Interval positroid varieties and a deformation of the ring of symmetric functions

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Abstract. Define the **interval rank** $r_{[i,j]} : Gr_k(\mathbb{C}^n) \to \mathbb{N}$ of a k-plane V as the dimension of the orthogonal projection $\pi_{[i,j]}(V)$ of V to the $(j-i+1)$-dimensional subspace that uses the coordinates $i, i+1, \ldots, j$. By measuring all these ranks, we define the **interval rank stratification** of the Grassmannian $Gr_k(\mathbb{C}^n)$. It is finer than the Schubert and Richardson stratifications, and coarser than the positroid stratification studied by Lusztig, Postnikov, and others, so we call the closures of these strata **interval positroid varieties**.

We connect Vakil's "geometric Littlewood-Richardson rule", in which he computed the homology classes of Richardson varieties (Schubert varieties intersected with opposite Schubert varieties), to Erdős-Ko-Rado shifting, and show that all of Vakil's varieties are interval positroid varieties. We build on his work in three ways: (1) we extend it to arbitrary interval positroid varieties, (2) we use it to compute in equivariant K-theory, not just homology, and (3) we simplify Vakil's $(2+1)$-dimensional "checker games" to 2-dimensional diagrams we call "IP pipe dreams".

The ring $Symm$ of symmetric functions and its basis of Schur functions is well-known to be very closely related to the ring $\bigoplus_{a,b} H_*(Gr_a(\mathbb{C}^{a+b}))$ and its basis of Schubert classes. We extend the latter ring to equivariant K-theory (with respect to a circle action on each $\mathbb{C}^{a+b}$), and compute the structure constants of this two-parameter deformation of $Symm$ using the interval positroid technology above.

Résumé. Le rang d'intervalle $r_{[i,j]} : Gr_k(\mathbb{C}^n) \to \mathbb{N}$ d'un sous–espace $V \subset \mathbb{C}^n$ de dimension k est la dimension de la projection orthogonale $\pi_{[i,j]}(V)$ de V sur le sous–espace de dimension $(j-i+1)$ paramétré par les coordonnées $i, i+1, \ldots j$. En considérant tous les rangs $[i,j]$ nous définissons la **stratification selon le rang d'intervalle** de la Grassmannienne $Gr_k(\mathbb{C}^n)$. Cette stratification est plus fine que les stratifications de Schubert et de Richardson, mais plus grossière que la stratification de positroïde étudiée entre autres par Lusztig et Postnikov. Nous appelons donc **variétés d'intervalle positroïde** les clôtures de ces strates.

Nous relions la "règle de Littlewood–Richardson géométrique" de Vakil, qui calcule les classes d'homologie des variétés de Richardson (intersections des variétés de Schubert et des variétés de Schubert opposées) au déplacement d'Erdős-Ko-Rado. Nous prouvons que toutes les variétés de Vakil sont des variétés d'intervalle positroïde. Nous étendons la théorie de Vakil de trois fa cons: (1) nous l'étendons aux variétés d'intervalle positroïde quelconques, (2) nous l'utilisons pour del calculs en K–théorie équivariante, plutôt que simplement en homologie, et (3) nous simplifions les "jeux de dames" de Vakil de dimension $(2+1)$ en introduisant des diagrammes bidimensionnels que nous appelons "tuyauteries IP" (pour $n > 1$ ceci n'est pas une pipe).

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Il est bien connu que l'anneau $Symm$ des fonctions symétriques et sa base des fonctions de Schur sont étroitement liées à l'anneau $\bigoplus_{a,b} H_*(Gr_a(\mathbb{C}^{a+b}))$ et à sa base des classes de Schubert. Nous étendons cet anneau à la K–théorie équivariante (pour une action du cercle sur chaque $\mathbb{C}^{a+b}$) et nous calculons les constantes de structure de cette déformation à deux paramètres de $Symm$, par le biais des techniques d'intervalles positroïdes précédentes.

Keywords: Schubert varieties, matroids, positroids, shifting, symmetric functions

1 Interval positroid varieties and IP pipe dreams

1.1 The varieties

Let $Gr_k(\mathbb{C}^n)$ denote the Grassmannian of k-planes in n-space. Given a k-plane

$$V = \text{row span of } \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix} \qquad \text{a } k \times n \text{ matrix of full rank } k$$

define $r_{[i,j]}(V)$ as the rank of the submatrix $\begin{bmatrix} \vec{c}_i & \vec{c}_{i+1} & \cdots & \vec{c}_j \end{bmatrix}$. Then one can prove that there is a unique upper triangular matrix $f(V)$ such that

$$r_{[i,j]}(V) = \big|[i,j]\big| - \sum_{i \le k \le l \le j} f(V)_{kl}$$

and this $f(V)$ is a partial permutation matrix with $n-k$ 1s, which we will refer to as its **dots**.

If f is an upper triangular partial permutation $n \times n$ matrix with $n-k$ dots, define its **interval positroid variety**

$$\Pi_f := \left\{ V \in Gr_k(\mathbb{C}^n) \quad : \quad \forall i \le j, \quad r_{[i,j]}(V) \le \big|[i,j]\big| - \sum_{i \le k \le l \le j} f(V)_{kl} \right\}$$

These are special cases of *positroid subvarieties* of the Grassmannian, for whose definition and the following results we refer to Knutson et al. (2013):

Theorem 1.1 *1. Each Π_f is nonempty, irreducible, and Cohen-Macaulay with rational singularities.*

2. The intersection of any set of interval positroid varieties is a reduced union of others.

These rank conditions also define a matroid, and Π_R is the closure of the corresponding matroid stratum. (Note that the matroid stratifcation does *not* enjoy the good properties above, by Mnëv's universality theorem.)

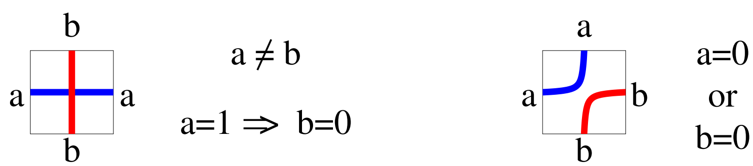
Theorem 1.2 *Let $\Pi_f \subseteq Gr_k(\mathbb{C}^n)$ be an interval positroid variety, and B, B_- the groups of upper and lower triangular matrices, respectively. Say that f is* **NW/SE** *if its dots run Northwest-to-Southeast.*

1. *Π_f is B_--invariant $\iff$ f is NW/SE and has its dots in the last $n-k$ columns. In this case, call Π_f the* **Schubert variety** *X_μ corresponding to the set $\mu \in \binom{n}{k}$ of rows containing dots.*
2. *Π_f is B-invariant $\iff$ f is NW/SE and has its dots in the first $n-k$ rows. In this case, call Π_f the* **opposite Schubert variety** *X^ν corresponding to the set $\nu \in \binom{n}{k}$ of columns containing dots.*
3. *f is NW/SE $\iff$ $\Pi_f = X_\mu \cap X^\nu$ where $\mu \in \binom{n}{k}$, $\nu \in \binom{n}{k}$ are the sets of rows and columns containing dots, respectively.*

1.2 IP pipe dreams

In this section we give a combinatorial formula for the equivariant homology class $[\Pi_f]$ of an interval positroid variety[i] as a positive combination of the classes of opposite Schubert varieties: the coefficients count certain "IP pipe dreams". When $[\Pi_f]$ is a Richardson variety, these IP pipe dreams correspond easily to the puzzles of Knutson and Tao (2003), and less easily to the checker games of Vakil (2006).

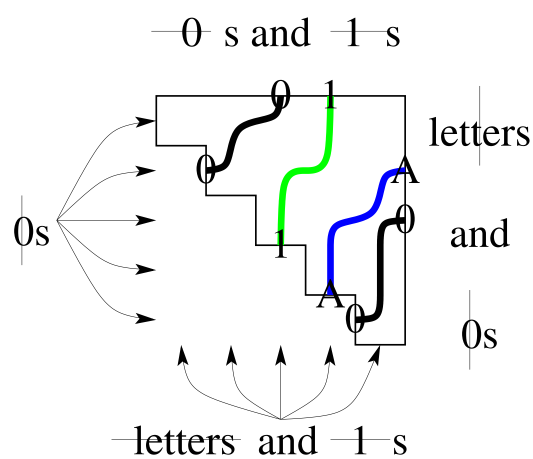
Consider an alphabet $\{0, 1\} \cup \{A, B, C, \dots\}$ of **pipe labels** and the following two kinds of **tiles**:



Call these the **crossing** and **elbows** tiles, and the $a = b = 0$ elbows the **equivariant tile**. We will often want to determine a tile from its South and East labels, and this can be done uniquely unless both are 0.

We will tile these together, such that the boundary labels of adjoining tiles match up, making continuous "pipes" from boundary to boundary bearing well-defined labels. Define an **IP pipe dream** (the IP for "interval positroid") to be a filling of the upper triangle of an $n \times n$ matrix, such that

- on the East edges (of each (i, n) square), there are no 1 labels,
- on the South edges (below each (i, i) square), there are no 0 labels,
- on the North edges (above each $(i, 1)$ square), there are only 0s and 1s,
- no two pipes of the same label cross, and finally,
- **no two *lettered* pipes cross twice.** This is the only nonlocal condition.



Each lettered pipe connects a horizontal edge below the diagonal to a vertical edge on the East side. Since $a \neq b$ in crossing tiles, the ith B from the left must connect to the ith B from the top. By the nonlocal condition, the ith A pipe will cross the jth B pipe either once or not at all, and one can predict which of these two possibilities occurs from the boundary and the Jordan curve theorem.

(i) In Knutson et al. (2013) we show that affine Stanley symmetric functions give representatives for the classes of arbitrary positroid varieties. However, as those symmetric functions are not Schur-positive it seems very difficult to use that result to compute the coefficients we seek here.

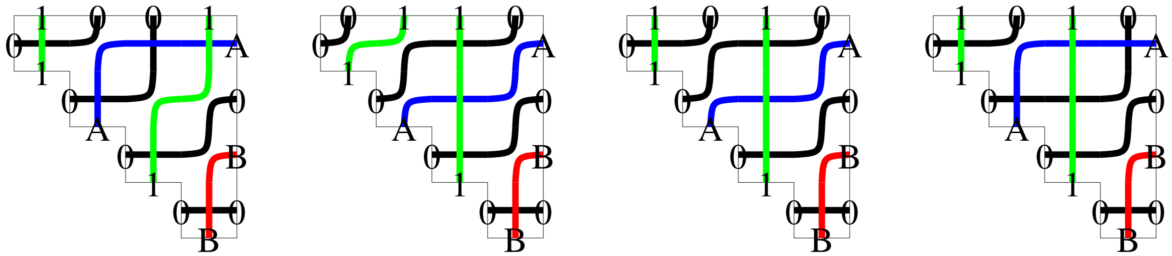


Fig. 1: The IP pipe dreams whose partial permutation is $1 \mapsto 2, 3 \mapsto 4$. (The lettered pipes connect the 1st and 3rd East edges to the 2nd and 4th South edges.) Note that in the first and fourth figures, the A pipe crosses a 0 pipe twice, but that's permissible because the 0 isn't a lettered pipe.

We think of two IP pipe dreams as equivalent if they differ only in the letter labels. This *includes* the possibility of folding two letters into the same letter (only allowed if those pipes don't cross, which as just explained can be predicted from the boundary).

To an IP pipe dream P, we associate two objects:

- $f(P)$, an upper triangular partial permutation depending on the South and East labels of P, and
- $\lambda(P)$, a partition depending on only the North labels.

The partial permutation $f(P)$ is induced by the lettered pipes (i.e. those not labeled $0, 1$), as follows. For each lettered pipe in P, place a dot in $f(P)$ above the South end of the pipe, and left of the East end. In particular, the dots coming from pipes of a given letter are arranged NW/SE (since such pipes don't cross). The IP pipe dreams in figure 1 are all those with the partial permutation $1 \mapsto 2, 3 \mapsto 4$.

The partition $\lambda(P)$ is read as follows. Start at the point $(0, \#0\text{s across the North side})$ in the fourth quadrant of the Cartesian plane, and reading the North side of P from left to right, move down for each 1, and left for each 0. The region above the resulting path is the partition $\lambda(P)$, and $\dim X^\lambda = |\lambda|$. In the IP pipe dreams in figure 1, the partitions are $(2) = \square\square, (1,1) = \begin{smallmatrix}\square\\\square\end{smallmatrix}, (1) = \square, (1) = \square$ respectively.

Theorem 1.3 *As elements of $H_*(Gr_k(\mathbb{C}^n))$, the expansion of $[\Pi_f]$ in the $\mathbb{Z}$-basis of opposite Schubert classes is*

$$[\Pi_f] = \sum_{\substack{P : f(P) = f \\ P \text{ has no equivariant tiles}}} [X^{\lambda(P)}] \quad = \quad \sum_\lambda \# \left\{ P \ : \ \begin{matrix} f(P) = f, \ \lambda(P) = \lambda \\ P \text{ has no equivariant tiles} \end{matrix} \right\} [X^\lambda].$$

Let $T \leq GL(n)$ denote the group of diagonal matrices. Plainly each Π_f is T-invariant, as T acts separately on the columns $(\vec{c}_i)$ from the definition of $r_{[i,j]}$, so Π_f defines a class in $H_*^T(Gr_k(\mathbb{C}^n))$ again denoted $[\Pi_f]$. The corresponding expansion in the basis requires coefficients from $H_T^*(pt) \cong Sym(T^*) \cong \mathbb{Z}[y_1, \ldots, y_n]$, where y_i is the character $y_i(\mathrm{diag}(t_1, \ldots, t_n)) = t_i$.

Define $wt(P) \in H_T^*(pt)$ (for "weight") as the product of $y_{row(t)} - y_{col(t)}$, over all equivariant tiles t. In the IP pipe dreams in figure 1, the weights are $1, 1, y_1 - y_2, y_2 - y_4$ respectively.

Theorem 1.4 *As elements of $H_*^T(Gr_k(\mathbb{C}^n))$, the expansion of $[\Pi_f]$ in the $H_T^*(pt)$-basis of opposite Schubert classes is*

$$[\Pi_f] = \sum_{P:\ f(P)=f} wt(P)\ [X^{\lambda(P)}].$$

Specializing each y_i to 0 recovers the previous theorem.

In Graham (2001) the coefficients are shown (abstractly, without a formula) to lie in $\mathbb{N}[y_1-y_2,\ldots,y_{n-1}-y_n]$. Theorem 1.4 is manifestly Graham-positive[ii] . In the figure 1 example, it says

$$[\Pi_{1\mapsto 2,\ 3\mapsto 4}] = [X^{(2)}] + [X^{(1,1)}] + (y_1 - \cancel{y_2} + \cancel{y_2} - y_4)[X^{(2,1)}] \qquad \in H_T^*(Gr_k(\mathbb{C}^n)).$$

The analogue of positivity in equivariant K-theory was (again, abstractly) proven in Anderson et al. (2008). With slightly more complicated tiles, omitted here, we can give a formula for the K_T-class of Π_f that is again manifestly positive in this richer sense.

The geometry and connection to Erdős-Ko-Rado shifting in §3.1-3.3 was laid out in the unpublished preprint Knutson (2010), which dealt only with Richardson varieties and their shifts. The primary novelties in this section are the extension to arbitrary interval positroid varieties, and the introduction of IP pipe dreams to compute their classes.

2 The K^S-deformation of $Symm$

The "direct sum" map

$$Gr_a(\mathbb{C}^{a+b}) \times Gr_c(\mathbb{C}^{c+d}) \overset{\oplus}{\to} Gr_{a+c}(\mathbb{C}^{a+b+c+d})$$

induces a map on homology, defining a trigraded multiplication on

$$R^H := \bigoplus_{a,b} H_*(Gr_a(\mathbb{C}^{a+b})).$$

It is easy to see that R^H is commutative and associative, and has a basis given by the Schubert classes of the individual Grassmannians, indexed by the set $\{(\lambda,a,b) : \lambda \text{ a partition inside } [a]\times[b]\}$. The identity 1 is $[Gr_0(\mathbb{C}^{0+0})]$. The following result is essentially standard:

Theorem 2.1 $R^H \big/ \langle [Gr_0(\mathbb{C}^{0+1})] - 1, [Gr_1(\mathbb{C}^{1+0})] - 1\rangle$ *is isomorphic to the ring $Symm$ of symmetric functions, under the correspondence $[X^\lambda] \mapsto Schur_\lambda$.*

In the opposite direction, R^H is isomorphic to the biRees algebra $\bigoplus_{a,b} Symm_{a,b}\ t^a s^b \leq Symm[t,s]$ constructed from the bifiltration $Symm_{a,b} :=$ the span of $\{Schur_\lambda : \lambda \subseteq [a]\times[b]\}$.

We will deform this ring R^H in two ways. The simpler one, R^K, comes by replacing homology with the K-homology group defined using complexes of coherent[iii] sheaves on the Grassmannian. The more complicated one,

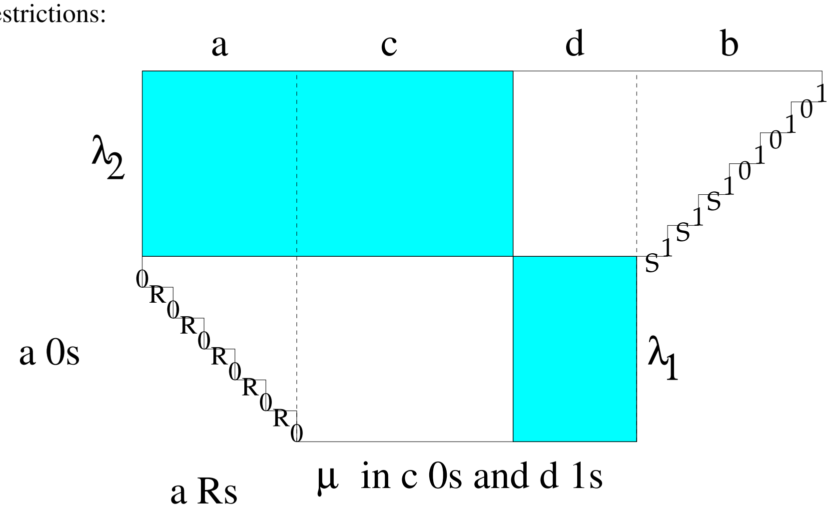
$$R^{H^S} := \bigoplus_{a,b} H_*^S(Gr_a(\mathbb{C}^{a+b}))$$

[ii] Moreover, Graham's derivation shows that if $X \subseteq G/P$ is a subvariety, and $[X] = \sum_\pi c_\pi [X^\pi]$, $c_\pi \in H_T^*$ is the expansion in opposite Schubert classes, then each coefficient c_π is not only a sum of products of simple roots, but can be written as a sum of products of *distinct, positive* roots. This formula for $[\Pi_f]$ also does this.

[iii] The more usual K-cohomology is defined using vector bundles, which pull back instead of pushing forward as this construction requires.

requires we introduce a circle S acting on each $\mathbb{C}^{a+b}$, acting with weight 1 on $\mathbb{C}^a$ and weight 0 on $\mathbb{C}^b$.

It is *not* true that the direct sum of two Schubert varieties is an interval positroid variety. As such, one cannot directly apply theorem 1.4 to compute the structure constants of R^{H^S}. Nonetheless the same techniques, plus a crucial dualization explained in §3.4, give a formula. Define a **DS pipe dream** (for direct sum[(iv)]) of type $(\lambda \in \binom{a+b}{a}, \mu \in \binom{c+d}{c})$ as a tiling of the following region, with the following additional restrictions:



- The tiles in the lower half are those from IP pipe dreams, on the alphabet $\{0, 1, R, S\}$.
- The tiles in the upper half are flipped horizontally from those from IP pipe dreams, with the roles of 0 and 1 exchanged.
- Any equivariant tiles only appear in the shaded regions.
- Write λ as a string of a 1s and b 0s. Then
  - λ_1 is the last b letters of λ, with the 1s turned into Rs, and
  - λ_2 is the first a letters of λ, with the 0s turned into Ss, and reversed.
- The number of Ss below the "b" equals the number of Ss in λ_2.

Theorem 2.2 *As elements of $H_*^S(Gr_{a+c}(\mathbb{C}^{a+b+c+d}))$,*

$$[X^\lambda \oplus X^\mu] = \sum_{\textit{DS pipe dreams}} y^{\#\textit{equivariant pieces}} \left[X^{\textit{top labels}}\right].$$

The statement for equivariant K-theory is only slightly more involved.

(iv) Pipe dreams come up in seemingly unrelated contexts, so it seems safest to include a modifier.

3 Ingredients of the proofs

3.1 Geometric shifting

Throughout this section X will denote a T-invariant subscheme of $Gr_k(\mathbb{C}^n)$.

Given $i < j$, define the **shift** $Ш_{i\to j}$ and **sweep** $\Psi_{i\to j}$ of a T-invariant subscheme $X \subseteq Gr_k(\mathbb{C}^n)$ as

$$Ш_{i\to j}X := \lim_{t\to\infty} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & t & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \cdot X, \quad \Psi_{i\to j}X := \overline{\bigcup_{t\in\mathbb{C}} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & t & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \cdot X}$$

where the t is in matrix position (i, j). We define the shift and sweep for $i = j$ to be the identity: $Ш_{i\to i}X = \Psi_{i\to i}X = X$.

Say that X is $Ш_{i\to j}$**-invariant** if $Ш_{i\to j}X = X$. Some results about the shift and sweep are very easy:

Theorem 3.1 *1.* $\Psi_{i\to j}X \supseteq X, Ш_{i\to j}X$.

2. *The shift and sweep are again T-invariant.*
3. $\dim Ш_{i\to j}X = \dim X$. *If X is $Ш_{i\to j}$-invariant, then $\Psi_{i\to j}X = X$, and otherwise* $\dim \Psi_{i\to j}X = \dim X + 1$.
4. $[X] = [Ш_{i\to j}X]$ *as classes in homology or K-homology.*
5. $[X] = [Ш_{i\to j}X] + d\,(y_i - y_j)[\Psi_{i\to j}X]$ *in equivariant homology, for some $d \in \mathbb{N}$.*
6. *(harder) If X is irreducible and not $Ш_{i\to j}$-invariant, this coefficient d is* 1.
7. *A similar formula holds in equivariant K-homology if $\Psi_{i\to j}X$ has rational singularities (a property known to hold for positroid varieties).*

To avoid confusion, we refer to subsets of $\{S \subseteq [1, n] : |S| = k\}$ as **collections**. If we identify the T-fixed points $Gr_k(\mathbb{C}^n)^T$ with $\binom{n}{k}$, taking a coordinate subspace to the coordinates it uses, then each X gives us a collection X^T.

Following Erdős et al. (1961), we define the **combinatorial shifts** of an element $k \in [1, n]$, a subset $S \subseteq [1, n]$, and a collection C by

- $Ш_{i\to j}(k) = k$ unless $k = i$, in which case $Ш_{i\to j}(i) = j$
- $Ш_{i\to j}(S) = \bigcup_{k\in S}\{Ш_{i\to j}(k)$ unless $Ш_{i\to j}(k) \in S$, in which case $k\}$
- $Ш_{i\to j}(C) = \bigcup_{S\in C}\{Ш_{i\to j}(S)$ unless $Ш_{i\to j}(S) \in C$, in which case $S\}$

with the mantra "move i to j, unless something's in the way".

The geometric and combinatorial shifts are most closely related when X is T**-convex**, meaning that if $L \subseteq Gr_k(\mathbb{C}^n)$ is a T-invariant $\mathbb{CP}^1$ connecting two T-fixed points in X, then $L \subseteq X$.

Theorem 3.2 *Let $X \subseteq Gr_k(\mathbb{C}^n)$ be a T-invariant subscheme.*

1. *$Ш_{i\to j}(X^T) \subseteq (Ш_{i\to j}X)^T$, where the left side uses the combinatorial shift, the right side the geometric shift.*
2. *If X is T-convex, then this containment is an equality.*
3. *If X is irreducible, then X is T-convex.*
4. *If X is defined by the vanishing of a collection of Plücker coordinates, then X is T-convex.*

3.2 The Vakil sequence of shifts

Following Vakil (2006) (which does not explicitly discuss shifts), we consider a sequence of $\binom{n+1}{2}$ shifts,

$$\begin{gathered} Ш_{n\to n}, \\ Ш_{n-1\to n}, \quad Ш_{n-1\to n-1}, \\ Ш_{n-2\to n}, \quad Ш_{n-2\to n-1}, \quad Ш_{n-2\to n-2} \\ \vdots \\ Ш_{1\to n}, \quad Ш_{1\to n-1}, \quad \cdots, \quad Ш_{1\to 1}. \end{gathered}$$

(Of course each $Ш_{i\to i}$ operation is trivial.) In our language, Vakil proves

Theorem 3.3 *Let $X = X_\mu \cap X^\nu$. Apply the shifts in the order above. At each step, if the result is reducible, break into geometric components and continue separately.*

1. *Each "mid-sort" variety Y met along the way can be encoded by a "checkerboard".*
2. *Each $Ш_{i\to j}Y$ is generically reduced, with one or two components.*
3. *There is a somewhat involved rule, given a checkerboard for Y, to determine the checkerboards of the components of $Ш_{i\to j}Y$.*
4. *After the last shift, each component Y is some opposite Schubert variety X^ν.*

Consequently, one can compute $[X_\mu \cap X^\nu]$ as a sum of $[X^\nu]$, with one summand per "checker game" (a sequence of compatible checkerboards).

One can simplify the rule alluded to in part (3) of this theorem by first determining the unique checkerboard for $\Psi_{i\to j}Y$, and then constructing from *that* the checkerboards for the components of $Ш_{i\to j}Y$, working from the divisorial containments $Y \subset \Psi_{i\to j}Y \supset Ш_{i\to j}Y$. (Vakil's rule is the composite of these two steps.) This also makes possible the computation of the class $[X_\mu \cap X^\nu]$ in *equivariant* homology, which Vakil does not address, using theorem 3.1 parts (5,6).

If one attempts these shifts in a different order, conclusion (2) of the theorem often fails. (Conclusions (1) and (3) fail as well, but while one could hope that they could be repaired with different definitions of "checkerboard", conclusion (2) depends only on the order of shifts.) So one would like to elucidate what makes this particular sequence so felicitous.

Let $X_{[i,j]\le r}$ denote the interval positroid variety defined by a single rank condition. Plainly any Π_f is the intersection of the $\{X_{[i,j]\le r}\}$ containing it. The following lemma is easy to prove:

Lemma 3.1 1. *$X_{[k,l]\leq r}$ is $Ш_{i\to j}$-invariant unless $[k,l]$ is $Ш_{j\to i}$-invariant, i.e. $i \notin [k,l] \ni j$.*

2. *$Ш_{i\to j} X_{[i+1,j]\leq r} = X_{[i,j-1]\leq r}$ ("the subsets shift backwards").*

3. *If $X, Y \subseteq Gr_k(\mathbb{C}^n)$ are T-invariant subschemes, and X is $Ш_{i\to j}$-invariant, then $Ш_{i\to j}(X\cap Y) \subseteq X \cap Ш_{i\to j} Y$.*

4. *Hence, if Π_f is $Ш_{i\to j}$-invariant, and $\Pi_g = \Pi_f \cap X_{[i+1,j]\leq r}$, then $Ш_{i\to j}\Pi_g \subseteq \Pi_f \cap X_{[i,j-1]\leq r}$. Note that this last intersection is a reduced union of interval positroid varieties, by theorem 1.1 (2).*

Rather than defining checkerboards, we describe Vakil's varieties as the interval positroid varieties they are. Define an upper triangular partial permutation f (or its variety Π_f) to be (i,j)**-mid-sort** if the dots in rows $[i,n]$ of f (except perhaps a dot in row i, column $\leq j$) are NW/SE, and the m dots in rows $[i+1,n]$ are in rows $[i+1,i+m]$. If in addition, the dots in rows $[1,i]$ (except perhaps a dot in row i, column $> j$) are NW/SE, call f and Π_f (i,j)**-Vakil**.

Any interval positroid variety Π_g has its g being (n,n)-mid-sort, but only Richardson varieties are (n,n)-Vakil, by theorem 1.2. The two concepts agree for $(i,j) = (1,1)$, and say that Π_g must be an opposite Schubert variety.

Theorem 3.4 *The varieties that can occur just before the $Ш_{i\to j}$ stage of Vakil's process are exactly $\{\Pi_g : g \text{ is } (i,j)\text{-Vakil}\}$. In this case conclusion (4) of lemma 3.1 holds, and in its notation $Ш_{i\to j}\Pi_g = \Pi_f \cap X_{[i,j-1]\leq r}$, each component of which is (i',j')-Vakil. (Here $(i',j') = (i,j-1)$ unless $j = i$, in which case $(i',j') = (i-1,n)$.)*

3.3 The interval positroid variety of a slice

The theorem extends to (i,j)-mid-sort varieties Π_g; each component of $Ш_{i\to j}\Pi_g$ is (i',j')-mid-sort. It will be more convenient to encode these f by 1-dimensional diagrams rather than Vakil's checkerboards, so we can string them together into 2-dimensional diagrams (which will be the IP pipe dreams of theorem 1.4).

Given (i,j), let S be the following collection of edges between matrix entries:

- the Southern edges of each (k,k) square, $k \leq i$,
- the Southern edges of each (i,k) square, $k \leq j$,
- the **kink**, the Eastern edge of (i,j),
- the Southern edges of each $(i-1,k)$ square, $k > j$, and
- the Eastern edges of each (k,n) square, $k < i$.

We will label S from the alphabet $\{0,1\} \cup \{A,B,\ldots\}$, based on the choice of (i,j)-mid-sort g.

Call the dots in g above S the **upper dots** of g, and the others the **lower dots**. First pick letter labels for the upper dots, subject to the requirement that no two dots with the same letter can be NE/SW of one another. In particular it is valid to give every dot a different letter, and it is possible to use only one letter iff g is (i,j)-Vakil. Label all the lower dots with 0.

Now project the upper dots to the right, causing some of the Eastern edges (possibly including the kink) to acquire labels from the upper dots. All remaining Eastern edges are labeled 0.

Next project the dots vertically to S, causing some of the Southern edges of S to acquire letter labels or 0 labels. All remaining Southern edges are labeled 1.

Call S with its labeling a **slice** or (i,j)**-slice** (soon, of an IP pipe dream). It is easy to see that g can be reconstructed from the slice σ, so we could refer to Π_σ instead. Also, connecting each lettered dot to its edges South and East gives a diagram of pipes.

Consider an (i,j)-slice σ and an $(i,j-1)$-slice σ' that agree on their overlap, so their symmetric difference is a tile in position (i,j). If the diagram of pipes for σ' does not have more crossings than for σ, say that σ' **follows** σ. (This is how we implement the "no two pipes cross twice" condition of an IP pipe dream.)

Theorem 3.5 *Let σ be a slice, and Π_g its (i,j)-mid-sort interval positroid variety. The following are equivalent (the "boring case"):*

- *Π_σ is $\text{Ш}_{i\to j}$-invariant*
- *the kink, and Southern edge immediately to its left, of σ are not both labeled* 0
- *there is exactly one σ' that follows σ, and $\Pi_{\sigma'} = \Pi_\sigma$.*

Assume not. Then

- *$\Psi_{i\to j}\Pi_\sigma = \Pi_\rho$, where the tile recording the symmetric difference of σ and ρ is the equivariant tile.*
- *If $\{\sigma'\}$ are the other slices following σ, then $\text{Ш}_{i\to j}\Pi_\sigma = \bigcup \Pi_{\sigma'}$, and each σ' gives a component.*

This is proven using the analogue of Lascoux's transition formula for affine double Stanley symmetric functions from Lam and Shimozono (2008). The condition in that formula that $\rho > \sigma'$ is a covering relation in Bruhat order turns into the "no two pipes cross twice" condition.

Theorems 1.3 and 1.4 follow from this and theorem 3.1.

3.4 Positroid varieties and duality

To prove theorem 2.2, we start by rewriting the direct sum map as

$$\left(\begin{smallmatrix}\text{row}\\ \text{span}\end{smallmatrix}\begin{bmatrix} A & B\end{bmatrix},\quad \begin{smallmatrix}\text{row}\\ \text{span}\end{smallmatrix}\begin{bmatrix} C & D\end{bmatrix}\right) \overset{\oplus}{\mapsto} \begin{smallmatrix}\text{row}\\ \text{span}\end{smallmatrix}\begin{bmatrix} A' & 0 & 0 & B' \\ 0 & C & D & 0\end{bmatrix} \quad \overset{\text{rot}}{\mapsto} \begin{smallmatrix}\text{row}\\ \text{span}\end{smallmatrix}\begin{bmatrix} B' & A' & 0 & 0 \\ 0 & 0 & C & D\end{bmatrix}$$

where A,B,C,D have a,b,c,d columns respectively, and A',B' are A,B flipped left-right. This flipping is equivariant with respect to the S-action. The second map, that rotates b columns, is only S-equivariant if we also rotate the action. The benefit of flipping A,B and rotating is that $\text{rot}(X^\lambda \oplus X^\mu)$ is then a Richardson variety.

Now we run the Vakil degeneration. Since we only care about computing in S-equivariant cohomology, we only permit equivariant tiles at locations (i,j) where coordinates i and j have different S-weights. To avoid introducing terms that are not Graham-positive, *we stop the degeneration after the first $d+c+a$ rows* from the bottom. Nothing happens during the first $d+c$ rows, as $\text{rot}(X^\lambda \oplus X^\mu)$ is invariant under those shifts. The remaining a rows give the lower half of the pipe dreams in theorem 2.2.

A miracle occurs at this point: the interval positroid varieties we are dealing with are, after another rotation, also *dual interval* positroid, which we now stop to explain.

If we generalize the interval rank functions to allow *cyclic* intervals $[i,n]\coprod[1,j]$, we get the finer positroid stratification, whose strata are indexed by affine permutations $f:\mathbb{Z}\to\mathbb{Z}$, where $f(i)-i$ lies in $[0,n]$ and is periodic with period n; see e.g. Knutson et al. (2013). One of the many benefits of this finer stratification is that it is closed under Grassmann duality, taking Π_f to Π_{f^*} where $f^*(i)=n+f^{-1}(i)$.

Draw f as a $\mathbb{Z}\times\mathbb{Z}$ permutation matrix, and let f_L be the $[1,n]\times[1,n]$ submatrix, automatically an upper triangular partial permutation. Then $\Pi_{f_L}\supseteq\Pi_f$ is the smallest *interval* positroid variety containing Π_f, and moreover $\Pi_f=\Pi_{f_L}\cap\Pi_{((f^*)_L)^*}$.

Returning to the varieties at hand; we rotate them, dualize, recognize them as again being interval positroid varieties, and continue the degeneration. (That recognition step involves the subtle definitions of λ_1,λ_2.) The record of this latter degeneration gives the upper half of the pipe dreams in theorem 2.2.

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