

Given an arrangement  $\mathcal{A}$ , let  $L(\mathcal{A})$  denote the intersection lattice of  $\mathcal{A}$ . By convention, the maximal element of  $\mathcal{A}$  is the center  $\bigcap_{H \in \mathcal{A}} H$  of  $\mathcal{A}$ . A coatom is a flat of  $L(\mathcal{A})$  of rank one more than the center. If  $X \in L(\mathcal{A})$ , the localization of  $\mathcal{A}$  at  $X$  is the arrangement  $\mathcal{A}_X$  containing all hyperplanes  $H$  of  $\mathcal{A}$  such that  $X \subset H$ . We let  $\mathcal{A}^X$  denote the restriction of  $\mathcal{A}$  to  $X$ , and  $\mathcal{A} \setminus H$  the deletion by  $H$ , which is the arrangement containing all hyperplanes of  $\mathcal{A}$  except  $H$ . An arrangement is said to be inductively free if either (a)  $\mathcal{A}$  contains no hyperplanes (in which case all coexponents are zero) or (b) there is some hyperplane  $H \subset \mathcal{A}$  such that  $\mathcal{A} \setminus H$  is inductively free with coexponents  $d_1, \dots, d_l - 1$ , and  $\mathcal{A}^H$  is inductively free with coexponents  $d_1, \dots, d_{l-1}$ . Any inductively free arrangement is free. The following lemma gives a useful sufficient criterion for  $\mathcal{A}$  to be inductively free.

**Lemma 4.1.** *If  $X$  is a modular coatom of  $L(\mathcal{A})$ , and  $\mathcal{A}_X$  is inductively free with coexponents  $0, m_1, \dots, m_{l-1}$ , then  $\mathcal{A}$  is inductively free with coexponents  $m_1, \dots, m_{l-1}, m_l = |\mathcal{A}| - |\mathcal{A}_X|$ .*

Here  $|\mathcal{A}|$  denotes the number of hyperplanes in  $\mathcal{A}$ . We use the following characterization of modular coatoms:

**Lemma 4.2** ([CDF<sup>+</sup>09], Lemma 3.20). *Let  $\mathcal{A}$  be an arrangement, and for each hyperplane  $H \in \mathcal{A}$  let  $\alpha_H$  be a normal vector to  $H$ . Let  $X \subset L(\mathcal{A})$  be a coatom. Then  $X$  is modular if and only if for every distinct pair  $H_1, H_2 \notin \mathcal{A}_X$ , there is  $H_3 \in \mathcal{A}_X$  such that  $\alpha_{H_1}, \alpha_{H_2}, \alpha_{H_3}$  are linearly dependent.*

Recall that  $V$  is the ambient Euclidean space containing  $R$ . Given  $J \subset S$ , let  $V_J \subset V$  be the subspace spanned by  $\{\alpha_s : s \in J\}$ , and let  $R_J = R \cap V_J$  be the root system for  $W_J$ . The following lemma is easy to prove:

**Lemma 4.3.** *The linear span of the inversion set  $I(w)$  in  $V$  is  $V_{S(w)}$ , where  $S(w)$  is the support set of  $w$ . The center of the inversion hyperplane arrangement  $\mathcal{I}(w)$  is the orthogonal complement of  $V_{S(w)}$ .*

Lemma 4.3 implies that the rank of  $\mathcal{I}(w)$  is the size of the support set  $S(w)$ , so a coatom of  $\mathcal{I}(w)$  is an element of  $L(\mathcal{I}(w))$  of rank  $|S(w)| - 1$ . If  $w = uv$  is a left parabolic decomposition, then the inversion set  $I(w)$  is the disjoint union of  $I(u)$  and  $uI(v)$ , and

$$X = \bigcap_{\alpha \in I(u)} \ker \alpha \tag{2}$$

is a flat of  $L(\mathcal{I}(w))$ . This flat has rank  $|S(u)|$ , and hence  $X$  will be a coatom if and only if  $|S(u)| = |S(w)| - 1$ .

**Theorem 4.4.** *Suppose that  $w = uv$  is a left parabolic decomposition with respect to  $J$ , so  $u \in W_J$  and  $v \in {}^J W$ . Let  $X \in L(\mathcal{I}(w))$  be defined as in Equation (2). Then  $w = uv$  is a chain BP decomposition if and only if  $X$  is a modular coatom of  $L(\mathcal{I}(w))$ .*

Theorem 4.4 holds for any element  $w \in W$ , where  $W$  is an arbitrary finite Coxeter group. However, the proof is lengthy, and will be left for [Slo13b]. We give a shorter proof of Theorem 4.4 when  $w$  and  $u$  both satisfy the HLSS condition, as this assumption is sufficient to prove Theorem 1.3.

*Proof of Theorem 4.4.* If  $w = uv$  is a chain BP decomposition, then  $|S(w) \cap J| = |S(w)| - 1$ , and  $S(v) \cap J \subset S(u)$ , so  $|S(u)| = |S(w)| - 1$ . Hence we can assume throughout that  $X$  is a coatom. The set  $I(u) = I(w) \cap V_J$ , and  $w \in {}^JW$  if and only if  $I(w) \cap V_J$  is empty. The hyperplanes of  $\mathcal{I}(w)$  which do not contain  $X$  correspond precisely to the roots of  $I(w)$  in  $uI(v)$ .

Suppose that  $w = uv$  is a chain BP decomposition, and that  $u$  satisfies the HLSS condition. Order  $I(w)$  so that all the elements of  $I(u)$  come after the elements of  $uI(v)$ . Every element  $\alpha \in uI(v)$  is independent from the span of  $I(u)$ . Thus if  $\{\gamma_1, \dots, \gamma_k\}$  is an nbc-set for  $I(u)$ , then  $\{\alpha, \gamma_1, \dots, \gamma_k\}$  is an nbc-set for  $I(w)$ . Consequently

$$|\text{nbc}(I(w))| \geq (1 + \ell(v)) \cdot |\text{nbc}(I(u))|.$$

Since  $u$  satisfies the HLSS condition,  $|\text{nbc}(I(u))| = |[e, u]|$ , while  $|\text{nbc}(I(w))| \leq |[e, w]|$ . But  $w = uv$  is a chain BP decomposition, so

$$|[e, w]| = P_w(1) = (1 + \ell(v))|[e, u]| = (1 + \ell(v)) \cdot |\text{nbc}(I(u))|.$$

We conclude that all nbc-sets of  $I(w)$  are either nbc-sets of  $I(u)$ , or of the form  $\{\alpha, \gamma_1, \dots, \gamma_k\}$  for  $\alpha \in uI(v)$  and  $\{\gamma_1, \dots, \gamma_k\}$  an nbc-set of  $I(u)$ .

In particular, if  $\alpha, \beta \in uI(v)$ ,  $\alpha < \beta$ , then  $\{\alpha, \beta\}$  is not an nbc-set, and hence there must be some  $\gamma \in I(w)$ ,  $\gamma > \beta$ , such that  $\alpha, \beta, \gamma$  is linearly dependent. If  $\gamma \notin I(u)$  we can repeat this process by replacing  $\alpha, \beta$  with  $\beta, \gamma$  until we find  $\gamma' \in I(u)$  such that  $\beta, \gamma, \gamma'$  is linearly dependent. This implies that  $\gamma'$  is in the span of  $\beta$  and  $\gamma$ , and since  $\gamma$  is in the span of  $\alpha$  and  $\beta$ , we get that  $\alpha, \beta, \gamma'$  is linearly dependent. By Lemma 4.2,  $X$  is modular.

Now suppose that  $X$  is modular. If we assume that  $u$  and  $w$  satisfy the HLSS condition, then

$$|[e, w]| = |\text{nbc}(I(w))| = |\text{nbc}(I(u))| \cdot (|\mathcal{A}| - |\mathcal{A}_X| + 1) = |[e, u]| \cdot (\ell(v) + 1).$$

On the other hand,  $[e, w] \geq |[e, u]| \cdot |[e, v] \cap {}^JW|$ , and  $[e, v] \cap {}^JW \geq \ell(v) + 1$ . So we must have  $[e, v] \cap {}^JW = \ell(v) + 1$ . But  $[e, v] \cap {}^JW$  contains a chain of size  $\ell(v) + 1$ , so  $[e, v] \cap {}^JW$  is a chain. Furthermore, the multiplication map  $[e, u] \times ([e, v] \times {}^JW) \rightarrow [e, w]$  will be surjective, so  $w = uv$  is a BP decomposition. □

We now prove one direction of Theorem 1.3:

**Corollary 4.5.** *If  $w \in W$  is rationally smooth then  $\mathcal{I}(w)$  is inductively free, and the coexponents of  $\mathcal{I}(w)$  are equal to the exponents of  $w$ .*

*Proof.* The proof is by induction on  $|S(w)|$ . Clearly the corollary is true if  $|S(w)| \leq 1$ . Suppose  $w$  has a chain BP decomposition. The element  $w$  is rationally smooth if and only if  $w^{-1}$  is rationally smooth, and since  $I(w^{-1}) = -w^{-1}I(w)$ , the arrangements  $\mathcal{I}(w)$  and  $\mathcal{I}(w^{-1})$  are linearly equivalent. Thus we can assume without loss of generality that  $w$  has a left chain BP decomposition  $w = uv$ . Then  $u$  is also rationally smooth, and  $P_w(q) = [\ell(v) + 1]_q P_u(q)$ , so if the exponents of  $u$  are  $0, m_1, \dots, m_{l-1}$ , then the exponents of  $w$  are  $m_1, \dots, m_{l-1}, m_l = \ell(v)$ . The coatom  $X$  corresponding to  $\mathcal{I}(u)$  is modular by



Theorem 4.4, and the arrangement  $\mathcal{I}(w)_X$  is simply  $\mathcal{I}(u)$ , which by induction is inductively free with coexponents  $0, m_1, \dots, m_{l-1}$ . Finally,  $|\mathcal{I}(w)| - |\mathcal{I}(w)_X| = \ell(w) - \ell(u) = \ell(v)$ . By Lemma 4.1, the arrangement  $\mathcal{I}(w)$  is inductively free with coexponents equal to  $m_1, \dots, m_l$ .

This leaves the possibility that  $w$  is one of the elements listed in Theorem 2.2. If  $w$  is the maximal element of  $D_n$ ,  $E_n$ , or  $F_4$ , then Barakat and Cuntz have shown that  $\mathcal{I}(w)$  is inductively free [BC12]. Again, without loss of generality we only need to check that the corollary holds for the elements  $w_{kl}$ , and this is done on a computer (we defer to [Slo13b] for details of the computation).  $\square$

## 5 The flattening map

Let  $U$  be a subspace of  $V$ . The intersection  $R_U = R \cap U$  is also a root system, with positive and negative roots  $R_U^+ = R^+ \cap U$  and  $R_U^- = R^- \cap U$  respectively. Let  $W_U$  be the Weyl group of  $R_U$ . Note that  $W_U$  is the parabolic subgroup of  $W$  generated by the reflections  $t_\beta$  for  $\beta \in R_U^+$ , or equivalently is the subgroup of  $W$  which acts identically on the orthogonal complement of  $U$  in  $V$ .

A subset  $I \subset R^+$  is convex if  $\alpha, \beta \in I$ ,  $\alpha + \beta \in R^+$  implies that  $\alpha + \beta \in I$ . The subset  $I$  is coconvex if  $R^+ \setminus I$  is convex, and  $I$  is biconvex if it is both convex and coconvex. A subset  $I$  is biconvex if and only if it is the inversion set  $I(w)$  for some  $w \in W$ . Since biconvexity is a linear condition, the intersection  $I(w) \cap U$  is biconvex, and hence there is an element  $w' \in W_U$  such that  $I(w') = I(w) \cap U$ . The element  $w'$  is called the flattening of  $w$ , and is denoted by  $\text{fl}_U(w)$  [BP05]. If  $U = V_J$ , then  $\text{fl}_U(w) = u$ , where  $w = uv$  is the left parabolic decomposition of  $W$  with respect to  $J$ . We use the following lemma:

**Lemma 5.1** ([BB03]). *If  $u \in W_U$ ,  $w \in W$ , then  $\text{fl}_U(uw) = u \text{fl}_U(w)$ .*

Recall from the definition of the HLSS condition that a convex order on an inversion set  $I(w)$  is an order coming from a reduced expression for  $w$ . An arbitrary total order  $<$  on  $I(w)$  is convex if and only if it satisfies two conditions [Pap94]:

- if  $\alpha < \beta$  and  $\alpha + \beta \in R^+$ , then  $\alpha < \alpha + \beta < \beta$ , and
- if  $\alpha \in I(w)$ ,  $\beta \notin I(w)$ , and  $\alpha - \beta \in R^+$ , then  $\alpha - \beta < \alpha$ .

Because these conditions are linear, we immediately get the following lemma:

**Lemma 5.2.** *If  $<$  is a convex order on  $I(w)$ , then the induced order on  $I(\text{fl}_U(w)) = I(w) \cap U$  is also convex.*

**Proposition 5.3.** *Let  $U \subset V$  be any subspace. If  $w$  satisfies the HLSS condition, then so does  $\text{fl}_U(w)$ .*

*Proof.* The absolute length  $\ell'(w)$  of an element  $w \in W$  is the smallest integer  $k$  such that  $w$  can be written as a product of  $k$  reflections. If  $w = t_{\beta_1} \cdots t_{\beta_m}$ , then clearly the fixed point space of  $w$  contains the orthogonal complement of  $\text{span}\{\beta_1, \dots, \beta_m\}$ . A theorem of Carter states that the fixed point space of  $w$  is equal to the orthogonal complement of  $\text{span}\{\beta_1, \dots, \beta_m\}$  if and only if  $\ell'(w) = m$ , and furthermore  $\ell'(w) = m$  if and only if  $\beta_1, \dots, \beta_m$  are linearly independent [Car72].

Choose a convex order  $<$  on  $I(w)$ , and take the induced convex order on  $I(\text{fl}_U(w))$ . If  $x \in [e, \text{fl}_U(w)]$ , we can always find  $u = t_{\beta_1} \cdots t_{\beta_m}$ , where  $\beta_1 < \dots < \beta_m$  in  $I(\text{fl}_U(w))$  such that  $x = u \text{fl}_U(w)$ . To show that  $\text{fl}_U(w)$  satisfies the HLSS condition, we want to show that we can take  $\{\beta_1 < \dots < \beta_m\}$  to be an nbc-set with respect to the given convex order. Now  $x' = uw$  is less than  $w$  in Bruhat order, and since  $w$  satisfies the HLSS condition, we can find an nbc-set  $\{\gamma_1 < \dots < \gamma_m\}$  such that  $x' = yw$ , where

$y = t_{\gamma_1} \cdots t_{\gamma_k}$ . Let  $V_0$  denote the fixed point space of  $y$ . Then  $\{\gamma_1, \dots, \gamma_m\}$  is linearly independent, so  $V_0$  is the orthogonal complement of  $\text{span}\{\gamma_1, \dots, \gamma_m\}$ . Since  $yw = uw$ , we have  $y = u$ , so the orthogonal complement of  $\text{span}\{\beta_1, \dots, \beta_k\}$  is contained in  $V_0$ . It follows that  $\text{span}\{\gamma_1, \dots, \gamma_m\} \subset \text{span}\{\beta_1, \dots, \beta_k\}$ , and hence  $\gamma_1, \dots, \gamma_m \in R_U$ . We conclude that  $y \in W_U$ .

Now  $\{\gamma_1 < \dots < \gamma_k\}$  is an nbc-set in  $I(\text{fl}_U(w))$ , and  $x = u \text{fl}_U(w) = \text{fl}_U(uw) = \text{fl}_U(yw) = y \text{fl}_U(w)$  by Lemma 5.1. We conclude that the map from  $2^{I(\text{fl}_U(w))} \rightarrow [e, \text{fl}_U(w)]$  restricts to a surjective map on  $\text{nbc}(I(\text{fl}_U(w)))$ , and hence  $\text{fl}_U(w)$  satisfies the HLSS condition.  $\square$

Given an arrangement  $\mathcal{A}$  in  $V$  and a subspace  $U_0$  of the center of  $\mathcal{A}$ , we let  $\mathcal{A}/U_0$  denote the quotient arrangement in  $V/U_0$ . It is easy to see that  $\mathcal{A}$  is free if and only if  $\mathcal{A}/U_0$  is free.

**Proposition 5.4.** *Let  $U \subset V$  be any subspace. If  $\mathcal{I}(w)$  is free, then so is  $\mathcal{I}(\text{fl}_U(w))$ .*

*Proof.* Let  $U^\perp$  be the orthogonal complement to  $U$ , and let

$$X = \bigcap_{\alpha \in I(w) \cap U} \ker \alpha \in L(\mathcal{I}(w)).$$

Then  $U^\perp \subset X$ , and  $\mathcal{I}(\text{fl}_U(w))$  is isomorphic to the localization  $\mathcal{I}(w)_X/U^\perp$ . It is well-known that localization preserves freeness (see [OT92, Theorem 4.37]), so if  $\mathcal{I}(w)$  is free then  $\mathcal{I}(w)_X$  is free, and consequently  $\mathcal{I}(w)_X/U^\perp$  is free.  $\square$

Let  $R'$  be another root system with Weyl group  $W(R')$ , and let  $w' \in W(R')$ . An element  $w \in W$  is said to contain the pattern  $(w', R')$  if there is a subspace  $U \subset V$  such that  $R_U$  is isomorphic to  $R'$ , and  $\text{fl}_U(w) = w'$  when  $R_U$  is identified with  $R'$ . If this does not happen for any subspace  $U$ , then  $w$  is said to avoid  $(w', R')$ . This notion of root system pattern avoidance due to Billey and Postnikov generalizes the usual notion of pattern avoidance for permutations [BP05]. We can now prove the main theorem:

*Proof of Theorem 1.3.* One direction of the theorem has already been proved in Corollary 4.5. Suppose that  $\mathcal{I}(w)$  is free, and the product of the coexponents  $\prod_i (1 + d_i)$  is equal to the size of the Bruhat interval. This latter condition implies that  $w$  satisfies the HLSS condition. We want to show that  $w$  is rationally smooth. The main result of [BP05] states that  $w$  is rationally smooth if and only if  $w$  avoids a finite list of bad patterns in the root systems  $R' = A_3, B_3, C_3$ , and  $D_4$ . If  $(w', R')$  is a pattern in this list (there are 17 bad patterns, since the patterns for  $B_3$  and  $C_3$  are equivalent), then either  $\mathcal{I}(w')$  is not free, or  $w'$  does not satisfy the HLSS condition. By Propositions 5.3 and 5.4,  $\mathcal{I}(\text{fl}_U(w))$  is free and  $\text{fl}_U(w)$  satisfies the HLSS condition for any subspace  $U \subset V$ . We conclude that  $w$  must avoid all the bad patterns, and hence  $w$  is rationally smooth.  $\square$

## 6 Peterson translation

In this section we assume that  $R$  has no components of type  $C$  or  $F_4$ . Given  $\alpha \in R^+$ , an  $\alpha$ -string is a subset of  $R^+$  of the form  $\{\beta, \beta + \alpha, \dots, \beta + k\alpha\}$ , where  $\beta - \alpha \notin R^+$  and  $\beta + (k+1)\alpha \notin R^+$ . The set of  $\alpha$ -strings partitions  $R^+$ . The Peterson translate of a subset  $T \subset R^+$  compresses each  $\alpha$ -string:

**Definition 6.1.** *Given  $T \subset R^+$ ,  $\alpha \in R^+$ , we define the Peterson translate  $\tau(T, \alpha)$  of  $T$  by  $\alpha$  as follows:*

- *If  $T$  is a subset of an  $\alpha$ -string  $\{\beta, \beta + \alpha, \beta + k\alpha\}$ , so  $T = \{\beta + i_1\alpha, \dots, \beta + i_r\alpha\}$ , then  $\tau(T, \alpha) = \{\beta, \beta + \alpha, \dots, \beta + (r-1)\alpha\}$ .*

- For a general subset  $T$  of  $R^+$ , let  $T = \bigcup T_i$  be the partition of  $T$  induced by the partition of  $R^+$  into  $\alpha$ -strings. Then  $\tau(T, \alpha) = \bigcup \tau(T_i, \alpha)$ .

This definition is equivalent to the geometric Peterson translate defined by Carrell and Kuttler [CK03].

**Theorem 6.2.** *Let  $T$  be a coconvex set in  $R^+$ , where  $R$  has no components of type  $C$  or  $F_4$ . Then:*

- The Peterson translate  $\tau(T, \alpha)$  is coconvex for every  $\alpha \in R^+$ .*
- If  $\mathcal{A}(T)$  is free and  $\alpha \in R^+$  then  $\mathcal{A}(\tau(T, \alpha))$  is free with the same coexponents as  $\mathcal{A}(T)$ .*
- If  $T$  is not a lower order ideal, then there is  $\alpha \in T$  such that  $\tau(T, \alpha)$  is not equal to  $T$ .*

*Proof sketch.* Part (c) of the theorem follows from the definition of coconvex set and lower order ideal. We use a computer to check that parts (a) and (b) hold for the root systems  $R = A_3, B_3,$  and  $G_2$ . The key idea of the proof for general  $R$  is that Peterson translation is local, in the sense that if  $U$  is a subspace of  $V$ , and  $\alpha \in U$ , then  $\tau(T, \alpha) \cap U = \tau(T \cap U, \alpha)$ , where the latter refers to Peterson translation in  $R_U$ . Since part (a) holds for all root systems of rank  $\leq 3$ , and  $T \cap U$  is a coconvex subset of  $R_U$  for any subspace  $U$ , we conclude that  $\tau(T, \alpha) \cap U$  is coconvex for all subspaces  $U \subseteq V$  of rank 3 with  $\alpha \in U$ . But to check that  $\tau(T, \alpha)$  is coconvex, we only need to check that  $\tau(T, \alpha) \cap U$  is coconvex for subspaces  $U' \subset V$  of rank 2, so part (a) holds for all root systems.

If an arrangement  $\mathcal{A}$  is free, then the Ziegler multiarrangement  $\tilde{\mathcal{A}}^H$  is free for any hyperplane  $H \in \mathcal{A}$  [Zie89]. Abe and Yoshinaga have recently proved a converse to Zeigler's theorem: if  $\tilde{\mathcal{A}}^H$  is free for some  $H \in \mathcal{A}$ , then  $\mathcal{A}$  is free if and only if  $\mathcal{A}_X$  is free for every flat  $X \subset H$  of corank 3.

Now given a subspace  $U$  in  $V$  spanned by elements of  $T$ , we can take the flat  $X \in L(\mathcal{A}(T))$  cut out by the elements of  $U \cap T$ . The corank of  $X$  is the rank of  $U$ , and  $X \subset \ker \alpha$  if and only if  $\alpha \in U$ . If  $\mathcal{A}(T)$  is free, then the Zeigler multirestriction of  $\mathcal{A}(T)$  to  $\ker \alpha$  is also free. Since the elements of  $T$  and  $\tau(T, \alpha)$  only differ by translation by  $\alpha$ , the Zeigler multirestriction of  $\mathcal{A}(T)$  to  $\ker \alpha$  is isomorphic to the Zeigler multirestriction of  $\mathcal{A}(\tau(T, \alpha))$ . Since part (b) holds for every root system of rank  $\leq 3$ , we know that  $\mathcal{A}(\tau(T, \alpha))_X = \mathcal{A}(\tau(T, \alpha) \cap U)$  is free for every corank 3 flat  $X \subset \ker \alpha$ . Hence we can apply Abe and Yoshinaga's theorem to prove that  $\mathcal{A}(\tau(T, \alpha))$  is free. The coexponents of  $\mathcal{A}(\tau(T, \alpha))$  can easily be recovered from the coexponents of the Ziegler multirestriction, so  $\mathcal{A}(\tau(T, \alpha))$  has the same coexponents as  $\mathcal{A}(T)$ .  $\square$

Peterson translation decreases the heights of roots, so part (c) of Theorem 6.2 implies that we can repeatedly translate any coconvex set  $T$  until we get an order ideal  $T'$ . As mentioned in the introduction,  $\mathcal{A}(T')$  is free with coexponents  $\text{Exp}(T')$  [ST06] [ABC<sup>+</sup>13], and Theorem 6.2 implies that if  $\mathcal{A}(T)$  is free, then  $\mathcal{A}(T)$  and  $\mathcal{A}(T')$  have the same exponents. If  $T = I(w)$  is the inversion set of a rationally smooth element  $w$ , then  $T$  is a coconvex set and  $\mathcal{A}(T)$  is free, so we get an analogue of the Shapiro-Steinberg-Kostant rule for calculating the exponents of  $w$ .

**Example 6.3.** *Let  $w = s_1 s_2 s_3 s_2$  in the Weyl group of  $A_3$ , where  $s_1$  and  $s_3$  are the simple generators corresponding to the leaves of the Dynkin diagram. If  $\alpha_i$  is the simple root corresponding to  $s_i$ , then*

$$I(w) = \{\alpha_1, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\},$$

so  $I(w)$  is not an order ideal. But

$$T = \tau(I(w), \alpha_1) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_3\}$$

is an order ideal, with  $\text{Exp}(T) = \{1, 1, 2\}$ . Since  $w$  is rationally smooth, we conclude that the exponents of  $w$  are 1, 1, 2.

If  $X(w)$  is smooth, it follows from [CK03] that there is a sequence of translations that sends  $I(w)$  to the lower order ideal  $-\Omega T_e X(w)$ , and thus we can use Theorem 6.2 to directly compare  $\mathcal{I}(w)$  and  $\mathcal{A}(-\Omega T_e X(w))$ . In [Slo13a], Theorem 6.2 is used to give a root-system pattern avoidance criterion for the arrangement  $\mathcal{A}(T)$  to be free, assuming that  $T$  is coconvex.

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