

0-Hecke algebra action on the Stanley-Reisner ring of the Boolean algebra

Jia Huang

► **To cite this version:**

Jia Huang. 0-Hecke algebra action on the Stanley-Reisner ring of the Boolean algebra. Louis J. Billera and Isabella Novik. 26th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2014), 2014, Chicago, United States. Discrete Mathematics and Theoretical Computer Science, DMTCS Proceedings vol. AT, 26th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2014), pp.11-22, 2014, DMTCS Proceedings. <hal-01207589>

HAL Id: hal-01207589

<https://hal.inria.fr/hal-01207589>

Submitted on 1 Oct 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

0-Hecke algebra action on the Stanley-Reisner ring of the Boolean algebra

Jia Huang*

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

Abstract. We define an action of the 0-Hecke algebra of type A on the Stanley-Reisner ring of the Boolean algebra. By studying this action we obtain a family of multivariate noncommutative symmetric functions, which specialize to the noncommutative Hall-Littlewood symmetric functions and their (q, t) -analogues introduced by Bergeron and Zabrocki. We also obtain multivariate quasisymmetric function identities, which specialize to a result of Garsia and Gessel on the generating function of the joint distribution of five permutation statistics.

Résumé. Nous définissons une action de l'algèbre de Hecke-0 de type A sur l'anneau Stanley-Reisner de l'algèbre de Boole. En étudiant cette action, on obtient une famille de fonctions symétriques non commutatives multivariées, qui se spécialisent pour les non commutatives fonctions de Hall-Littlewood symétriques et leur (q, t) -analogues introduits par Bergeron et Zabrocki. Nous obtenons également des identités de fonction quasisymétrique multivariées, qui se spécialisent à la suite de Garsia et Gessel sur la fonction génératrice de la distribution conjointe de cinq statistiques de permutation.

Keywords: 0-Hecke algebra, Stanley-Reisner ring, Boolean algebra, noncommutative Hall-Littlewood symmetric function, multivariate quasisymmetric function.

1 Introduction

Let \mathbb{F} be any field. The symmetric group \mathfrak{S}_n naturally acts on the polynomial ring $\mathbb{F}[X] := \mathbb{F}[x_1, \dots, x_n]$ by permuting the variables x_1, \dots, x_n . The invariant algebra $\mathbb{F}[X]^{\mathfrak{S}_n}$, which consists of all the polynomials fixed by this \mathfrak{S}_n -action, is a polynomial algebra generated by the elementary symmetric functions e_1, \dots, e_n . The coinvariant algebra $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$, with $(\mathbb{F}[X]_+^{\mathfrak{S}_n}) = (e_1, \dots, e_n)$, is a vector space of dimension $n!$ over \mathbb{F} , and when \mathbb{F} has characteristic larger than n the coinvariant algebra carries the regular representation of \mathfrak{S}_n . A well known basis for $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ consists of the descent monomials. Garsia [7] obtained this basis by transferring a natural basis from the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ of the Boolean algebra \mathcal{B}_n to the polynomial ring $\mathbb{F}[X]$. Here the *Boolean algebra* \mathcal{B}_n is the set of all subsets of $[n] := \{1, 2, \dots, n\}$ partially ordered by inclusion, and the *Stanley-Reisner ring* $\mathbb{F}[\mathcal{B}_n]$ is the quotient of the polynomial algebra $\mathbb{F}[y_A : A \subseteq [n]]$ by the ideal $(y_A y_B : A \text{ and } B \text{ are incomparable in } \mathcal{B}_n)$.

The 0-Hecke algebra $H_n(0)$ (of type A) is a deformation of the group algebra of \mathfrak{S}_n . It acts on $\mathbb{F}[X]$ by the Demazure operators, also known as the isobaric divided difference operators, having the same

*Email: jiahuangustc@gmail.com. Supported by NSF grant DMS-1001933.

invariant algebra as the \mathfrak{S}_n -action on $\mathbb{F}[X]$. In our earlier work [13], we showed that the coinvariant algebra $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ is also isomorphic to the regular representation of $H_n(0)$, for any field \mathbb{F} , by constructing another basis for $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ which consists of certain polynomials whose leading terms are the descent monomials. This and the previously mentioned connection between the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ and the polynomial ring $\mathbb{F}[X]$ motivate us to define an $H_n(0)$ -action on $\mathbb{F}[\mathcal{B}_n]$.

It turns out that our $H_n(0)$ -action on $\mathbb{F}[\mathcal{B}_n]$ has similar properties to the $H_n(0)$ -action on $\mathbb{F}[X]$. It preserves an \mathbb{N}^{n+1} -multigrading of $\mathbb{F}[\mathcal{B}_n]$ and has invariant algebra equal to a polynomial algebra $\mathbb{F}[\Theta]$, where Θ is the set of *rank polynomials* θ_i (the usual analogue of e_i in $\mathbb{F}[\mathcal{B}_n]$). We show that the $H_n(0)$ -action is Θ -linear and thus descends to the coinvariant algebra $\mathbb{F}[\mathcal{B}_n]/(\Theta)$. Using the descent monomials in $\mathbb{F}[\mathcal{B}_n]$ it is not hard to see that $\mathbb{F}[\mathcal{B}_n]/(\Theta)$ carries the regular representation of $H_n(0)$.

It is well known that every finite dimensional (complex) \mathfrak{S}_n -representation is a direct sum of simple (i.e. irreducible) \mathfrak{S}_n -modules, and the simple \mathfrak{S}_n -modules are indexed by partitions λ of n , which correspond to the Schur functions s_λ via the *Frobenius characteristic map*. Hotta-Springer [11] and Garsia-Procesi [9] discovered that the cohomology ring of the *Springer fiber* indexed by a partition μ of n is isomorphic to certain quotient ring R_μ of $\mathbb{F}[X]$, which admits a graded \mathfrak{S}_n -module structure corresponding to the *modified Hall-Littlewood symmetric function* $\tilde{H}_\mu(x; t)$ via the Frobenius characteristic map. The coinvariant algebra of \mathfrak{S}_n is nothing but R_{1^n} .

In our previous work [13] we established a partial analogue of the above result by showing that the $H_n(0)$ -action on $\mathbb{F}[X]$ descends to R_μ if and only if $\mu = (1^k, n - k)$ is a hook, and if so then R_μ has graded quasisymmetric characteristic equal to $\tilde{H}_\mu(x; t)$ and graded noncommutative characteristic $\tilde{\mathbf{H}}_\mu(\mathbf{x}; t)$. Here $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$ is the noncommutative modified Hall-Littlewood symmetric function introduced by Bergeron and Zabrocki [3] for any composition α of n . Using an analogue of the nabla operator Bergeron and Zabrocki [3] also introduced a (q, t) -analogue $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$ for any composition α . Now we provide in Theorem 1.1 below a complete representation theoretic interpretation for $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$ and $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$ by the $H_n(0)$ -action on $\mathbb{F}[\mathcal{B}_n]$.

To state our result, we first recall the two characteristic maps for representations of $H_n(0)$ introduced by Krob and Thibon [14], which we call the *quasisymmetric characteristic* and the *noncommutative characteristic*. The simple $H_n(0)$ -modules are indexed by compositions α of n and correspond to the *fundamental quasisymmetric functions* F_α via the quasisymmetric characteristic; the projective indecomposable $H_n(0)$ -modules are also indexed by compositions α of n and correspond to the *noncommutative ribbon Schur functions* \mathbf{s}_α via the noncommutative characteristic. See §2 for details.

Theorem 1.1 *Let α be a composition of n . Then there exists a homogeneous $H_n(0)$ -invariant ideal I_α of the multigraded algebra $\mathbb{F}[\mathcal{B}_n]$ such that the quotient algebra $\mathbb{F}[\mathcal{B}_n]/I_\alpha$ becomes a projective $H_n(0)$ -module with multigraded noncommutative characteristic equal to*

$$\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \dots, t_{n-1}) := \sum_{\beta \preceq \alpha} \underline{t}^{D(\beta)} \mathbf{s}_\beta \quad \text{inside} \quad \mathbf{NSym}[t_1, \dots, t_{n-1}].$$

One has $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t, t^2, \dots, t^{n-1}) = \tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$, and obtains $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$ from $\tilde{\mathbf{H}}_{1^n}(\mathbf{x}; t_1, \dots, t_{n-1})$ by taking $t_i = t^i$ for all $i \in D(\alpha)$, and $t_i = q^{n-i}$ for all $i \in [n-1] \setminus D(\alpha)$.

Here $D(\alpha)$ is the set of partial sums of α , the notation $\beta \preceq \alpha$ means α and β are compositions of n with $D(\beta) \subseteq D(\alpha)$, and \underline{t}^S denotes the product $\prod_{i \in S} t_i$ over all elements i in a multiset S , including the repeated ones. Taking $\alpha = (1^n)$ shows that $\mathbb{F}[\mathcal{B}_n]/(\Theta)$ carries the regular representation of $H_n(0)$.

Specializations of $\tilde{\mathbf{H}}_{1^n}(\mathbf{x}; t_1, \dots, t_{n-1})$ include not only $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$, but also a more general family of noncommutative symmetric functions depending on parameters associated with paths in binary trees introduced recently by Lascoux, Novelli, and Thibon [15].

Next we study the quasisymmetric characteristic of $\mathbb{F}[\mathcal{B}_n]$. We combine the usual \mathbb{N}^{n+1} -multigrading of $\mathbb{F}[\mathcal{B}_n]$ (recorded by $\underline{t} := t_0, \dots, t_n$) with the length filtration of $H_n(0)$ (recorded by q) and obtain an $\mathbb{N} \times \mathbb{N}^{n+1}$ -multigraded quasisymmetric characteristic for $\mathbb{F}[\mathcal{B}_n]$.

Theorem 1.2 *The $\mathbb{N} \times \mathbb{N}^{n+1}$ -multigraded quasisymmetric characteristic of $\mathbb{F}[\mathcal{B}_n]$ is*

$$\begin{aligned} \text{Ch}_{q, \underline{t}}(\mathbb{F}[\mathcal{B}_n]) &= \sum_{k \geq 0} \sum_{\alpha \in \text{Com}(n, k+1)} \underline{t}^{D(\alpha)} \sum_{w \in \mathfrak{S}^\alpha} q^{\text{inv}(w)} F_{D(w^{-1})} \\ &= \sum_{w \in \mathfrak{S}_n} \frac{q^{\text{inv}(w)} \underline{t}^{D(w)} F_{D(w^{-1})}}{\prod_{0 \leq i \leq n} (1 - t_i)} \\ &= \sum_{k \geq 0} \sum_{\mathbf{p} \in [k+1]^n} t_{p'_1} \cdots t_{p'_k} q^{\text{inv}(\mathbf{p})} F_{D(\mathbf{p})}. \end{aligned}$$

Here we identify F_I with F_α if $D(\alpha) = I \subseteq [n-1]$. The set $\text{Com}(n, k)$ consists of all *weak compositions of n with length k* , i.e. all the sequences $\alpha = (\alpha_1, \dots, \alpha_k)$ of k nonnegative integers with $|\alpha| := \sum_{i=1}^k \alpha_i = n$. The *descent multiset* of the weak composition α is the *multiset*

$$D(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \cdots + \alpha_{k-1}\}.$$

We also define $\mathfrak{S}^\alpha := \{w \in \mathfrak{S}_n : D(w) \subseteq D(\alpha)\}$. The set $[k+1]^n$ consists of all words of length n on the alphabet $[k+1]$. Given $\mathbf{p} = (p_1, \dots, p_n) \in [k+1]^n$, we write $p'_i := \#\{j : p_j \leq i\}$, $\text{inv}(\mathbf{p}) := \#\{(i, j) : 1 \leq i < j \leq n : p_i > p_j\}$, and $D(\mathbf{p}) := \{i : p_i > p_{i+1}\}$.

Let $\mathbf{ps}_{q; \ell}(F_\alpha) := F_\alpha(1, q, q^2, \dots, q^{\ell-1}, 0, 0, \dots)$. Applying the linear transformation $\sum_{\ell \geq 0} u_1^\ell \mathbf{ps}_{q_1; \ell+1}$ and the specialization $t_i = q_2^i u_2$ for all $i = 0, 1, \dots, n$ to Theorem 1.2, we recover a result of Garsia and Gessel [8, Theorem 2.2] on the generating function of the joint distribution of five permutation statistics:

$$\frac{\sum_{w \in \mathfrak{S}_n} q_0^{\text{inv}(w)} q_1^{\text{maj}(w^{-1})} u_1^{\text{des}(w^{-1})} q_2^{\text{maj}(w)} u_2^{\text{des}(w)}}{(u_1; q_1)_n (u_2; q_2)_n} = \sum_{\ell, k \geq 0} u_1^\ell u_2^k \sum_{(\lambda, \mu) \in B(\ell, k)} q_0^{\text{inv}(\mu)} q_1^{|\lambda|} q_2^{|\mu|} \quad (1)$$

Here $(u; q)_n := \prod_{0 \leq i \leq n} (1 - q^i u)$, the set $B(\ell, k)$ consists of pairs of weak compositions $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ satisfying the conditions $\ell \geq \lambda_1 \geq \cdots \geq \lambda_n$, $\max\{\mu_i : 1 \leq i \leq n\} \leq k$, and $\lambda_i = \lambda_{i+1} \Rightarrow \mu_i \geq \mu_{i+1}$ (such pairs (λ, μ) are sometimes called *bipartite partitions*), and $\text{inv}(\mu)$ is the number of inversion pairs in μ . Some further specializations of Theorem 1.2 imply identities of Carlitz-MacMahon [6, 17] and Adin-Brenti-Roichman [1].

The structure of this paper is as follows. Section 2 reviews the representation theory of the 0-Hecke algebra. Section 3 studies the Stanley-Reisner ring of the Boolean algebra. Section 4 defines a 0-Hecke algebra action on the Stanley-Reisner ring of the Boolean algebra. The noncommutative and quasisymmetric characteristics are discussed in Section 5 and Section 6. Finally we give some remarks and questions for future research in Section 7, including a generalization to an action of the Hecke algebra of any finite Coxeter group on the Stanley-Reisner ring of the Coxeter complex.

2 Representation theory of the 0-Hecke algebra

We review the representation theory of the 0-Hecke algebra in this section. The (type A) *Hecke algebra* $H_n(q)$ is the associative $\mathbb{F}(q)$ -algebra generated by T_1, \dots, T_{n-1} with relations

$$\begin{cases} (T_i + 1)(T_i - q) = 0, & 1 \leq i \leq n-1, \\ T_i T_j = T_j T_i, & 1 \leq i, j \leq n-1, |i-j| > 1, \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, & 1 \leq i \leq n-2. \end{cases} \quad (2)$$

It has an $\mathbb{F}(q)$ -basis $\{T_w : w \in \mathfrak{S}_n\}$ where $T_w := T_{i_1} \cdots T_{i_k}$ if $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression.

Specializing $q = 1$ gives the group algebra of \mathfrak{S}_n , with $s_i = T_i|_{q=1}$ and $w = T_w|_{q=1}$. Let $w \in \mathfrak{S}_n$. The *length* of w equals $\text{inv}(w) := \#\{(i, j) : 1 \leq i < j \leq n, w(i) > w(j)\}$, and the *descent set* of w is $D(w) = \{i : 1 \leq i \leq n-1, w(i) > w(i+1)\}$. We write $\text{des}(w) := |D(w)|$ and $\text{maj}(w) := \sum_{i \in D(w)} i$.

Let α be a (weak) composition of n , and let α^c be the composition of n with $D(\alpha^c) = [n-1] \setminus D(\alpha)$. The *parabolic subgroup* \mathfrak{S}_α is the subgroup of \mathfrak{S}_n generated by $\{s_i : i \in D(\alpha^c)\}$. The set of all minimal \mathfrak{S}_α -coset representatives is $\mathfrak{S}^\alpha := \{w \in \mathfrak{S}_n : D(w) \subseteq D(\alpha)\}$. The *descent class* of α consists of the permutations in \mathfrak{S}_n with descent set equal to $D(\alpha)$, and turns out to be an interval under the (left) weak order of \mathfrak{S}_n , denoted by $[w_0(\alpha), w_1(\alpha)]$. One sees that $w_0(\alpha)$ is the longest element of the parabolic subgroup \mathfrak{S}_{α^c} , and $w_1(\alpha)$ is the longest element in \mathfrak{S}^α (c.f. Björner and Wachs [5, Theorem 6.2]).

Another interesting specialization of $H_n(q)$ is the *0-Hecke algebra* $H_n(0)$, with generators $\bar{\pi}_i = T_i|_{q=0}$ for $i = 1, \dots, n-1$, and an \mathbb{F} -basis $\{\bar{\pi}_w = T_w|_{q=0} : w \in \mathfrak{S}_n\}$. Let $\pi_i := \bar{\pi}_i + 1$. Then π_1, \dots, π_{n-1} form another generating set for $H_n(0)$, with the same relations as (2) except $\pi_i^2 = \pi_i$, $1 \leq i \leq n-1$. The element $\pi_w := \pi_{i_1} \cdots \pi_{i_k}$ is well defined for any $w \in \mathfrak{S}_n$ with a reduced expression $w = s_{i_1} \cdots s_{i_k}$, and $\{\pi_w : w \in \mathfrak{S}_n\}$ is another \mathbb{F} -basis for $H_n(0)$. One can check that π_w equals the sum of $\bar{\pi}_u$ over all u less than or equal to w in the Bruhat order of \mathfrak{S}_n . In particular, $\pi_{w_0(\alpha)}$ is the sum of $\bar{\pi}_u$ for all $u \in \mathfrak{S}_{\alpha^c}$.

Norton [18] decomposed the 0-Hecke algebra $H_n(0)$ into a direct sum of projective indecomposable submodules $\mathbf{P}_\alpha := H_n(0) \cdot \bar{\pi}_{w_0(\alpha)} \pi_{w_0(\alpha^c)}$ for all $\alpha \models n$ (i.e. compositions of n). Each \mathbf{P}_α has an \mathbb{F} -basis $\{\bar{\pi}_w \pi_{w_0(\alpha^c)} : w \in [w_0(\alpha), w_1(\alpha)]\}$. Its *radical* $\text{rad } \mathbf{P}_\alpha$ is the unique maximal $H_n(0)$ -submodule spanned by $\{\bar{\pi}_w \pi_{w_0(\alpha^c)} : w \in (w_0(\alpha), w_1(\alpha)]\}$. Although \mathbf{P}_α itself is not necessarily simple, its *top* $\mathbf{C}_\alpha := \mathbf{P}_\alpha / \text{rad } \mathbf{P}_\alpha$ is a one-dimensional simple $H_n(0)$ -module with the action of $H_n(0)$ given by

$$\bar{\pi}_i = \begin{cases} -1, & \text{if } i \in D(\alpha), \\ 0, & \text{if } i \notin D(\alpha). \end{cases}$$

It follows from general representation theory of algebras (see e.g. [2, §I.5]) that $\{\mathbf{P}_\alpha : \alpha \models n\}$ and $\{\mathbf{C}_\alpha : \alpha \models n\}$ are the complete lists of pairwise non-isomorphic projective indecomposable and simple $H_n(0)$ -modules, respectively.

Krob and Thibon [14] introduced a correspondence between $H_n(0)$ -representations and the dual Hopf algebras QSym and NSym , which we review next. The Hopf algebra QSym has a free \mathbb{Z} -basis of *fundamental quasisymmetric functions* F_α , and the dual Hopf algebra NSym has a dual basis of *non-commutative ribbon Schur functions* \mathfrak{s}_α , for all compositions α .

Let $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_k \supseteq M_{k+1} = 0$ be a composition series of $H_n(0)$ -modules with simple factors $M_i/M_{i+1} \cong \mathbf{C}_{\alpha^{(i)}}$ for $i = 0, 1, \dots, k$. Then the *quasisymmetric characteristic* of M is

$$\text{Ch}(M) := F_{\alpha^{(0)}} + \cdots + F_{\alpha^{(k)}}.$$

The *noncommutative characteristic* of a projective $H_n(0)$ -module $M \cong \mathbf{P}_{\alpha(1)} \oplus \cdots \oplus \mathbf{P}_{\alpha(k)}$ is

$$\mathbf{ch}(M) := \mathbf{s}_{\alpha(1)} + \cdots + \mathbf{s}_{\alpha(k)}.$$

It is not hard to extend these characteristic maps to $H_n(0)$ -modules with gradings and filtrations.

3 Stanley-Reisner ring of the Boolean algebra

In this section we study the Stanley-Reisner ring of the Boolean algebra. The *Boolean algebra* \mathcal{B}_n is the ranked poset of all subsets of $[n] := \{1, 2, \dots, n\}$ ordered by inclusion, with minimum element \emptyset and maximum element $[n]$. The rank of a subset of $[n]$ is defined as its cardinality. The *Stanley-Reisner ring* $\mathbb{F}[\mathcal{B}_n]$ of the Boolean algebra \mathcal{B}_n is the quotient of the polynomial algebra $\mathbb{F}[y_A : A \subseteq [n]]$ by the ideal $(y_A y_B : A \text{ and } B \text{ are incomparable in } \mathcal{B}_n)$. It has an \mathbb{F} -basis $\{y_M\}$ indexed by the multichains M in \mathcal{B}_n , and is multigraded by the rank multisets $r(M)$ of the multichains M .

The symmetric group \mathfrak{S}_n acts on the Boolean algebra \mathcal{B}_n by permuting the integers $1, \dots, n$. This induces an \mathfrak{S}_n -action on the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$, preserving its multigrading. The *invariant algebra* $\mathbb{F}[\mathcal{B}_n]^{\mathfrak{S}_n}$ consists of all elements in $\mathbb{F}[\mathcal{B}_n]$ invariant under this \mathfrak{S}_n -action. One can show that $\mathbb{F}[\mathcal{B}_n]^{\mathfrak{S}_n} = \mathbb{F}[\Theta]$, where $\Theta := \{\theta_0, \dots, \theta_n\}$. Garsia [7] showed that $\mathbb{F}[\mathcal{B}_n]$ is a free $\mathbb{F}[\Theta]$ -module on the basis of descent monomials

$$Y_w := \prod_{i \in D(w)} y_{\{w(1), \dots, w(i)\}}, \quad \forall w \in \mathfrak{S}_n. \quad (3)$$

There is an analogy between the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ and the polynomial ring $\mathbb{F}[X]$ via the *transfer map* $\tau : \mathbb{F}[\mathcal{B}_n] \rightarrow \mathbb{F}[X]$ defined by

$$\tau(y_M) := \prod_{1 \leq i \leq k} \prod_{j \in A_i} x_j$$

for all multichains $M = (A_1 \subseteq \cdots \subseteq A_k)$ in \mathcal{B}_n . It is *not* a ring homomorphism (e.g. $y_{\{1\}} y_{\{2\}} = 0$ but $x_1 x_2 \neq 0$). Nevertheless, it induces an isomorphism $\tau : \mathbb{F}[\mathcal{B}_n]/(\theta_0) \cong \mathbb{F}[X]$ of \mathfrak{S}_n -modules. Moreover, it sends the rank polynomials $\theta_1, \dots, \theta_n$ to the elementary symmetric polynomials e_1, \dots, e_n , and sends the descent monomials Y_w in $\mathbb{F}[\mathcal{B}_n^*]$ defined by (3) to the corresponding descent monomials in $\mathbb{F}[X]$ for all $w \in \mathfrak{S}_n$.

Example 3.1 *The Boolean algebra \mathcal{B}_3 consists of all subsets of $\{1, 2, 3\}$. Its Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_3]$ is a free $\mathbb{F}[\Theta]$ -module with a basis of descent monomials $Y_1 = 1$, $Y_{s_1} = y_2$, $Y_{s_2 s_1} = y_3$, $Y_{s_2} = y_{13}$, $Y_{s_1 s_2} = y_{23}$, $Y_{s_1 s_2 s_1} = y_{23} y_3$, where Θ consists of the rank polynomials $\theta_0 = y_\emptyset$, $\theta_1 := y_1 + y_2 + y_3$, $\theta_2 := y_{12} + y_{13} + y_{23}$, $\theta_3 := y_{123}$. The transfer map τ sends $\theta_1, \theta_2, \theta_3$ to e_1, e_2, e_3 , and sends the six descent monomials in $\mathbb{F}[\mathcal{B}_3]$ to the six descent monomials $1, x_2, x_3, x_1 x_3, x_2 x_3, x_2 x_3^2$ in $\mathbb{F}[x_1, x_2, x_3]$.*

The homogeneous components of $\mathbb{F}[\mathcal{B}_n]$ are indexed by multisets with elements in $\{0, \dots, n\}$, or equivalently by weak compositions α of n . The α -homogeneous component $\mathbb{F}[\mathcal{B}_n]_\alpha$ has an \mathbb{F} -basis $\{y_M : r(M) = D(\alpha)\}$. Denote by $\text{Com}(n, k)$ the set of all weak compositions of n with length k . If $M = (A_1 \subseteq \cdots \subseteq A_k)$ is a multichain of length k in \mathcal{B}_n then we set $A_0 := \emptyset$ and $A_{k+1} := [n]$ by convention. Define $\alpha(M) := (\alpha_1, \dots, \alpha_{k+1})$, where $\alpha_i = |A_i| - |A_{i-1}|$ for all $i \in [k+1]$. Then

$\alpha(M) \in \text{Com}(n, k+1)$ and $D(\alpha(M)) = r(M)$, i.e. $\alpha(M)$ indexes the homogeneous component containing y_M . Define $\sigma(M)$ to be the minimal element in \mathfrak{S}_n which sends the standard multichain $[\alpha_1] \subseteq [\alpha_1 + \alpha_2] \subseteq \cdots \subseteq [\alpha_1 + \cdots + \alpha_k]$ with rank multiset $D(\alpha(M))$ to M . Then $\sigma(M) \in \mathfrak{S}^{\alpha(M)}$.

The map $M \mapsto (\alpha(M), \sigma(M))$ is a bijection between multichains of length k in \mathcal{B}_n and the pairs (α, σ) of $\alpha \in \text{Com}(n, k+1)$ and $\sigma \in \mathfrak{S}^\alpha$. A short way to write down this encoding of M is to insert bars at the descent positions of $\sigma(M)$. For example, the length-4 multichain $\{2\} \subseteq \{2\} \subseteq \{1, 2, 4\} \subseteq [4]$ in \mathcal{B}_4 is encoded by $2||14|3|$.

There is another way to encode the multichain M . Let $p_i(M) := \min\{j \in [k+1] : i \in A_j\}$, $1 \leq i \leq n$. So $p_i(M)$ is the first position where i appears in M . One checks that

$$\begin{cases} p_i(M) > p_{i+1}(M) \Leftrightarrow i \in D(\sigma(M)^{-1}), \\ p_i(M) = p_{i+1}(M) \Leftrightarrow i \notin D(\sigma(M)^{-1}), D(s_i\sigma(M)) \not\subseteq D(\alpha(M)), \\ p_i(M) < p_{i+1}(M) \Leftrightarrow i \notin D(\sigma(M)^{-1}), D(s_i\sigma(M)) \subseteq D(\alpha(M)). \end{cases} \quad (4)$$

The map $M \mapsto p(M) := (p_1(M), \dots, p_n(M))$ is an bijection between the set of multichains with length k in \mathcal{B}_n and the set $[k+1]^n$ of all words of length n on the alphabet $[k+1]$, for any fixed integer $k \geq 0$.

Let $p(M) = \mathbf{p} = (p_1, \dots, p_n) \in [k+1]^n$. Then $\text{inv}(p(M)) = \text{inv}(\sigma(M))$. Let $\mathbf{p}' := (p'_1, \dots, p'_k)$ where $p'_i := |\{j : p_j(M) \leq i\}| = |A_i|$. Then the rank multiset of M consists of p'_1, \dots, p'_k . Define $D(\mathbf{p}) := \{i \in [n-1] : p_i > p_{i+1}\}$. For example, the multichain $3|14||2|5$ corresponds to $\mathbf{p} = (2, 4, 1, 2, 5) \in [5]^5$, and one has $\mathbf{p}' = (1, 3, 3, 4)$, $D(2, 5, 1, 2, 4) = \{2\}$.

These two encodings (with slightly different notation) were already used by Garsia and Gessel [8] in their work on generating functions of multivariate distributions of permutation statistics.

4 0-Hecke algebra action

We saw an analogy between $\mathbb{F}[\mathcal{B}_n]$ and $\mathbb{F}[X]$ in the last section. The usual $H_n(0)$ -action on the polynomial ring $\mathbb{F}[X]$ is via the *Demazure operators*

$$\bar{\pi}_i(f) := \frac{x_{i+1}f - x_{i+1}s_i f}{x_i - x_{i+1}}, \quad \forall f \in \mathbb{F}[X], 1 \leq i \leq n-1. \quad (5)$$

The above definition is equivalent to

$$\bar{\pi}_i(x_i^a x_{i+1}^b m) = \begin{cases} (x_i^{a-1} x_{i+1}^{b+1} + x_i^{a-2} x_{i+1}^{b+2} \cdots + \underline{x_i^b x_{i+1}^a})m, & \text{if } a > b, \\ 0, & \text{if } a = b, \\ -(x_i^a x_{i+1}^b + x_i^{a+1} x_{i+1}^{b-1} + \cdots + x_i^{b-1} x_{i+1}^{a+1})m, & \text{if } a < b. \end{cases} \quad (6)$$

Here m is any monomial in $\mathbb{F}[X]$ containing neither x_i nor x_{i+1} . Denote by $\bar{\pi}'_i$ the operator obtained from (6) by taking only the leading term (underlined) in the lexicographic order of the result. Then $\bar{\pi}'_1, \dots, \bar{\pi}'_{n-1}$ realize another $H_n(0)$ -action on $\mathbb{F}[X]$. We call it the *transferred $H_n(0)$ -action* because it can be obtained by applying the transfer map τ to our $H_n(0)$ -action on $\mathbb{F}[\mathcal{B}_n]$, which we now define.

Let $M = (A_1 \subseteq \cdots \subseteq A_k)$ be a multichain in \mathcal{B}_n . Recall that $p_i(M) := \min\{j \in [k+1] : i \in A_j\}$, $1 \leq i \leq n$. We define

$$\bar{\pi}_i(y_M) := \begin{cases} -y_M, & p_i(M) > p_{i+1}(M), \\ 0, & p_i(M) = p_{i+1}(M), \\ s_i(y_M), & p_i(M) < p_{i+1}(M) \end{cases} \quad (7)$$

for $i = 1, \dots, n - 1$. Applying the transfer map τ one recovers $\bar{\pi}'_i$. For instance, when $n = 4$ one has

$$\begin{aligned} \bar{\pi}_1(y_{1|34||2|}) &= y_{2|34||1|}, & \bar{\pi}'_1(x_1^4 x_2 x_3^3 x_4^3) &= x_1 x_2^4 x_3^3 x_4^3, \\ \bar{\pi}_2(y_{1|34||2|}) &= -y_{1|34||2|}, & \bar{\pi}'_2(x_1^4 x_2 x_3^3 x_4^3) &= -x_1^4 x_2 x_3^3 x_4^3, \\ \bar{\pi}_3(y_{1|34||2|}) &= 0, & \bar{\pi}'_3(x_1^4 x_2 x_3^3 x_4^3) &= 0. \end{aligned}$$

One can check that $\bar{\pi}_1, \dots, \bar{\pi}_{n-1}$ realize an $H_n(0)$ -action on $\mathbb{F}[\mathcal{B}_n]$ preserving the multigrading of $\mathbb{F}[\mathcal{B}_n]$. If one set $t_i = t^i$ for $i = 1, \dots, n$, then there is an isomorphism $\mathbb{F}[\mathcal{B}_n]/(\emptyset) \cong \mathbb{F}[X]$ of graded $H_n(0)$ -modules (which can be given explicitly, but *not* via the transfer map τ).

It is not hard to show that $\mathbb{F}[\mathcal{B}_n]^{H_n(0)} = \mathbb{F}[\Theta]$, where $\mathbb{F}[\mathcal{B}_n]^{H_n(0)}$ is the *invariant algebra* of the $H_n(0)$ -action on $\mathbb{F}[\mathcal{B}_n]$, defined as

$$\mathbb{F}[\mathcal{B}_n]^{H_n(0)} := \{f \in \mathbb{F}[\mathcal{B}_n] : \pi_i f = f, i = 1, \dots, n - 1\}.$$

Proposition 4.1 *The $H_n(0)$ -action on $\mathbb{F}[\mathcal{B}_n]$ is Θ -linear.*

Therefore the *coinvariant algebra* $\mathbb{F}[\mathcal{B}_n]/(\Theta)$ is a multigraded $H_n(0)$ -module, which is isomorphic to the regular representation of $H_n(0)$ by Theorem 1.1. This cannot be obtained simply by applying the transfer map τ , since τ is *not* a map of $H_n(0)$ -modules.

5 Noncommutative characteristic

In this section we use the $H_n(0)$ -action on the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ of the Boolean algebra \mathcal{B}_n to provide a noncommutative analogue of the following remarkable result.

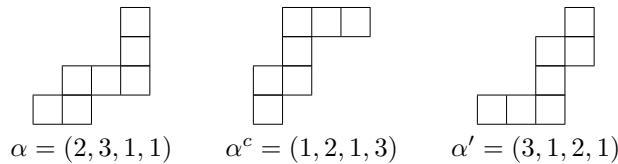
Theorem 5.1 (Hotta-Springer [11], Garsia-Procesi [9]) *For any partition $\mu = (0 < \mu_1 \leq \dots \leq \mu_k)$ of n , there exists an \mathfrak{S}_n -invariant ideal J_μ of $\mathbb{C}[X]$ such that $\mathbb{C}[X]/J_\mu$ is isomorphic to the cohomology ring of the Springer fiber indexed by μ and has graded Frobenius characteristic equal to the modified Hall-Littlewood symmetric function*

$$\tilde{H}_\mu(X; t) = \sum_\lambda t^{n(\mu)} K_{\lambda\mu}(t^{-1}) s_\lambda \quad \text{inside } \text{Sym}[t]$$

where $n(\mu) := \mu_{k-1} + 2\mu_{k-2} + \dots + (k-1)\mu_1$ and $K_{\lambda\mu}(t)$ is the Kostka-Foulkes polynomial.

Example 5.2 *Tanisaki [19] gives a construction for the ideal J_μ . If $\mu = (1^k, n - k)$ is a hook then $J_{1^k, n-k}$ is generated by e_1, \dots, e_k and all monomials $x_{i_1} \dots x_{i_{k+1}}$ with $1 \leq i_1 < \dots < i_{k+1} \leq n$.*

Now consider a composition $\alpha = (\alpha_1, \dots, \alpha_\ell)$. The major index of α is $\text{maj}(\alpha) := \sum_{i \in D(\alpha)} i$. Viewing a partition $\mu = (0 < \mu_1 \leq \mu_2 \leq \dots)$ as a composition one has $\text{maj}(\mu) = n(\mu)$. Recall that $\overleftarrow{\alpha} := (\alpha_\ell, \dots, \alpha_1)$ and α^c is the composition of n with $D(\alpha^c) = [n-1] \setminus D(\alpha)$. We define $\alpha' := \overleftarrow{\alpha^c} = (\overleftarrow{\alpha})^c$. One can identify α with a *ribbon diagram*, i.e. a connected skew Young diagram without 2 by 2 boxes, which has row lengths $\alpha_1, \dots, \alpha_\ell$, ordered from bottom to top. Note that a ribbon diagram is a Young diagram if and only if it is a hook. One can check that α' is the transpose of α ; see the example below.



Bergeron and Zabrocki [3] introduced a noncommutative modified Hall-Littlewood symmetric function

$$\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t) := \sum_{\beta \preceq \alpha} t^{\text{maj}(\beta)} \mathbf{s}_\beta \quad \text{inside } \mathbf{NSym}[t] \quad (8)$$

and a (q, t) -analogue

$$\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t) := \sum_{\beta \models n} t^{c(\alpha, \beta)} q^{c(\alpha', \bar{\beta})} \mathbf{s}_\beta \quad \text{inside } \mathbf{NSym}[q, t] \quad (9)$$

for every composition α , where \mathbf{s}_β is the noncommutative ribbon Schur function indexed by β , and $c(\alpha, \beta) := \sum_{i \in D(\alpha) \cap D(\beta)} i$. In our earlier work [13] we provided a partial representation theoretic interpretation for $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$ when $\alpha = (1^k, n - k)$ is a hook, using the $H_n(0)$ -action on the polynomial ring $\mathbb{F}[X]$ by the Demazure operators.

Theorem 5.3 ([13]) *The ideal J_μ of $\mathbb{F}[X]$ is $H_n(0)$ -invariant if and only if $\mu = (1^{n-k}, k)$ is a hook, and if that holds then $\mathbb{F}[X]/J_\mu$ becomes a graded projective $H_n(0)$ -module with*

$$\begin{aligned} \text{ch}_t(\mathbb{F}[X]/J_\mu) &= \tilde{\mathbf{H}}_\mu(\mathbf{x}; t), \\ \text{Ch}_t(\mathbb{F}[X]/J_\mu) &= \tilde{H}_\mu(\mathbf{x}; t). \end{aligned}$$

Now we switch to the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ and define I_α to be its ideal generated by

$$\Theta_\alpha := \{\theta_i : i \in D(\alpha) \cup \{n\}\} \quad \text{and} \quad \{y_A : A \subseteq [n], |A| \notin D(\alpha) \cup \{n\}\}$$

for any composition α of n . The following result is a restatement of Theorem 1.1.

Theorem 5.4 *Let α be a composition of n . Then $\mathbb{F}[\mathcal{B}_n]/I_\alpha$ is a projective $H_n(0)$ -module with multi-graded noncommutative characteristic equal to*

$$\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \dots, t_{n-1}) := \sum_{\beta \preceq \alpha} t^{D(\beta)} \mathbf{s}_\beta \quad \text{inside } \mathbf{NSym}[t_1, \dots, t_{n-1}].$$

One has $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t, t^2, \dots, t^{n-1}) = \tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$, and one obtains $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$ from $\tilde{\mathbf{H}}_{1^n}(\mathbf{x}; t_1, \dots, t_{n-1})$ by taking $t_i = t^i$ for all $i \in D(\alpha)$, and $t_i = q^{n-i}$ for all $i \in [n-1] \setminus D(\alpha)$.

Proof: There is an \mathbb{F} -basis for $\mathbb{F}[\mathcal{B}_n]/(\Theta_\alpha)$ given by the descent monomials Y_w defined in (3) for all $w \in \mathfrak{S}^\alpha$. The result follows from the $H_n(0)$ -action on this basis and (4). \square

The proof of this theorem is actually simpler than the proof of our partial interpretation for $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$ in [13]. This is because $\bar{\pi}_i$ sends a descent monomial in $\mathbb{F}[\mathcal{B}_n]$ to either 0 or ± 1 times a descent monomial, but sends a descent monomial in $\mathbb{F}[X]$ to a polynomial in general (whose leading term is still a descent monomial). We view the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ (or $\mathbb{F}[\mathcal{B}_n]/(\emptyset)$) as a $q = 0$ analogue of the polynomial ring $\mathbb{F}[X]$. For an odd (i.e. $q = -1$) analogue, see Lauda and Russell [16].

Remark 5.5 *If $\alpha = (1^k, n - k)$ is a hook, one can check that the ideal $I_{1^k, n-k}$ of $\mathbb{F}[\mathcal{B}_n]$ has generators $\theta_1, \dots, \theta_k$ and all y_A with $A \subseteq [n]$ and $|A| \notin [k]$. By Example 5.2, the images of these generators under the transfer map τ are the Tanisaki generators for the ideal $J_{1^k, n-k}$ of $\mathbb{F}[X]$, but $\tau(I_{1^k, n-k}) \neq J_{1^k, n-k}$.*

For any composition $\alpha \models n$, one can view $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \dots, t_{n-1})$ as a modified version of

$$\mathbf{H}_\alpha = \mathbf{H}_\alpha(\mathbf{x}; t_1, \dots, t_{n-1}) := \sum_{\beta \preceq \alpha} \underline{t}^{D(\alpha) \setminus D(\beta)} \mathbf{s}_\beta.$$

Below are some properties satisfied by \mathbf{H}_α , generalizing the properties of $\mathbf{H}_\alpha(\mathbf{x}; t)$ given in [3].

Proposition 5.6 *Let α and β be two compositions.*

(i) $\mathbf{H}_\alpha(0, \dots, 0) = \mathbf{s}_\alpha$, $\mathbf{H}_\alpha(1, \dots, 1) = \mathbf{h}_\alpha$.

(ii) $\bigcup_{n \geq 0} \{\mathbf{H}_\alpha : \alpha \models n\}$ is a basis for $\mathbf{NSym}[t_1, t_2, \dots]$.

(iii) $\langle \mathbf{H}_\alpha, \mathbf{H}_\beta \rangle = (-1)^{|\alpha| + \ell(\alpha)} \delta_{\alpha, \beta^c}$ for any pair of compositions α and β .

(iv)

$$\mathbf{H}_\alpha \cdot \mathbf{H}_\beta = \sum_{\gamma \preceq \beta} \left(\prod_{i \in D(\beta) \setminus D(\gamma)} (t_i - t_{|\alpha|+i}) \right) (\mathbf{H}_{\alpha\gamma} + (1 - t_{|\alpha|}) \mathbf{H}_{\alpha \triangleright \gamma}).$$

(v) If $n = |\alpha|$ and $t|n := (t_1, \dots, t_{n-1}, 1, t_1, \dots, t_{n-1}, 1, \dots)$ then

$$\mathbf{H}_\alpha(\mathbf{x}; t_1, \dots, t_{n-1}) \mathbf{H}_\beta(\mathbf{x}; t|n) = \mathbf{H}_{\alpha\beta}(t|n).$$

6 Quasisymmetric characteristic

Now we study the quasisymmetric characteristic of $\mathbb{F}[\mathcal{B}_n]$. The following lemma follows easily from (4).

Lemma 6.1 *Let α be a weak composition of n . Then the α -homogeneous component $\mathbb{F}[\mathcal{B}_n]_\alpha$ of the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ is an $H_n(0)$ -submodule of $\mathbb{F}[\mathcal{B}_n]$ with homogeneous multigrading $\underline{t}^{D(\alpha)}$ and isomorphic to the cyclic module $H_n(0)\pi_{w_0(\alpha^c)}$.*

Since $\mathbb{F}[\mathcal{B}_n]_\alpha$ is a cyclic multigraded $H_n(0)$ -module, we get an $\mathbb{N} \times \mathbb{N}^{n+1}$ -multigraded quasisymmetric characteristic

$$\text{Ch}_{q, \underline{t}}(\mathbb{F}[\mathcal{B}_n]_\alpha) = \sum_{w \in \mathfrak{S}^\alpha} q^{\text{inv}(w)} \underline{t}^{D(\alpha)} F_{D(w^{-1})} \quad (10)$$

where q keeps track of the length filtration and \underline{t} keeps track of the multigrading of $\mathbb{F}[\mathcal{B}_n]_\alpha$. This defines an $\mathbb{N} \times \mathbb{N}^{n+1}$ -multigraded quasisymmetric characteristic for the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$, which is explicitly given in Theorem 1.2 and restated below.

Theorem 6.2 *The $\mathbb{N} \times \mathbb{N}^{n+1}$ -multigraded quasisymmetric characteristic of $\mathbb{F}[\mathcal{B}_n]$ is*

$$\begin{aligned} \text{Ch}_{q, \underline{t}}(\mathbb{F}[\mathcal{B}_n]) &= \sum_{k \geq 0} \sum_{\alpha \in \text{Com}(n, k+1)} \underline{t}^{D(\alpha)} \sum_{w \in \mathfrak{S}^\alpha} q^{\text{inv}(w)} F_{D(w^{-1})} \\ &= \sum_{w \in \mathfrak{S}_n} \frac{q^{\text{inv}(w)} \underline{t}^{D(w)} F_{D(w^{-1})}}{\prod_{0 \leq i \leq n} (1 - t_i)} \\ &= \sum_{k \geq 0} \sum_{\mathbf{p} \in [k+1]^n} t_{p'_1} \cdots t_{p'_k} q^{\text{inv}(\mathbf{p})} F_{D(\mathbf{p})}. \end{aligned}$$

Proof: Use the two encodings of the multichains in \mathcal{B}_n as well as the free $\mathbb{F}[\Theta]$ -basis $\{Y_w : w \in \mathfrak{S}_n\}$ of descent monomials for $\mathbb{F}[\mathcal{B}_n]$ discussed in Section 3. \square

Next we explain here how this theorem specializes to (1), a result of Garsia and Gessel [8, Theorem 2.2] on the multivariate generating function of the permutation statistics $\text{inv}(w)$, $\text{maj}(w)$, $\text{des}(w)$, $\text{maj}(w^{-1})$, and $\text{des}(w^{-1})$ for all $w \in \mathfrak{S}_n$. First recall that

$$F_\alpha = \sum_{\substack{i_1 \geq \dots \geq i_n \geq 1 \\ i \in D(\alpha) \Rightarrow i_j > i_{j+1}}} x_{i_1} \cdots x_{i_n}, \quad \forall \alpha \models n.$$

Let $\mathbf{ps}_{q;\ell}(F_\alpha) := F_\alpha(1, q, q^2, \dots, q^{\ell-1}, 0, 0, \dots)$, and $(u; q)_n := (1-u)(1-qu)(1-q^2u) \cdots (1-q^nu)$. It is not hard to check (see Gessel and Reutenauer [10, Lemma 5.2]) that

$$\sum_{\ell \geq 0} u^\ell \mathbf{ps}_{q;\ell+1}(F_\alpha) = \frac{q^{\text{maj}(\alpha)} u^{\text{des}(\alpha)}}{(u; q)_n}.$$

Then applying the linear transformation $\sum_{\ell \geq 0} u_1^\ell \mathbf{ps}_{q_1;\ell+1}$ and the specialization $t_i = q_2^i u_2$ for all $i = 0, 1, \dots, n$ to Theorem 6.2 we recover (1).

A further specialization of Theorem 6.2 gives a well known result which is often attributed to Carlitz [6] but actually dates back to MacMahon [17, Volume 2, Chapter 4].

Corollary 6.3 (Carlitz-MacMahon) *Let $[k+1]_q := 1 + q + q^2 + \dots + q^k$. Then*

$$\frac{\sum_{w \in \mathfrak{S}_n} q^{\text{maj}(w)} u^{\text{des}(w)}}{(u; q)_n} = \sum_{k \geq 0} ([k+1]_q)^n u^k.$$

Theorem 6.2 also implies the following result, which was obtained by Adin, Brenti, and Roichman [1] from the Hilbert series of the coinvariant algebra $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$.

Corollary 6.4 (Adin-Brenti-Roichman) *Let $\text{Par}(n)$ be the set of weak partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \dots \geq \lambda_n \geq 0$, and let $m(\lambda) = (m_0(\lambda), m_1(\lambda), \dots)$, where $m_j(\lambda) := \#\{1 \leq i \leq n : \lambda_i = j\}$. Then*

$$\sum_{\lambda \in \text{Par}(n)} \binom{n}{m(\lambda)} \prod_{i=1}^n q_i^{\lambda_i} = \frac{\sum_{w \in \mathfrak{S}_n} \prod_{i \in D(w)} q_1 \cdots q_i}{(1-q_1)(1-q_1q_2) \cdots (1-q_1 \cdots q_n)}.$$

7 Remarks and questions for future research

7.1 Hecke algebra action

It is well known that the symmetric group \mathfrak{S}_n is the Coxeter group of type A_{n-1} . The Stanley-Reisner ring of \mathcal{B}_n is essentially the Stanley-Reisner ring of the Coxeter complex of \mathfrak{S}_n . The Hecke algebra $H_W(q)$ can be defined for any finite Coxeter group W . We can generalize our action $H_n(0)$ -action on $\mathbb{F}[\mathcal{B}_n]$ to an $H_W(q)$ -action on the Stanley-Reisner ring $\mathbb{F}(q)[\Delta(W)]$ of the Coxeter complex $\Delta(W)$ of any finite Coxeter group W . We show similar results for this $H_W(q)$ -action.

7.2 Gluing the group algebra and the 0-Hecke algebra

The group algebra $\mathbb{F}W$ of a finite Coxeter group W naturally admits both actions of W and $H_W(0)$. Hivert and Thiéry [12] defined the *Hecke group algebra* of W by gluing these two actions. In type A , one can also glue the usual actions of \mathfrak{S}_n and $H_n(0)$ on the polynomial ring $\mathbb{F}[X]$, but the resulting algebra is different from the Hecke group algebra of \mathfrak{S}_n .

Now one has a W -action and an $H_W(0)$ -action on the Stanley-Reisner ring $\mathbb{F}[\Delta(W)]$. What can we say about the algebra generated by the operators s_i and $\bar{\pi}_i$ on $\mathbb{F}[\Delta(W)]$? Is it the same as the Hecke group algebra of W ? If not, what properties (dimension, bases, presentation, simple and projective indecomposable modules, etc.) does it have?

7.3 Tits Building

Let $\Delta(G)$ be the Tits building of the general linear group $G = GL(n, \mathbb{F}_q)$ and its usual BN-pair over a finite field \mathbb{F}_q ; see e.g. Björner [4]. The Stanley-Reisner ring $\mathbb{F}[\Delta(G)]$ is a q -analogue of $\mathbb{F}[\mathcal{B}_n]$. The nonzero monomials in $\mathbb{F}[\Delta(G)]$ are indexed by multiflags of subspaces of \mathbb{F}_q^n , and there are $q^{\text{inv}(w)}$ many multiflags corresponding to a given multichain M in \mathcal{B}_n , where $w = \sigma(M)$. Can one obtain the multivariate quasisymmetric function identities in Theorem 1.2 by defining a nice $H_n(0)$ -action on $\mathbb{F}[\Delta(G)]$?

Acknowledgements

The author is grateful to Victor Reiner for providing valuable suggestions. He also thanks Ben Braun and Jean-Yves Thibon for helpful conversations and email correspondence.

References

- [1] R. Adin, F. Brenti, and Y. Roichman, Descent representations and multivariate statistics, *Trans. Amer. Math. Soc.* 357 (2005) 3051–3082.
- [2] I. Assem, D. Simson, and A. Skowroński, *Elements of the representation theory of associative algebras*, vol. 1: Techniques of representation theory, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006.
- [3] N. Bergeron and M. Zabrocki, q and q, t -analogs of non-commutative symmetric functions, *Discrete Math.* 298 (2005) 79–103.
- [4] A. Björner, Some combinatorial and algebraic properties of Coxeter complexes and Tits buildings, *Adv. Math.* 52 (1984) 173–212.
- [5] A. Björner and M. Wachs, Generalized quotients in Coxeter groups, *Trans. Amer. Math. Soc.* 308 (1988) 1–37.
- [6] L. Carlitz, A combinatorial property of q -Eulerian numbers, *Amer. Monthly*, 82 (1975) 51–54.
- [7] A. Garsia, Combinatorial methods in the theory of Cohen-Macaulay rings, *Adv. Math.* 38 (1980) 229–266.

- [8] A. Garsia and I. Gessel, Permutation statistics and partitions, *Adv. Math.* 31 (1979) 288–305.
- [9] A. Garsia and C. Procesi, On certain graded S_n -modules and the q -Kostka polynomials, *Adv. Math.* 94 (1992) 82–138.
- [10] I. Gessel and C. Reutenauer, Counting permutations with given cycle structure and descent set, *J. Combin. Theory Ser. A* 64 (1993) 189–215.
- [11] R. Hotta and T. A. Springer, A specialization theorem for certain Weyl group representations and an application to the Green polynomials of unitary groups, *Invent. Math.* 41 (1977), 113–127.
- [12] F. Hivert and N. Thiéry, The Hecke group algebra of a Coxeter group and its representation theory, *J. Algebra*, 321(2009) 2230–2258.
- [13] J. Huang, 0-Hecke algebra actions on coinvariants and flags, to appear in *J. Algebraic Combin.*.
- [14] D. Krob and J.-Y. Thibon, Noncommutative symmetric functions IV: Quantum linear groups and Hecke algebras at $q = 0$, *J. Algebraic Combin.* 6 (1997) 339–376.
- [15] A. Lascoux, J.-C. Novelli, and J.-Y. Thibon, Noncommutative symmetric functions with matrix parameters, *J. Algebr. Comb.* 37 (2013) 621–642.
- [16] A. Lauda and H. Russell, Oddification of the cohomology of type A Springer varieties, [arXiv:1203.0797](https://arxiv.org/abs/1203.0797).
- [17] P.A. MacMahon, *Combinatorial Analysis*, Vols. 1 and 2, Cambridge University Press 1915–1916; reprinted by Chelsea 1960.
- [18] P.N. Norton, 0-Hecke algebras, *J. Austral. Math. Soc. A* 27 (1979) 337–357.
- [19] T. Tanisaki, Defining ideals of the closures of conjugacy classes and representations of the Weyl groups, *Tohoku Math. J.* 34 (1982), 575–585.