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0-Hecke algebra action on the Stanley-Reisner ring of the Boolean algebra

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Abstract. We define an action of the 0-Hecke algebra of type A on the Stanley-Reisner ring of the Boolean algebra. By studying this action we obtain a family of multivariate noncommutative symmetric functions, which specialize to the noncommutative Hall-Littlewood symmetric functions and their (q, t) -analogues introduced by Bergeron and Zabrocki. We also obtain multivariate quasisymmetric function identities, which specialize to a result of Garsia and Gessel on the generating function of the joint distribution of five permutation statistics.

Résumé. Nous définissons une action de l'algèbre de Hecke-0 de type A sur l'anneau Stanley-Reisner de l'algèbre de Boole. En étudiant cette action, on obtient une famille de fonctions symétriques non commutatives multivariées, qui se spécialisent pour les non commutatives fonctions de Hall-Littlewood symétriques et leur (q, t) -analogues introduits par Bergeron et Zabrocki. Nous obtenons également des identités de fonction quasisymétrique multivariées, qui se spécialisent à la suite de Garsia et Gessel sur la fonction génératrice de la distribution conjointe de cinq statistiques de permutation.

Keywords: 0-Hecke algebra, Stanley-Reisner ring, Boolean algebra, noncommutative Hall-Littlewood symmetric function, multivariate quasisymmetric function.

1 Introduction

Let \mathbb{F} be any field. The symmetric group \mathfrak{S}_n naturally acts on the polynomial ring $\mathbb{F}[X] := \mathbb{F}[x_1, \dots, x_n]$ by permuting the variables x_1, \dots, x_n . The invariant algebra $\mathbb{F}[X]^{\mathfrak{S}_n}$, which consists of all the polynomials fixed by this \mathfrak{S}_n -action, is a polynomial algebra generated by the elementary symmetric functions e_1, \dots, e_n . The coinvariant algebra $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$, with $(\mathbb{F}[X]_+^{\mathfrak{S}_n}) = (e_1, \dots, e_n)$, is a vector space of dimension $n!$ over \mathbb{F} , and when \mathbb{F} has characteristic larger than n the coinvariant algebra carries the regular representation of \mathfrak{S}_n . A well known basis for $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ consists of the descent monomials. Garsia [7] obtained this basis by transferring a natural basis from the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ of the Boolean algebra \mathcal{B}_n to the polynomial ring $\mathbb{F}[X]$. Here the *Boolean algebra* \mathcal{B}_n is the set of all subsets of $[n] := \{1, 2, \dots, n\}$ partially ordered by inclusion, and the *Stanley-Reisner ring* $\mathbb{F}[\mathcal{B}_n]$ is the quotient of the polynomial algebra $\mathbb{F}[y_A : A \subseteq [n]]$ by the ideal $(y_A y_B : A \text{ and } B \text{ are incomparable in } \mathcal{B}_n)$.

The 0-Hecke algebra $H_n(0)$ (of type A) is a deformation of the group algebra of \mathfrak{S}_n . It acts on $\mathbb{F}[X]$ by the Demazure operators, also known as the isobaric divided difference operators, having the same

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invariant algebra as the \mathfrak{S}_n -action on $\mathbb{F}[X]$. In our earlier work [13], we showed that the coinvariant algebra $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ is also isomorphic to the regular representation of $H_n(0)$, for any field \mathbb{F} , by constructing another basis for $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ which consists of certain polynomials whose leading terms are the descent monomials. This and the previously mentioned connection between the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ and the polynomial ring $\mathbb{F}[X]$ motivate us to define an $H_n(0)$ -action on $\mathbb{F}[\mathcal{B}_n]$.

It turns out that our $H_n(0)$ -action on $\mathbb{F}[\mathcal{B}_n]$ has similar properties to the $H_n(0)$ -action on $\mathbb{F}[X]$. It preserves an \mathbb{N}^{n+1} -multigrading of $\mathbb{F}[\mathcal{B}_n]$ and has invariant algebra equal to a polynomial algebra $\mathbb{F}[\Theta]$, where Θ is the set of *rank polynomials* θ_i (the usual analogue of e_i in $\mathbb{F}[\mathcal{B}_n]$). We show that the $H_n(0)$ -action is Θ -linear and thus descends to the coinvariant algebra $\mathbb{F}[\mathcal{B}_n]/(\Theta)$. Using the descent monomials in $\mathbb{F}[\mathcal{B}_n]$ it is not hard to see that $\mathbb{F}[\mathcal{B}_n]/(\Theta)$ carries the regular representation of $H_n(0)$.

It is well known that every finite dimensional (complex) \mathfrak{S}_n -representation is a direct sum of simple (i.e. irreducible) \mathfrak{S}_n -modules, and the simple \mathfrak{S}_n -modules are indexed by partitions λ of n , which correspond to the Schur functions s_λ via the *Frobenius characteristic map*. Hotta-Springer [11] and Garsia-Procesi [9] discovered that the cohomology ring of the *Springer fiber* indexed by a partition μ of n is isomorphic to certain quotient ring R_μ of $\mathbb{F}[X]$, which admits a graded \mathfrak{S}_n -module structure corresponding to the *modified Hall-Littlewood symmetric function* $\tilde{H}_\mu(x; t)$ via the Frobenius characteristic map. The coinvariant algebra of \mathfrak{S}_n is nothing but R_{1^n} .

In our previous work [13] we established a partial analogue of the above result by showing that the $H_n(0)$ -action on $\mathbb{F}[X]$ descends to R_μ if and only if $\mu = (1^k, n - k)$ is a hook, and if so then R_μ has graded quasisymmetric characteristic equal to $\tilde{H}_\mu(x; t)$ and graded noncommutative characteristic $\tilde{\mathbf{H}}_\mu(\mathbf{x}; t)$. Here $\tilde{\mathbf{H}}_\mu(\mathbf{x}; t)$ is the noncommutative modified Hall-Littlewood symmetric function introduced by Bergeron and Zabrocki [3] for any composition α of n . Using an analogue of the nabla operator Bergeron and Zabrocki [3] also introduced a (q, t) -analogue $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$ for any composition α . Now we provide in Theorem 1.1 below a complete representation theoretic interpretation for $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$ and $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$ by the $H_n(0)$ -action on $\mathbb{F}[\mathcal{B}_n]$.

To state our result, we first recall the two characteristic maps for representations of $H_n(0)$ introduced by Krob and Thibon [14], which we call the *quasisymmetric characteristic* and the *noncommutative characteristic*. The simple $H_n(0)$ -modules are indexed by compositions α of n and correspond to the *fundamental quasisymmetric functions* F_α via the quasisymmetric characteristic; the projective indecomposable $H_n(0)$ -modules are also indexed by compositions α of n and correspond to the *noncommutative ribbon Schur functions* \mathbf{s}_α via the noncommutative characteristic. See §2 for details.

Theorem 1.1 *Let α be a composition of n . Then there exists a homogeneous $H_n(0)$ -invariant ideal I_α of the multigraded algebra $\mathbb{F}[\mathcal{B}_n]$ such that the quotient algebra $\mathbb{F}[\mathcal{B}_n]/I_\alpha$ becomes a projective $H_n(0)$ -module with multigraded noncommutative characteristic equal to*

$$\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \dots, t_{n-1}) := \sum_{\beta \preceq \alpha} \underline{t}^{D(\beta)} \mathbf{s}_\beta \quad \text{inside} \quad \mathbf{NSym}[t_1, \dots, t_{n-1}].$$

One has $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t, t^2, \dots, t^{n-1}) = \tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$, and obtains $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$ from $\tilde{\mathbf{H}}_{1^n}(\mathbf{x}; t_1, \dots, t_{n-1})$ by taking $t_i = t^i$ for all $i \in D(\alpha)$, and $t_i = q^{n-i}$ for all $i \in [n-1] \setminus D(\alpha)$.

Here $D(\alpha)$ is the set of partial sums of α , the notation $\beta \preceq \alpha$ means α and β are compositions of n with $D(\beta) \subseteq D(\alpha)$, and \underline{t}^S denotes the product $\prod_{i \in S} t_i$ over all elements i in a multiset S , including the repeated ones. Taking $\alpha = (1^n)$ shows that $\mathbb{F}[\mathcal{B}_n]/(\Theta)$ carries the regular representation of $H_n(0)$.

Specializations of $\tilde{\mathbf{H}}_{1^n}(\mathbf{x}; t_1, \dots, t_{n-1})$ include not only $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$, but also a more general family of noncommutative symmetric functions depending on parameters associated with paths in binary trees introduced recently by Lascoux, Novelli, and Thibon [15].

Next we study the quasisymmetric characteristic of $\mathbb{F}[\mathcal{B}_n]$. We combine the usual \mathbb{N}^{n+1} -multigrading of $\mathbb{F}[\mathcal{B}_n]$ (recorded by $\underline{t} := t_0, \dots, t_n$) with the length filtration of $H_n(0)$ (recorded by q) and obtain an $\mathbb{N} \times \mathbb{N}^{n+1}$ -multigraded quasisymmetric characteristic for $\mathbb{F}[\mathcal{B}_n]$.

Theorem 1.2 *The $\mathbb{N} \times \mathbb{N}^{n+1}$ -multigraded quasisymmetric characteristic of $\mathbb{F}[\mathcal{B}_n]$ is*

$$\begin{aligned} \text{Ch}_{q, \underline{t}}(\mathbb{F}[\mathcal{B}_n]) &= \sum_{k \geq 0} \sum_{\alpha \in \text{Com}(n, k+1)} \underline{t}^{D(\alpha)} \sum_{w \in \mathfrak{S}^\alpha} q^{\text{inv}(w)} F_{D(w^{-1})} \\ &= \sum_{w \in \mathfrak{S}_n} \frac{q^{\text{inv}(w)} \underline{t}^{D(w)} F_{D(w^{-1})}}{\prod_{0 \leq i \leq n} (1 - t_i)} \\ &= \sum_{k \geq 0} \sum_{\mathbf{p} \in [k+1]^n} t_{p'_1} \cdots t_{p'_k} q^{\text{inv}(\mathbf{p})} F_{D(\mathbf{p})}. \end{aligned}$$

Here we identify F_I with F_α if $D(\alpha) = I \subseteq [n-1]$. The set $\text{Com}(n, k)$ consists of all *weak compositions of n with length k* , i.e. all the sequences $\alpha = (\alpha_1, \dots, \alpha_k)$ of k nonnegative integers with $|\alpha| := \sum_{i=1}^k \alpha_i = n$. The *descent multiset* of the weak composition α is the *multiset*

$$D(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \cdots + \alpha_{k-1}\}.$$

We also define $\mathfrak{S}^\alpha := \{w \in \mathfrak{S}_n : D(w) \subseteq D(\alpha)\}$. The set $[k+1]^n$ consists of all words of length n on the alphabet $[k+1]$. Given $\mathbf{p} = (p_1, \dots, p_n) \in [k+1]^n$, we write $p'_i := \#\{j : p_j \leq i\}$, $\text{inv}(\mathbf{p}) := \#\{(i, j) : 1 \leq i < j \leq n : p_i > p_j\}$, and $D(\mathbf{p}) := \{i : p_i > p_{i+1}\}$.

Let $\mathbf{ps}_{q; \ell}(F_\alpha) := F_\alpha(1, q, q^2, \dots, q^{\ell-1}, 0, 0, \dots)$. Applying the linear transformation $\sum_{\ell \geq 0} u_1^\ell \mathbf{ps}_{q_1; \ell+1}$ and the specialization $t_i = q_2^i u_2$ for all $i = 0, 1, \dots, n$ to Theorem 1.2, we recover a result of Garsia and Gessel [8, Theorem 2.2] on the generating function of the joint distribution of five permutation statistics:

$$\frac{\sum_{w \in \mathfrak{S}_n} q_0^{\text{inv}(w)} q_1^{\text{maj}(w^{-1})} u_1^{\text{des}(w^{-1})} q_2^{\text{maj}(w)} u_2^{\text{des}(w)}}{(u_1; q_1)_n (u_2; q_2)_n} = \sum_{\ell, k \geq 0} u_1^\ell u_2^k \sum_{(\lambda, \mu) \in B(\ell, k)} q_0^{\text{inv}(\mu)} q_1^{|\lambda|} q_2^{|\mu|} \quad (1)$$

Here $(u; q)_n := \prod_{0 \leq i \leq n} (1 - q^i u)$, the set $B(\ell, k)$ consists of pairs of weak compositions $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ satisfying the conditions $\ell \geq \lambda_1 \geq \cdots \geq \lambda_n$, $\max\{\mu_i : 1 \leq i \leq n\} \leq k$, and $\lambda_i = \lambda_{i+1} \Rightarrow \mu_i \geq \mu_{i+1}$ (such pairs (λ, μ) are sometimes called *bipartite partitions*), and $\text{inv}(\mu)$ is the number of inversion pairs in μ . Some further specializations of Theorem 1.2 imply identities of Carlitz-MacMahon [6, 17] and Adin-Brenti-Roichman [1].

The structure of this paper is as follows. Section 2 reviews the representation theory of the 0-Hecke algebra. Section 3 studies the Stanley-Reisner ring of the Boolean algebra. Section 4 defines a 0-Hecke algebra action on the Stanley-Reisner ring of the Boolean algebra. The noncommutative and quasisymmetric characteristics are discussed in Section 5 and Section 6. Finally we give some remarks and questions for future research in Section 7, including a generalization to an action of the Hecke algebra of any finite Coxeter group on the Stanley-Reisner ring of the Coxeter complex.

2 Representation theory of the 0-Hecke algebra

We review the representation theory of the 0-Hecke algebra in this section. The (type A) *Hecke algebra* $H_n(q)$ is the associative $\mathbb{F}(q)$ -algebra generated by T_1, \dots, T_{n-1} with relations

$$\begin{cases} (T_i + 1)(T_i - q) = 0, & 1 \leq i \leq n-1, \\ T_i T_j = T_j T_i, & 1 \leq i, j \leq n-1, |i-j| > 1, \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, & 1 \leq i \leq n-2. \end{cases} \quad (2)$$

It has an $\mathbb{F}(q)$ -basis $\{T_w : w \in \mathfrak{S}_n\}$ where $T_w := T_{i_1} \cdots T_{i_k}$ if $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression.

Specializing $q = 1$ gives the group algebra of \mathfrak{S}_n , with $s_i = T_i|_{q=1}$ and $w = T_w|_{q=1}$. Let $w \in \mathfrak{S}_n$. The *length* of w equals $\text{inv}(w) := \#\{(i, j) : 1 \leq i < j \leq n, w(i) > w(j)\}$, and the *descent set* of w is $D(w) = \{i : 1 \leq i \leq n-1, w(i) > w(i+1)\}$. We write $\text{des}(w) := |D(w)|$ and $\text{maj}(w) := \sum_{i \in D(w)} i$.

Let α be a (weak) composition of n , and let α^c be the composition of n with $D(\alpha^c) = [n-1] \setminus D(\alpha)$. The *parabolic subgroup* \mathfrak{S}_α is the subgroup of \mathfrak{S}_n generated by $\{s_i : i \in D(\alpha^c)\}$. The set of all minimal \mathfrak{S}_α -coset representatives is $\mathfrak{S}^\alpha := \{w \in \mathfrak{S}_n : D(w) \subseteq D(\alpha)\}$. The *descent class* of α consists of the permutations in \mathfrak{S}_n with descent set equal to $D(\alpha)$, and turns out to be an interval under the (left) weak order of \mathfrak{S}_n , denoted by $[w_0(\alpha), w_1(\alpha)]$. One sees that $w_0(\alpha)$ is the longest element of the parabolic subgroup \mathfrak{S}_{α^c} , and $w_1(\alpha)$ is the longest element in \mathfrak{S}^α (c.f. Björner and Wachs [5, Theorem 6.2]).

Another interesting specialization of $H_n(q)$ is the *0-Hecke algebra* $H_n(0)$, with generators $\bar{\pi}_i = T_i|_{q=0}$ for $i = 1, \dots, n-1$, and an \mathbb{F} -basis $\{\bar{\pi}_w = T_w|_{q=0} : w \in \mathfrak{S}_n\}$. Let $\pi_i := \bar{\pi}_i + 1$. Then π_1, \dots, π_{n-1} form another generating set for $H_n(0)$, with the same relations as (2) except $\pi_i^2 = \pi_i$, $1 \leq i \leq n-1$. The element $\pi_w := \pi_{i_1} \cdots \pi_{i_k}$ is well defined for any $w \in \mathfrak{S}_n$ with a reduced expression $w = s_{i_1} \cdots s_{i_k}$, and $\{\pi_w : w \in \mathfrak{S}_n\}$ is another \mathbb{F} -basis for $H_n(0)$. One can check that π_w equals the sum of $\bar{\pi}_u$ over all u less than or equal to w in the Bruhat order of \mathfrak{S}_n . In particular, $\pi_{w_0(\alpha)}$ is the sum of $\bar{\pi}_u$ for all $u \in \mathfrak{S}_{\alpha^c}$.

Norton [18] decomposed the 0-Hecke algebra $H_n(0)$ into a direct sum of projective indecomposable submodules $\mathbf{P}_\alpha := H_n(0) \cdot \bar{\pi}_{w_0(\alpha)} \pi_{w_0(\alpha^c)}$ for all $\alpha \models n$ (i.e. compositions of n). Each \mathbf{P}_α has an \mathbb{F} -basis $\{\bar{\pi}_w \pi_{w_0(\alpha^c)} : w \in [w_0(\alpha), w_1(\alpha)]\}$. Its *radical* $\text{rad } \mathbf{P}_\alpha$ is the unique maximal $H_n(0)$ -submodule spanned by $\{\bar{\pi}_w \pi_{w_0(\alpha^c)} : w \in (w_0(\alpha), w_1(\alpha)]\}$. Although \mathbf{P}_α itself is not necessarily simple, its *top* $\mathbf{C}_\alpha := \mathbf{P}_\alpha / \text{rad } \mathbf{P}_\alpha$ is a one-dimensional simple $H_n(0)$ -module with the action of $H_n(0)$ given by

$$\bar{\pi}_i = \begin{cases} -1, & \text{if } i \in D(\alpha), \\ 0, & \text{if } i \notin D(\alpha). \end{cases}$$

It follows from general representation theory of algebras (see e.g. [2, §I.5]) that $\{\mathbf{P}_\alpha : \alpha \models n\}$ and $\{\mathbf{C}_\alpha : \alpha \models n\}$ are the complete lists of pairwise non-isomorphic projective indecomposable and simple $H_n(0)$ -modules, respectively.

Krob and Thibon [14] introduced a correspondence between $H_n(0)$ -representations and the dual Hopf algebras QSym and NSym , which we review next. The Hopf algebra QSym has a free \mathbb{Z} -basis of *fundamental quasisymmetric functions* F_α , and the dual Hopf algebra NSym has a dual basis of *non-commutative ribbon Schur functions* \mathfrak{s}_α , for all compositions α .

Let $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_k \supseteq M_{k+1} = 0$ be a composition series of $H_n(0)$ -modules with simple factors $M_i/M_{i+1} \cong \mathbf{C}_{\alpha^{(i)}}$ for $i = 0, 1, \dots, k$. Then the *quasisymmetric characteristic* of M is

$$\text{Ch}(M) := F_{\alpha^{(0)}} + \cdots + F_{\alpha^{(k)}}.$$

The *noncommutative characteristic* of a projective $H_n(0)$ -module $M \cong \mathbf{P}_{\alpha(1)} \oplus \cdots \oplus \mathbf{P}_{\alpha(k)}$ is

$$\mathbf{ch}(M) := \mathbf{s}_{\alpha(1)} + \cdots + \mathbf{s}_{\alpha(k)}.$$

It is not hard to extend these characteristic maps to $H_n(0)$ -modules with gradings and filtrations.

3 Stanley-Reisner ring of the Boolean algebra

In this section we study the Stanley-Reisner ring of the Boolean algebra. The *Boolean algebra* \mathcal{B}_n is the ranked poset of all subsets of $[n] := \{1, 2, \dots, n\}$ ordered by inclusion, with minimum element \emptyset and maximum element $[n]$. The rank of a subset of $[n]$ is defined as its cardinality. The *Stanley-Reisner ring* $\mathbb{F}[\mathcal{B}_n]$ of the Boolean algebra \mathcal{B}_n is the quotient of the polynomial algebra $\mathbb{F}[y_A : A \subseteq [n]]$ by the ideal $(y_A y_B : A \text{ and } B \text{ are incomparable in } \mathcal{B}_n)$. It has an \mathbb{F} -basis $\{y_M\}$ indexed by the multichains M in \mathcal{B}_n , and is multigraded by the rank multisets $r(M)$ of the multichains M .

The symmetric group \mathfrak{S}_n acts on the Boolean algebra \mathcal{B}_n by permuting the integers $1, \dots, n$. This induces an \mathfrak{S}_n -action on the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$, preserving its multigrading. The *invariant algebra* $\mathbb{F}[\mathcal{B}_n]^{\mathfrak{S}_n}$ consists of all elements in $\mathbb{F}[\mathcal{B}_n]$ invariant under this \mathfrak{S}_n -action. One can show that $\mathbb{F}[\mathcal{B}_n]^{\mathfrak{S}_n} = \mathbb{F}[\Theta]$, where $\Theta := \{\theta_0, \dots, \theta_n\}$. Garsia [7] showed that $\mathbb{F}[\mathcal{B}_n]$ is a free $\mathbb{F}[\Theta]$ -module on the basis of descent monomials

$$Y_w := \prod_{i \in D(w)} y_{\{w(1), \dots, w(i)\}}, \quad \forall w \in \mathfrak{S}_n. \quad (3)$$

There is an analogy between the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ and the polynomial ring $\mathbb{F}[X]$ via the *transfer map* $\tau : \mathbb{F}[\mathcal{B}_n] \rightarrow \mathbb{F}[X]$ defined by

$$\tau(y_M) := \prod_{1 \leq i \leq k} \prod_{j \in A_i} x_j$$

for all multichains $M = (A_1 \subseteq \cdots \subseteq A_k)$ in \mathcal{B}_n . It is *not* a ring homomorphism (e.g. $y_{\{1\}} y_{\{2\}} = 0$ but $x_1 x_2 \neq 0$). Nevertheless, it induces an isomorphism $\tau : \mathbb{F}[\mathcal{B}_n]/(\theta_0) \cong \mathbb{F}[X]$ of \mathfrak{S}_n -modules. Moreover, it sends the rank polynomials $\theta_1, \dots, \theta_n$ to the elementary symmetric polynomials e_1, \dots, e_n , and sends the descent monomials Y_w in $\mathbb{F}[\mathcal{B}_n^*]$ defined by (3) to the corresponding descent monomials in $\mathbb{F}[X]$ for all $w \in \mathfrak{S}_n$.

Example 3.1 *The Boolean algebra \mathcal{B}_3 consists of all subsets of $\{1, 2, 3\}$. Its Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_3]$ is a free $\mathbb{F}[\Theta]$ -module with a basis of descent monomials $Y_1 = 1$, $Y_{s_1} = y_2$, $Y_{s_2 s_1} = y_3$, $Y_{s_2} = y_{13}$, $Y_{s_1 s_2} = y_{23}$, $Y_{s_1 s_2 s_1} = y_{23} y_3$, where Θ consists of the rank polynomials $\theta_0 = y_\emptyset$, $\theta_1 := y_1 + y_2 + y_3$, $\theta_2 := y_{12} + y_{13} + y_{23}$, $\theta_3 := y_{123}$. The transfer map τ sends $\theta_1, \theta_2, \theta_3$ to e_1, e_2, e_3 , and sends the six descent monomials in $\mathbb{F}[\mathcal{B}_3]$ to the six descent monomials $1, x_2, x_3, x_1 x_3, x_2 x_3, x_2 x_3^2$ in $\mathbb{F}[x_1, x_2, x_3]$.*

The homogeneous components of $\mathbb{F}[\mathcal{B}_n]$ are indexed by multisets with elements in $\{0, \dots, n\}$, or equivalently by weak compositions α of n . The α -homogeneous component $\mathbb{F}[\mathcal{B}_n]_\alpha$ has an \mathbb{F} -basis $\{y_M : r(M) = D(\alpha)\}$. Denote by $\text{Com}(n, k)$ the set of all weak compositions of n with length k . If $M = (A_1 \subseteq \cdots \subseteq A_k)$ is a multichain of length k in \mathcal{B}_n then we set $A_0 := \emptyset$ and $A_{k+1} := [n]$ by convention. Define $\alpha(M) := (\alpha_1, \dots, \alpha_{k+1})$, where $\alpha_i = |A_i| - |A_{i-1}|$ for all $i \in [k+1]$. Then

$\alpha(M) \in \text{Com}(n, k+1)$ and $D(\alpha(M)) = r(M)$, i.e. $\alpha(M)$ indexes the homogeneous component containing y_M . Define $\sigma(M)$ to be the minimal element in \mathfrak{S}_n which sends the standard multichain $[\alpha_1] \subseteq [\alpha_1 + \alpha_2] \subseteq \cdots \subseteq [\alpha_1 + \cdots + \alpha_k]$ with rank multiset $D(\alpha(M))$ to M . Then $\sigma(M) \in \mathfrak{S}^{\alpha(M)}$.

The map $M \mapsto (\alpha(M), \sigma(M))$ is a bijection between multichains of length k in \mathcal{B}_n and the pairs (α, σ) of $\alpha \in \text{Com}(n, k+1)$ and $\sigma \in \mathfrak{S}^\alpha$. A short way to write down this encoding of M is to insert bars at the descent positions of $\sigma(M)$. For example, the length-4 multichain $\{2\} \subseteq \{2\} \subseteq \{1, 2, 4\} \subseteq [4]$ in \mathcal{B}_4 is encoded by $2||14|3|$.

There is another way to encode the multichain M . Let $p_i(M) := \min\{j \in [k+1] : i \in A_j\}$, $1 \leq i \leq n$. So $p_i(M)$ is the first position where i appears in M . One checks that

$$\begin{cases} p_i(M) > p_{i+1}(M) \Leftrightarrow i \in D(\sigma(M)^{-1}), \\ p_i(M) = p_{i+1}(M) \Leftrightarrow i \notin D(\sigma(M)^{-1}), D(s_i\sigma(M)) \not\subseteq D(\alpha(M)), \\ p_i(M) < p_{i+1}(M) \Leftrightarrow i \notin D(\sigma(M)^{-1}), D(s_i\sigma(M)) \subseteq D(\alpha(M)). \end{cases} \quad (4)$$

The map $M \mapsto p(M) := (p_1(M), \dots, p_n(M))$ is an bijection between the set of multichains with length k in \mathcal{B}_n and the set $[k+1]^n$ of all words of length n on the alphabet $[k+1]$, for any fixed integer $k \geq 0$.

Let $p(M) = \mathbf{p} = (p_1, \dots, p_n) \in [k+1]^n$. Then $\text{inv}(p(M)) = \text{inv}(\sigma(M))$. Let $\mathbf{p}' := (p'_1, \dots, p'_k)$ where $p'_i := |\{j : p_j(M) \leq i\}| = |A_i|$. Then the rank multiset of M consists of p'_1, \dots, p'_k . Define $D(\mathbf{p}) := \{i \in [n-1] : p_i > p_{i+1}\}$. For example, the multichain $3|14||2|5$ corresponds to $\mathbf{p} = (2, 4, 1, 2, 5) \in [5]^5$, and one has $\mathbf{p}' = (1, 3, 3, 4)$, $D(2, 5, 1, 2, 4) = \{2\}$.

These two encodings (with slightly different notation) were already used by Garsia and Gessel [8] in their work on generating functions of multivariate distributions of permutation statistics.

4 0-Hecke algebra action

We saw an analogy between $\mathbb{F}[\mathcal{B}_n]$ and $\mathbb{F}[X]$ in the last section. The usual $H_n(0)$ -action on the polynomial ring $\mathbb{F}[X]$ is via the *Demazure operators*

$$\bar{\pi}_i(f) := \frac{x_{i+1}f - x_{i+1}s_i f}{x_i - x_{i+1}}, \quad \forall f \in \mathbb{F}[X], 1 \leq i \leq n-1. \quad (5)$$

The above definition is equivalent to

$$\bar{\pi}_i(x_i^a x_{i+1}^b m) = \begin{cases} (x_i^{a-1} x_{i+1}^{b+1} + x_i^{a-2} x_{i+1}^{b+2} \cdots + \underline{x_i^b x_{i+1}^a})m, & \text{if } a > b, \\ 0, & \text{if } a = b, \\ -(x_i^a x_{i+1}^b + x_i^{a+1} x_{i+1}^{b-1} + \cdots + x_i^{b-1} x_{i+1}^{a+1})m, & \text{if } a < b. \end{cases} \quad (6)$$

Here m is any monomial in $\mathbb{F}[X]$ containing neither x_i nor x_{i+1} . Denote by $\bar{\pi}'_i$ the operator obtained from (6) by taking only the leading term (underlined) in the lexicographic order of the result. Then $\bar{\pi}'_1, \dots, \bar{\pi}'_{n-1}$ realize another $H_n(0)$ -action on $\mathbb{F}[X]$. We call it the *transferred $H_n(0)$ -action* because it can be obtained by applying the transfer map τ to our $H_n(0)$ -action on $\mathbb{F}[\mathcal{B}_n]$, which we now define.

Let $M = (A_1 \subseteq \cdots \subseteq A_k)$ be a multichain in \mathcal{B}_n . Recall that $p_i(M) := \min\{j \in [k+1] : i \in A_j\}$, $1 \leq i \leq n$. We define

$$\bar{\pi}_i(y_M) := \begin{cases} -y_M, & p_i(M) > p_{i+1}(M), \\ 0, & p_i(M) = p_{i+1}(M), \\ s_i(y_M), & p_i(M) < p_{i+1}(M) \end{cases} \quad (7)$$

for $i = 1, \dots, n - 1$. Applying the transfer map τ one recovers $\bar{\pi}'_i$. For instance, when $n = 4$ one has

$$\begin{aligned} \bar{\pi}_1(y_{1|34||2|}) &= y_{2|34||1|}, & \bar{\pi}'_1(x_1^4 x_2 x_3^3 x_4^3) &= x_1 x_2^4 x_3^3 x_4^3, \\ \bar{\pi}_2(y_{1|34||2|}) &= -y_{1|34||2|}, & \bar{\pi}'_2(x_1^4 x_2 x_3^3 x_4^3) &= -x_1^4 x_2 x_3^3 x_4^3, \\ \bar{\pi}_3(y_{1|34||2|}) &= 0, & \bar{\pi}'_3(x_1^4 x_2 x_3^3 x_4^3) &= 0. \end{aligned}$$

One can check that $\bar{\pi}_1, \dots, \bar{\pi}_{n-1}$ realize an $H_n(0)$ -action on $\mathbb{F}[\mathcal{B}_n]$ preserving the multigrading of $\mathbb{F}[\mathcal{B}_n]$. If one set $t_i = t^i$ for $i = 1, \dots, n$, then there is an isomorphism $\mathbb{F}[\mathcal{B}_n]/(\emptyset) \cong \mathbb{F}[X]$ of graded $H_n(0)$ -modules (which can be given explicitly, but *not* via the transfer map τ).

It is not hard to show that $\mathbb{F}[\mathcal{B}_n]^{H_n(0)} = \mathbb{F}[\Theta]$, where $\mathbb{F}[\mathcal{B}_n]^{H_n(0)}$ is the *invariant algebra* of the $H_n(0)$ -action on $\mathbb{F}[\mathcal{B}_n]$, defined as

$$\mathbb{F}[\mathcal{B}_n]^{H_n(0)} := \{f \in \mathbb{F}[\mathcal{B}_n] : \pi_i f = f, i = 1, \dots, n - 1\}.$$

Proposition 4.1 *The $H_n(0)$ -action on $\mathbb{F}[\mathcal{B}_n]$ is Θ -linear.*

Therefore the *coinvariant algebra* $\mathbb{F}[\mathcal{B}_n]/(\Theta)$ is a multigraded $H_n(0)$ -module, which is isomorphic to the regular representation of $H_n(0)$ by Theorem 1.1. This cannot be obtained simply by applying the transfer map τ , since τ is *not* a map of $H_n(0)$ -modules.

5 Noncommutative characteristic

In this section we use the $H_n(0)$ -action on the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ of the Boolean algebra \mathcal{B}_n to provide a noncommutative analogue of the following remarkable result.

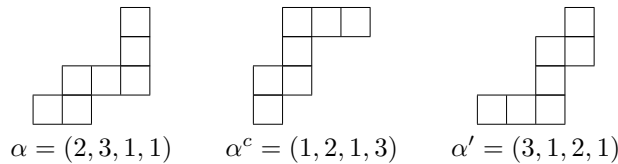
Theorem 5.1 (Hotta-Springer [11], Garsia-Procesi [9]) *For any partition $\mu = (0 < \mu_1 \leq \dots \leq \mu_k)$ of n , there exists an \mathfrak{S}_n -invariant ideal J_μ of $\mathbb{C}[X]$ such that $\mathbb{C}[X]/J_\mu$ is isomorphic to the cohomology ring of the Springer fiber indexed by μ and has graded Frobenius characteristic equal to the modified Hall-Littlewood symmetric function*

$$\tilde{H}_\mu(X; t) = \sum_{\lambda} t^{n(\mu)} K_{\lambda\mu}(t^{-1}) s_\lambda \quad \text{inside } \text{Sym}[t]$$

where $n(\mu) := \mu_{k-1} + 2\mu_{k-2} + \dots + (k-1)\mu_1$ and $K_{\lambda\mu}(t)$ is the Kostka-Foulkes polynomial.

Example 5.2 *Tanisaki [19] gives a construction for the ideal J_μ . If $\mu = (1^k, n - k)$ is a hook then $J_{1^k, n-k}$ is generated by e_1, \dots, e_k and all monomials $x_{i_1} \dots x_{i_{k+1}}$ with $1 \leq i_1 < \dots < i_{k+1} \leq n$.*

Now consider a composition $\alpha = (\alpha_1, \dots, \alpha_\ell)$. The major index of α is $\text{maj}(\alpha) := \sum_{i \in D(\alpha)} i$. Viewing a partition $\mu = (0 < \mu_1 \leq \mu_2 \leq \dots)$ as a composition one has $\text{maj}(\mu) = n(\mu)$. Recall that $\overleftarrow{\alpha} := (\alpha_\ell, \dots, \alpha_1)$ and α^c is the composition of n with $D(\alpha^c) = [n-1] \setminus D(\alpha)$. We define $\alpha' := \overleftarrow{\alpha^c} = (\overleftarrow{\alpha})^c$. One can identify α with a *ribbon diagram*, i.e. a connected skew Young diagram without 2 by 2 boxes, which has row lengths $\alpha_1, \dots, \alpha_\ell$, ordered from bottom to top. Note that a ribbon diagram is a Young diagram if and only if it is a hook. One can check that α' is the transpose of α ; see the example below.



Bergeron and Zabrocki [3] introduced a noncommutative modified Hall-Littlewood symmetric function

$$\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t) := \sum_{\beta \preceq \alpha} t^{\text{maj}(\beta)} \mathbf{s}_\beta \quad \text{inside } \mathbf{NSym}[t] \quad (8)$$

and a (q, t) -analogue

$$\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t) := \sum_{\beta \models n} t^{c(\alpha, \beta)} q^{c(\alpha', \bar{\beta})} \mathbf{s}_\beta \quad \text{inside } \mathbf{NSym}[q, t] \quad (9)$$

for every composition α , where \mathbf{s}_β is the noncommutative ribbon Schur function indexed by β , and $c(\alpha, \beta) := \sum_{i \in D(\alpha) \cap D(\beta)} i$. In our earlier work [13] we provided a partial representation theoretic interpretation for $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$ when $\alpha = (1^k, n - k)$ is a hook, using the $H_n(0)$ -action on the polynomial ring $\mathbb{F}[X]$ by the Demazure operators.

Theorem 5.3 ([13]) *The ideal J_μ of $\mathbb{F}[X]$ is $H_n(0)$ -invariant if and only if $\mu = (1^{n-k}, k)$ is a hook, and if that holds then $\mathbb{F}[X]/J_\mu$ becomes a graded projective $H_n(0)$ -module with*

$$\begin{aligned} \text{ch}_t(\mathbb{F}[X]/J_\mu) &= \tilde{\mathbf{H}}_\mu(\mathbf{x}; t), \\ \text{Ch}_t(\mathbb{F}[X]/J_\mu) &= \tilde{H}_\mu(\mathbf{x}; t). \end{aligned}$$

Now we switch to the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ and define I_α to be its ideal generated by

$$\Theta_\alpha := \{\theta_i : i \in D(\alpha) \cup \{n\}\} \quad \text{and} \quad \{y_A : A \subseteq [n], |A| \notin D(\alpha) \cup \{n\}\}$$

for any composition α of n . The following result is a restatement of Theorem 1.1.

Theorem 5.4 *Let α be a composition of n . Then $\mathbb{F}[\mathcal{B}_n]/I_\alpha$ is a projective $H_n(0)$ -module with multi-graded noncommutative characteristic equal to*

$$\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \dots, t_{n-1}) := \sum_{\beta \preceq \alpha} t^{D(\beta)} \mathbf{s}_\beta \quad \text{inside } \mathbf{NSym}[t_1, \dots, t_{n-1}].$$

One has $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t, t^2, \dots, t^{n-1}) = \tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$, and one obtains $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$ from $\tilde{\mathbf{H}}_{1^n}(\mathbf{x}; t_1, \dots, t_{n-1})$ by taking $t_i = t^i$ for all $i \in D(\alpha)$, and $t_i = q^{n-i}$ for all $i \in [n-1] \setminus D(\alpha)$.

Proof: There is an \mathbb{F} -basis for $\mathbb{F}[\mathcal{B}_n]/(\Theta_\alpha)$ given by the descent monomials Y_w defined in (3) for all $w \in \mathfrak{S}^\alpha$. The result follows from the $H_n(0)$ -action on this basis and (4). \square

The proof of this theorem is actually simpler than the proof of our partial interpretation for $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t)$ in [13]. This is because $\bar{\pi}_i$ sends a descent monomial in $\mathbb{F}[\mathcal{B}_n]$ to either 0 or ± 1 times a descent monomial, but sends a descent monomial in $\mathbb{F}[X]$ to a polynomial in general (whose leading term is still a descent monomial). We view the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ (or $\mathbb{F}[\mathcal{B}_n]/(\emptyset)$) as a $q = 0$ analogue of the polynomial ring $\mathbb{F}[X]$. For an odd (i.e. $q = -1$) analogue, see Lauda and Russell [16].

Remark 5.5 *If $\alpha = (1^k, n - k)$ is a hook, one can check that the ideal $I_{1^k, n-k}$ of $\mathbb{F}[\mathcal{B}_n]$ has generators $\theta_1, \dots, \theta_k$ and all y_A with $A \subseteq [n]$ and $|A| \notin [k]$. By Example 5.2, the images of these generators under the transfer map τ are the Tanisaki generators for the ideal $J_{1^k, n-k}$ of $\mathbb{F}[X]$, but $\tau(I_{1^k, n-k}) \neq J_{1^k, n-k}$.*

For any composition $\alpha \models n$, one can view $\tilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \dots, t_{n-1})$ as a modified version of

$$\mathbf{H}_\alpha = \mathbf{H}_\alpha(\mathbf{x}; t_1, \dots, t_{n-1}) := \sum_{\beta \preceq \alpha} \underline{t}^{D(\alpha) \setminus D(\beta)} \mathbf{s}_\beta.$$

Below are some properties satisfied by \mathbf{H}_α , generalizing the properties of $\mathbf{H}_\alpha(\mathbf{x}; t)$ given in [3].

Proposition 5.6 *Let α and β be two compositions.*

(i) $\mathbf{H}_\alpha(0, \dots, 0) = \mathbf{s}_\alpha$, $\mathbf{H}_\alpha(1, \dots, 1) = \mathbf{h}_\alpha$.

(ii) $\bigcup_{n \geq 0} \{\mathbf{H}_\alpha : \alpha \models n\}$ is a basis for $\mathbf{NSym}[t_1, t_2, \dots]$.

(iii) $\langle \mathbf{H}_\alpha, \mathbf{H}_\beta \rangle = (-1)^{|\alpha| + \ell(\alpha)} \delta_{\alpha, \beta^c}$ for any pair of compositions α and β .

(iv)

$$\mathbf{H}_\alpha \cdot \mathbf{H}_\beta = \sum_{\gamma \preceq \beta} \left(\prod_{i \in D(\beta) \setminus D(\gamma)} (t_i - t_{|\alpha|+i}) \right) (\mathbf{H}_{\alpha\gamma} + (1 - t_{|\alpha|}) \mathbf{H}_{\alpha \triangleright \gamma}).$$

(v) If $n = |\alpha|$ and $t|n := (t_1, \dots, t_{n-1}, 1, t_1, \dots, t_{n-1}, 1, \dots)$ then

$$\mathbf{H}_\alpha(\mathbf{x}; t_1, \dots, t_{n-1}) \mathbf{H}_\beta(\mathbf{x}; t|n) = \mathbf{H}_{\alpha\beta}(t|n).$$

6 Quasisymmetric characteristic

Now we study the quasisymmetric characteristic of $\mathbb{F}[\mathcal{B}_n]$. The following lemma follows easily from (4).

Lemma 6.1 *Let α be a weak composition of n . Then the α -homogeneous component $\mathbb{F}[\mathcal{B}_n]_\alpha$ of the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ is an $H_n(0)$ -submodule of $\mathbb{F}[\mathcal{B}_n]$ with homogeneous multigrading $\underline{t}^{D(\alpha)}$ and isomorphic to the cyclic module $H_n(0)\pi_{w_0(\alpha^c)}$.*

Since $\mathbb{F}[\mathcal{B}_n]_\alpha$ is a cyclic multigraded $H_n(0)$ -module, we get an $\mathbb{N} \times \mathbb{N}^{n+1}$ -multigraded quasisymmetric characteristic

$$\text{Ch}_{q, \underline{t}}(\mathbb{F}[\mathcal{B}_n]_\alpha) = \sum_{w \in \mathfrak{S}^\alpha} q^{\text{inv}(w)} \underline{t}^{D(\alpha)} F_{D(w^{-1})} \quad (10)$$

where q keeps track of the length filtration and \underline{t} keeps track of the multigrading of $\mathbb{F}[\mathcal{B}_n]_\alpha$. This defines an $\mathbb{N} \times \mathbb{N}^{n+1}$ -multigraded quasisymmetric characteristic for the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$, which is explicitly given in Theorem 1.2 and restated below.

Theorem 6.2 *The $\mathbb{N} \times \mathbb{N}^{n+1}$ -multigraded quasisymmetric characteristic of $\mathbb{F}[\mathcal{B}_n]$ is*

$$\begin{aligned} \text{Ch}_{q, \underline{t}}(\mathbb{F}[\mathcal{B}_n]) &= \sum_{k \geq 0} \sum_{\alpha \in \text{Com}(n, k+1)} \underline{t}^{D(\alpha)} \sum_{w \in \mathfrak{S}^\alpha} q^{\text{inv}(w)} F_{D(w^{-1})} \\ &= \sum_{w \in \mathfrak{S}_n} \frac{q^{\text{inv}(w)} \underline{t}^{D(w)} F_{D(w^{-1})}}{\prod_{0 \leq i \leq n} (1 - t_i)} \\ &= \sum_{k \geq 0} \sum_{\mathbf{p} \in [k+1]^n} t_{p'_1} \cdots t_{p'_k} q^{\text{inv}(\mathbf{p})} F_{D(\mathbf{p})}. \end{aligned}$$

Proof: Use the two encodings of the multichains in \mathcal{B}_n as well as the free $\mathbb{F}[\Theta]$ -basis $\{Y_w : w \in \mathfrak{S}_n\}$ of descent monomials for $\mathbb{F}[\mathcal{B}_n]$ discussed in Section 3. \square

Next we explain here how this theorem specializes to (1), a result of Garsia and Gessel [8, Theorem 2.2] on the multivariate generating function of the permutation statistics $\text{inv}(w)$, $\text{maj}(w)$, $\text{des}(w)$, $\text{maj}(w^{-1})$, and $\text{des}(w^{-1})$ for all $w \in \mathfrak{S}_n$. First recall that

$$F_\alpha = \sum_{\substack{i_1 \geq \dots \geq i_n \geq 1 \\ i \in D(\alpha) \Rightarrow i_j > i_{j+1}}} x_{i_1} \cdots x_{i_n}, \quad \forall \alpha \models n.$$

Let $\mathbf{ps}_{q;\ell}(F_\alpha) := F_\alpha(1, q, q^2, \dots, q^{\ell-1}, 0, 0, \dots)$, and $(u; q)_n := (1-u)(1-qu)(1-q^2u) \cdots (1-q^{n-1}u)$. It is not hard to check (see Gessel and Reutenauer [10, Lemma 5.2]) that

$$\sum_{\ell \geq 0} u^\ell \mathbf{ps}_{q;\ell+1}(F_\alpha) = \frac{q^{\text{maj}(\alpha)} u^{\text{des}(\alpha)}}{(u; q)_n}.$$

Then applying the linear transformation $\sum_{\ell \geq 0} u_1^\ell \mathbf{ps}_{q_1;\ell+1}$ and the specialization $t_i = q_2^i u_2$ for all $i = 0, 1, \dots, n$ to Theorem 6.2 we recover (1).

A further specialization of Theorem 6.2 gives a well known result which is often attributed to Carlitz [6] but actually dates back to MacMahon [17, Volume 2, Chapter 4].

Corollary 6.3 (Carlitz-MacMahon) *Let $[k+1]_q := 1 + q + q^2 + \dots + q^k$. Then*

$$\frac{\sum_{w \in \mathfrak{S}_n} q^{\text{maj}(w)} u^{\text{des}(w)}}{(u; q)_n} = \sum_{k \geq 0} ([k+1]_q)^n u^k.$$

Theorem 6.2 also implies the following result, which was obtained by Adin, Brenti, and Roichman [1] from the Hilbert series of the coinvariant algebra $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$.

Corollary 6.4 (Adin-Brenti-Roichman) *Let $\text{Par}(n)$ be the set of weak partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \dots \geq \lambda_n \geq 0$, and let $m(\lambda) = (m_0(\lambda), m_1(\lambda), \dots)$, where $m_j(\lambda) := \#\{1 \leq i \leq n : \lambda_i = j\}$. Then*

$$\sum_{\lambda \in \text{Par}(n)} \binom{n}{m(\lambda)} \prod_{i=1}^n q_i^{\lambda_i} = \frac{\sum_{w \in \mathfrak{S}_n} \prod_{i \in D(w)} q_1 \cdots q_i}{(1-q_1)(1-q_1q_2) \cdots (1-q_1 \cdots q_n)}.$$

7 Remarks and questions for future research

7.1 Hecke algebra action

It is well known that the symmetric group \mathfrak{S}_n is the Coxeter group of type A_{n-1} . The Stanley-Reisner ring of \mathcal{B}_n is essentially the Stanley-Reisner ring of the Coxeter complex of \mathfrak{S}_n . The Hecke algebra $H_W(q)$ can be defined for any finite Coxeter group W . We can generalize our action $H_n(0)$ -action on $\mathbb{F}[\mathcal{B}_n]$ to an $H_W(q)$ -action on the Stanley-Reisner ring $\mathbb{F}(q)[\Delta(W)]$ of the Coxeter complex $\Delta(W)$ of any finite Coxeter group W . We show similar results for this $H_W(q)$ -action.

7.2 Gluing the group algebra and the 0-Hecke algebra

The group algebra $\mathbb{F}W$ of a finite Coxeter group W naturally admits both actions of W and $H_W(0)$. Hivert and Thiéry [12] defined the *Hecke group algebra* of W by gluing these two actions. In type A , one can also glue the usual actions of \mathfrak{S}_n and $H_n(0)$ on the polynomial ring $\mathbb{F}[X]$, but the resulting algebra is different from the Hecke group algebra of \mathfrak{S}_n .

Now one has a W -action and an $H_W(0)$ -action on the Stanley-Reisner ring $\mathbb{F}[\Delta(W)]$. What can we say about the algebra generated by the operators s_i and $\bar{\pi}_i$ on $\mathbb{F}[\Delta(W)]$? Is it the same as the Hecke group algebra of W ? If not, what properties (dimension, bases, presentation, simple and projective indecomposable modules, etc.) does it have?

7.3 Tits Building

Let $\Delta(G)$ be the Tits building of the general linear group $G = GL(n, \mathbb{F}_q)$ and its usual BN-pair over a finite field \mathbb{F}_q ; see e.g. Björner [4]. The Stanley-Reisner ring $\mathbb{F}[\Delta(G)]$ is a q -analogue of $\mathbb{F}[\mathcal{B}_n]$. The nonzero monomials in $\mathbb{F}[\Delta(G)]$ are indexed by multiflags of subspaces of \mathbb{F}_q^n , and there are $q^{\text{inv}(w)}$ many multiflags corresponding to a given multichain M in \mathcal{B}_n , where $w = \sigma(M)$. Can one obtain the multivariate quasisymmetric function identities in Theorem 1.2 by defining a nice $H_n(0)$ -action on $\mathbb{F}[\Delta(G)]$?

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