Electrical network and Lie theory

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Electrical networks, electrical Lie algebras and Lie groups of finite Dynkin type

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Abstract Curtis-Ingerman-Morrow studied the space of circular planar electrical networks, and classified all possible response matrices for such networks. Lam and Pylyavskyy found a Lie group EL_{2n} whose positive part $(EL_{2n})_{\geq 0}$ naturally acts on the circular planar electrical network via some combinatorial description, where the action is inspired by the star-triangle transformation of the electrical networks. The Lie algebra el_{2n} is semisimple and isomorphic to the symplectic algebra. In the end of their paper, they suggest a generalization of electrical Lie algebras to all finite Dynkin types. We give the structure of the type B electrical Lie algebra $e_{b_{2n}}$. The nonnegative part $(E_{B_{2n}})_{\geq 0}$ of the corresponding Lie group conjecturally acts on a class of "mirror symmetric circular planar electrical networks". This class of networks has interesting combinatorial properties. Finally, we mention some partial results for type C and D electrical Lie algebras, where an analogous story needs to be developed.

Résumé Curtis, Ingerman et Morrow ont étudié l'espace des réseaux électriques circulaires plans et ont classifié toutes les matrices de réponses possibles pour ces réseaux. Lam et Pylyavskyy ont trouvé un groupe de Lie EL_{2n} dont la partie positive $(EL_{2n})_{\geq 0}$ agit naturellement sur le réseau électrique circulaire plan par une description combinatoire, où l'action est inspirée par la transformation étoile vers triangle des réseaux électriques. L'algèbre de Lie el_{2n} est semi-simple et isomorphe à l'algèbre symplectique. A la fin de leur article, ils proposent une généralisation des algèbres de Lie électriques pour tous les types de Dynkin finis. Nous donnons la structure de l'algèbre de Lie électrique eb_{2n} du type B. La partie positive $(EB_{2n})_{\geq 0}$ du groupe de Lie correspondant agit conjecturalement sur une famille de "miroirs réseaux électriques circulaires symétriques plans". Cette famille de réseaux a des propriétés combinatoires intéressantes. Nous donnons enfin quelques résultats partiels de l'algèbres de Lie électrique du type C et du type D, où une étude analogue doit être développée.

Keywords: network, electrical Lie algebra, finite Dynkin type

1 Introduction

In this paper we consider electrical networks consisting only of nodes and resistors. The response matrix $L(N)$ of an electrical network N captures all of the electrical properties of N,and is preserved by a number of combinatorial moves including series and parallel reductions, and star-triangle ($Y-\Delta$) relation. [CIM] and [VGV] developed a robust theory of circular planar networks, gave the criterions for two networks to have the same response matrix, and classified the set of response matrices as the set of symmetric, circular, totally positive square matrices.

In [CIM], Curtis-Ingerman-Morrow studied two operations on electrical networks, *adding a boundary spike* and *adding a boundary edge* to planar electrical networks. In [LP], Lam and Pylyavskyy introduced the electrical Lie algebra el_{2n} of type A_{2n}, and the nonnegative part $(EL_{2n})_{\geq 0}$ of the corresponding Lie group acts on the electrical networks exactly as the above operations. The star-triangle transformation translates to electrical Serra relations $[e,[e,e']] = -2e$. They showed that el_{2n} is isomorphic to symplectic Lie algebra $sp(2n)$, and showed the nonnegative Lie subsemigroup $(EL_{2n})_{\geq 0}$ admits a cell decomposition, which is an analogue of the results in the study of the totally nonnegative part of the unipotent subgroup of SL_n. In the end of [LP], Lam and Pylyavskyy suggested a generalization of electrical Lie algebras to all finite Dynkin types.

Working on this generalization, in Section 4, we manage to classify electrical Lie algebras of type A_{2n+1}, B_n, and C_n, and prove a partial result for type D_{2n+1}. It turns out that electrical Lie algebras of type A are the building blocks of the electrical Lie algebra of other finite Dynkin types. As for exceptional Lie type, in theory, the structure of the corresponding electrical Lie algebra is computable by any computer algebra system.

In Section 5, we focus on the study of type B electrical Lie theory. [LP] gives the electrical braid relation of electrical Lie group E_{B_n}. Inspired by the embedding of the Weyl group B_n into S_{2n}, we introduce a new class of *mirror symmetric circular planar electrical networks*, together with two operations: *mirror symmetrically adding a pair of boundary spikes*, and *adding a (pair of) boundary edge(s)*. These two operations form a new kind of electrical transformation, *mirror symmetric square move*, which corresponds to the braid relation in E_{B_n} . Thus the positive part $(E_{B_n})_{\geq 0}$ conjecturally acts on mirror symmetric circular planar electrical networks. This suggests a direction of developing the electrical Lie theory for type B.

Finally, in Section 6 we mention some of the possible work in the future.

2 Circular Planar Electrical Networks

Let $G = (V, V_B, E)$ be a planar undirected graph with the vertex set V, the boundary vertex set $V_B \subseteq V$ and the edge set E. Assume that V_B is nonempty. A *circular planar electrical network* $N = (G, \gamma)$ is the graph G together with a map $\gamma : E \to \mathbb{R}^{|E|}_{>0}$, called the *edge weights* of G. The weights of edges are conductances of resistors in an electrical network. Define the *Kirchhoff matrix* $K = K(N)$ to be the square matrix with rows and columns indexed by the vertices as follows:

- If $i \neq j$, then $K_{ij} = -\sum_e \gamma(e)$, where the sum is over all of the edges joining i and j.
- If $i = j$, then $K_{ii} = \sum_e \gamma(e)$, where the sum is over all of the edges that are incident with i.

If $I, J \subseteq V$, where I, J are in ascending order, and $|I| = |J|$, then denote K_{IJ} to be the submatrix of K with row indices I and column indices J. We have the following Lemma.

Lemma 1 ([CIM]) *Let $I \subseteq V$. If $J = V$, then K_{II} is a positive semi-definite matrix. If $I \subseteq V$ and $I \neq V$, then K_{II} is positive definite.*

Let $I = V \setminus V_B$. Since V_B is nonempty, I is a proper subset of V, so K in the following form:

$$K = \begin{pmatrix} K_{V_B V_B} & C \\ D & K_{II} \end{pmatrix}$$

By Lemma 1, K_{II} is positive definite. In particular, it is invertible. Then one can define the response matrix $L(N)$ of N as the Schur complement

$$L(N) = K/K_{II} := K_{V_B V_B} - C K_{II}^{-1} D$$

We can interpret the response matrix $L(N)$ as a linear transformation in the following sense: If one puts electrical potentials $p = \{p(v_i)\}$ on each of the boundary vertices (think of p as column vector), then $L(N).p$ will be the current flowing into or out of each boundary vertex resulting from this potential p. In other words, the response matrix $L(N)$ captures the electrical properties of the network N.

Two natural questions arise from the above observation:

(a) What are the relations between two networks N and N', so that $L(N) = L(N')$?

(b) What is the set of matrices $L(N)$, where N runs over the set of networks with n boundary vertices?

The first question question is answered in [CIM] and [VGV], and the second question is answered fully in [CIM].

If a connected component of G has no boundary vertices, then it will not affect the electrical properties of N, so one can delete it. Therefore, without loss of generality one can assume the underlying graph G has no internal connected component. Then define the following reductions of networks (see Figure 1):

R(1) Remove a loop connecting one vertex to itself.

R(2) Remove any non-boundary vertex of degree 1 and its incident edge.

R(3) If a non-boundary vertex has degree 2, and its incident edges have weight a and b and connecting to u and v, then one can delete this vertex and its incident edges, and join vertices u and v by a new edge with weight $ab/(a+b)$.

R(4) If there are parallel edges of weight a and b joining vertices u and v, then merge those two edges to a new edge with weight $a+b$.

It can be easily seen that the reduction above will not change the electrical properties of the network, *i.e.*, the response matrix remains unchanged, meanwhile the number of edges gets reduced. Define a *minimal network* if none of the reductions $R(1) - R(4)$ can be performed in this network. So every electrical network can be reduced to some minimal network.

We define the move M (see Figure 1):

M ($Y - \Delta$ or star-triangle transformation) If a vertex has exactly three incident edges with weights a, b and c, then replace this vertex and its incident edges by a triangle of the corresponding weight $\frac{bc}{a+b+c}$, $\frac{ac}{a+b+c}$, $\frac{ab}{a+b+c}$. On the other hand, if there is a triangle, with edge weights A, B, and C, then we remove these three edges, put a vertex in the middle of the triangle, and join the vertices of the triangle with this new vertex, with corresponding weights $\frac{AB+AC+BC}{A}$, $\frac{AB+AC+BC}{B}$, $\frac{AB+AC+BC}{C}$.

The star-triangle transformation doesn't change the response matrix either, and the number of edges doesn't change. If two minimal networks can be transformed to each other via the move M, then we call these two networks are *move-equivalent* to each other. We have the following theorem.

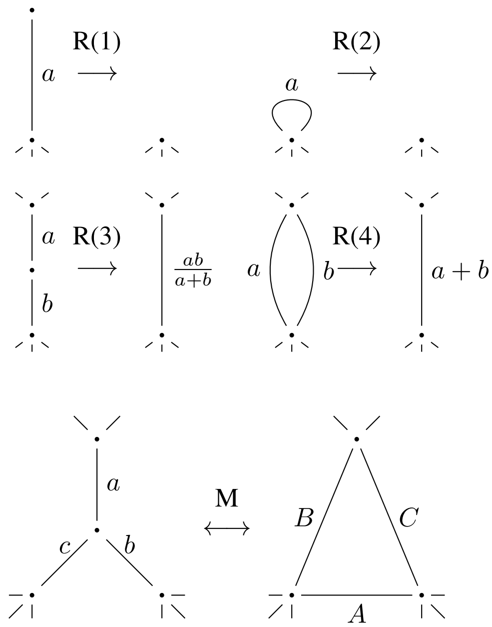

Fig. 1: Figure 1: Reduction and moves for circular planar electrical network

Theorem 2 ([CIM]) *Two electrical networks N and N' have the same response matrix, i.e., $L(N) = L(N')$ if and only if N and N' can be reduced through R(1)-R(4) to move-equivalent minimal networks.*

As for question (b) above, we need to introduce some terminology. Let $P = \{p_1, p_2, \ldots, p_k\}$, $Q = \{q_1, q_2, \ldots, q_k\}$ be two disjoint ordered subsets of the boundary vertices. We say (P, Q) is a *circular pair* if $p_1, p_2 \ldots, p_k, q_k, \ldots, q_2, q_1$ is in circular order. We say (P, Q) is *connected* if there is a collection of vertex-disjoint paths joining p_i with q_i. Define a *circular minor* M_{PQ} of a matrix M to be $(M_{PQ})_{ij} = M_{p_i q_j}$. Let $\mathcal{P}(n+1)$ be the set of symmetric matrices such that the row sum and column sum equal to zero, and all circular minors are nonnegative. Then we have the following theorem.

Theorem 3 ([CIM]) *Let L be an $(n+1) \times (n+1)$ matrix. Then $L = L(N)$, where N is an electrical network of $n+1$ boundary vertices if and only if $L \in \mathcal{P}(n+1)$.*

In Theorem 2, Curtis, Ingerman and Morrow studied the relations among electrical networks under which the response matrices are invariant. In the same paper, they also studied the "generators" of the electrical networks. Lam and Pylyavskyy studied this action in [LP] as well. We will follow the latter description. Say N has $n+1$ vertices, and label them cyclically as $\{1, 2, \ldots, n+1\}$. For a fixed positive real number t, and $k \in \{1, 2, , \ldots, n+1\}$, define two operations on N (see Figure 2):

Adding a boundary spike: Define $v_{2k-1}(t) \cdot N$ to be the action on N by adding a vertex u into N, joining an edge of weight $1/t$ between this vertex and boundary vertex k, and then treating u as the new boundary vertex k, and old boundary vertex k as an interior vertex.

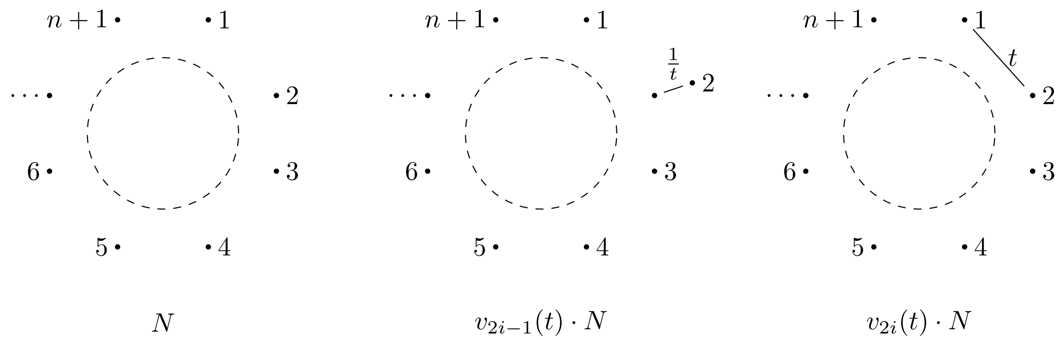

Fig. 2: The actions of $v_i(t)$ on network N

Adding a boundary edge: Define $v_{2k}(t)\cdot N$ to be the action on N by adding an edge of weight t between boundary vertices k and $k+1$.

It is shown in [CIM] that $L(v_i(t)\cdot N)$ depends only on $L(N)$, thus giving an action on response matrices $M = (x_{ij})_{i,j}^{n+1}$ by:

- $v_{2k-1}(t) : x_{ij} \mapsto x_{ij} - \frac{tx_{ik}x_{kj}}{tx_{kk}+1}$
- $v_{2k}(t) : x_{ij} \mapsto x_{ij} + (\delta_{ik} - \delta_{(i+1)k})(\delta_{jk} - \delta_{(j+1)k})t$

where δ_{ij} is the Kronecker delta, and the indices is taken modulo $n+1$.

This means that the generators also satisfy some of the move-reduction relations M and R(1)-R(4), which leads to the invention of electrical Lie algebra by Lam and Pylyavskyy introduced in the following section.

3 Electrical Lie Algebra and Lie group of Type A_{2n}

This section is based on [LP]. Define the *electrical Lie algebra* of type A_{2n}, denoted by el_{2n} by generators $\{e_1, e_2, \ldots, e_{2n}\}$ and relations over $\mathbb{R}$:

$$[e_i, e_j] = 0 \text{ if } |i-j| \geq 2, \qquad [e_i, [e_i, e_j]] = -2e_i \text{ if } |i-j| = 1$$

Theorem 4 ([LP]) *el_{2n} is isomorphic to the real symplectic Lie algebra $sp(2n)$.*

Let EL_{2n} be the electrical Lie group associated with Lie algebra el_{2n}. For $t > 0$, let $u_i(t) = \exp(te_i)$,

Theorem 5 ([LP]) *If $a, b, c > 0$, then the elements $u_i(a), u_i(b), u_i(c)$ satisfy the relations:*

(1) $u_i(a)u_j(b) = u_j(b)u_i(a)$ *if* $|i-j| \geq 2$

(2) $u_i(a)u_i(b) = u_i(a+b)$

(3) $u_i(a)u_j(b)u_i(c) = u_j\left(\dfrac{bc}{a+c+abc}\right)u_i(a+c+abc)u_j\left(\dfrac{ab}{a+c+abc}\right)$ *if* $|i-j| = 1$

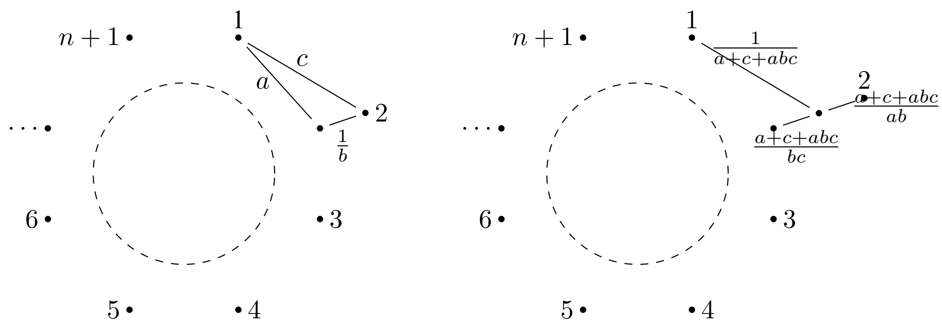

Fig. 3: Braid relation

Define the nonnegative part of EL_{2n} to be the Lie subsemigroup generated by $\{u_i(t) : t \geq 0, i = 1, 2, \dots, 2n\}$, denoted as $(EL_{2n})_{\geq 0}$. Let $\mathbf{i} = i_1 i_2 \dots i_l$ be a reduced word of ω in S_{2n+1}, consider the map:

$$\phi_{\mathbf{i}} : (t_1, t_2, \dots, t_l) \in \mathbb{R}^l_{>0} \mapsto u_{i_1}(t_1) u_{i_2}(t_2) \dots u_{i_l}(t_l)$$

Denote $E(\omega)$ as the image of the above map. By Theorem 5, the image of this map does not depend on which reduced word $\mathbf{i}$ one picks. Thus $E(\omega)$ is well-defined. Then we have the theorem:

Theorem 6 ([LP]) $(EL_{2n})_{\geq 0} = \bigsqcup_{\omega \in S_{2n+1}} E(\omega)$*, where the union is disjoint, and each of* $\phi_{\mathbf{i}}$ *is a homeomorphism.*

This theorem is an analogue of the totally nonnegative part of the unipotent group. Furthermore Lam and Pylyavskyy pointed out that the proof of this theorem in [LP] implies that the relations in Theorem 5 are all of the relations in the semigroup $(EL_{2n})_{\geq 0}$. Recall $\mathcal{P}(n+1)$ was defined in Section 2, as was the action of $v_i(t)'s$ on electrical networks N. We then have:

Theorem 7 ([LP]) *The semisubgroup* $(EL_{2n})_{\geq 0}$ *of the electrical group acts on* $\mathcal{P}(n+1)$ *by the equation:*

$$u_i(t) \cdot L(N) = L(v_i(t \cdot N))$$

It suffices to check the relations (1)-(3) in Theorem 5. Relation (1) and (2) are trivial. Figure 3 illustrates relation (3).

Let L_{null} be the zero matrix. Say a permutation $\omega = \omega(1)\omega(2)\dots\omega(2n+1) \in S_{2n+1}$ is *efficient* if

- $\omega(1) < \omega(3) < \dots < \omega(2n-1),\ \omega(2) < \omega(4) < \dots < \omega(2n)$
- $\omega(1) < \omega(2), \omega(3) < \omega(4), \dots, \omega(2n-1) < \omega(2n)$

Then, if $\omega \in S_{2n+1}$ we consider the map

$$\Phi_\omega : E(\omega) \to \mathcal{P}(n+1), \Phi_\omega(u) = u \cdot L_{null}$$

We have the following theorem:

Theorem 8 ([LP]) *Φ_ω is injective if and only if ω is efficient. For each non-efficient ω, there is unique efficient permutation ω_e such that the image of Φ_ω equals the image of Φ_{ω_e}*

Comparing this theorem with Theorem 2, it's easy to see that if ω is not efficient, then the network corresponding to the image of Φ_ω are exactly those which can be reduced by reduction $R(1) - R(4)$, and if ω is efficient, then the image of Φ_ω corresponds to the minimal networks defined in Section 2. The following theorem is the connection between $(El_{2n})_{\geq 0}$ with $\mathcal{P}(n+1)$

Theorem 9 ([LP]) *Let N_{null} be the empty network with $n+1$ boundary vertices. Then $L((EL_{2n})_{\geq 0} \cdot N_{null})$ is a dense subset of $\mathcal{P}(n+1)$.*

4 Electrical Lie Algebra of Other Finite Types

Note that the relation (3) in Theorem 5 is similar to the Coxeter relation of type A. One would naturally think of what the possible electrical Lie theory for other finite types are. Indeed in [LP], a definition of electrical Lie algebra of finite type is given.

Let X be a Dynkin diagram of finite type, and $A = (a_{ij})$ be the corresponding Cartan matrix. Define e_X to be the Lie algebra generated over $\mathbb{R}$ by the e_i's where i runs over the vertex indices in X, modulo the relations:

- $ad(e_i)^{1-a_{ij}}(e_j) = 0$, if $i \neq j, a_{ij} \neq -1$
- $ad(e_i)^{1-a_{ij}}(e_j) = -2e_i$, if $i \neq j, a_{ij} = -1$

Lemma 10 ([LP]) *e_X has a spanning set indexed by positive roots of the type X root system.*

This is an upper bound of the dimension of electrical Lie algebra.

Conjecture 11 ([LP]) *The above spanning set is a basis of the corresponding Lie algebra.*

Our work establishes the conjecture for types A_{2n+1}, B_n and C_n.

In the previous section, only the Dynkin diagram A_{2n} is considered. By the above lemma and dimension of sp_{2n}, one can deduce that the spanning set in the above lemma is actually a basis for el_{2n}. Since el_{2n+1} (*i.e.* $e_{A_{2n+1}}$) is a Lie subalgebra of el_{2n+2}, and its spanning set is a subset of a basis of el_{2n+2}, one knows its dimension is $(n+1)(2n+1)$. To translate the abstract Lie algebra el_{2n+1} to something familiar, we need the notion of the *odd symplectic* Lie algebra. In [GZ] Gelfand and Zelevinsky give one possible definition of odd symplectic Lie group and Lie algebra: Let $V = \mathbb{R}^{2n+2}$ with standard basis $\{e_i\}_{i=1}^{2n+2}$. Let $\{f_i\}_{i=1}^{2n+2}$ be the corresponding dual basis of V^*. Let B be a nondegenerate skew symmetric bilinear form of V, and β be one nonzero element in V^*. Let $GZ(V, \beta, B)$ be the Lie subgroup of $GL(V)$ which preserves both β and B, whose Lie algebra is denoted as $gz(V, \beta, B)$. Denote $GZ(V, \beta, B)$ as SP_{2n+1}, and $gz(V, \beta, B)$ as sp_{2n+1}. Since $SP_{2n+2} = \{M \in GL(V) | M$ preserves $B\}$, $sp_{2n+1} \subset sp_{2n+2}$. Fix $B = \begin{pmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{pmatrix}$ and $\beta = f_{n+1}$ in this definition.

Theorem 12 $el_{2n+1} \cong sp_{2n+1}$.

Similar to type A_{2n}, we can also define EL_{2n+1}, its nonnegative part $(EL_{2n+1})_{\geq 0}$, and an action on $\mathcal{P}(n+1)$, and it is straight-forward to see that Theorem 5, 6, 7, 8, 9 still hold.

We also have a full understanding of the stucture of e_{B_n}. It is actually a direct sum of two type A electrical Lie algebras.

Theorem 13 *e_{B_n} is isomorphic to $el_n \oplus el_{n-1}$ as Lie algebras.*

Before stating our results for types C and D, we need to make some definitions. Suppose L is a Lie algebra, and V is a representation of L. Then one can define a new Lie algebra to be the semi-directed product $L \ltimes V$ as follows: The underlying set is $L \oplus V$. Define the bracket $[\ ,\ ]_{\ltimes}$: Suppose $x_1, x_2 \in L$, $v_1, v_2 \in V$, then $[x_1, x_2]_{\ltimes} = [x_1, x_2]$, $[x_1, v_1]_{\ltimes} = x_1 \cdot v_1$, which is the result of action on v_1 by x_1, and $[v_1, v_2]_{\ltimes} = 0$, and then extend the Lie bracket by linearity. Let $\mathbf{0} = (0, 0, \ldots, 0)$, $\lambda = (1, 1, 0, \ldots, 0)$ be in the weight space of the root system of type C_{2n}. Let $V_\lambda, V_{\mathbf{0}}$ be the irreducible representations labelled by λ and $\mathbf{0}$ respectively. Then,

Theorem 14 *$e_{C_{2n}}$ is isomorphic to $el_{2n} \ltimes (V_\lambda \oplus V_{\mathbf{0}})$.*

Since $e_{C_{2n+1}}$ is a subalgebra of $e_{C_{2n+2}}$, similar argument as above shows that the dimension of $e_{C_{2n+1}}$ actually meets the upper bound in Lemma 10. Furthermore, $e_{C_{2n+1}}$ is also isomorphic to $el_{2n+1} \ltimes I$, where I is an abelian ideal. Since the representation theory for el_{2n+1} is not well understood, at this point I cannot be expressed as anything nicer.

Theorem 15 *$e_{D_{2n+1}}$ has a radical ideal I such that $[I, I] \neq 0$, and $[[I, I], [I, I]] = 0$ and $e_{D_{2n+1}}/I$ is isomorphic to el_{2n}.*

As for the exceptional Lie types, since their Dynkin diagram has a fixed number of vertices, in theory the structure should be computable by a computer algebra system. Lam and Pylyavskyy computed the dimension of e_{G_2}:

Theorem 16 ([LP]) $\dim(e_{G_2}) = 6$.

The following section will be devoted to the introduction of the type B electrical Lie theory.

5 Type B Electrical Lie Theory and Mirror Symmetric Circular Planar Networks

Let $\{e_1, e_2, \ldots, e_n\}$ be the generators of type B electrical Lie algebra e_{B_n}, satisfying the relation:

(a) If $[e_i, e_j] = 0$ when $|i - j| \geq 2$, $[e_i, [e_i, e_j]] = -2e_i$ when $|i - j| = 1$ and $i, j \neq 1$;

(b) $[e_1, [e_1, e_2]] = -2e_1$, $[e_2, [e_2, [e_2, e_1]]] = 0$.

Let E_{B_n} denote the real Lie group associated with e_{B_n} (by Theorem 13, we can think E_{B_n} as a product of SP_n and SP_{n-1}). Let $u_i(t) := \exp(te_i)$. Let $(E_{B_n})_{\geq 0}$ be the Lie semisubgroup of E_{B_n} generated by $u_i(t)$'s, where $t > 0$. Lam and Pylyavskyy show the following theorem:

Theorem 17 ([LP]) *If $t > 0$, then $u_i(t)$'s satisfy the following relation:*

(1) $u_i(a)u_j(b) = u_j(b)u_i(a)$ *if* $|i - j| \geq 2$

(2) $u_i(a)u_i(b) = u_i(a + b)$

(3) $u_i(a)u_j(b)u_i(c) = u_j\left(\frac{bc}{a + c + abc}\right)u_i(a + c + abc)u_j\left(\frac{ab}{a + c + abc}\right)$ *if* $|i - j| = 1, i, j \neq 1$

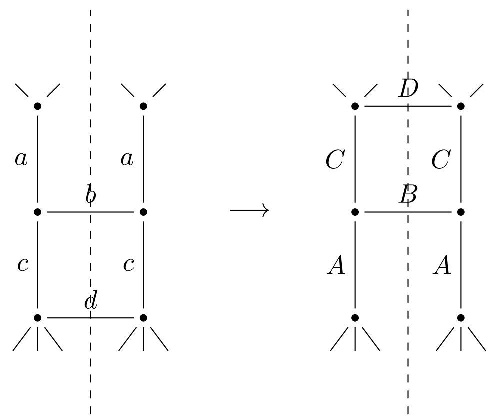

Fig. 4: Square mirror symmetric move

(4) $u_2(t_1)u_1(t_2)u_2(t_3)u_1(t_4) = u_1(p_1)u_2(p_2)u_1(p_3)u_2(p_4)$ *where*

$$p_1 = \frac{t_2t_3^2t_4}{\pi_2}, p_2 = \frac{\pi_2}{\pi_1}, p_3 = \frac{\pi_1^2}{\pi_2}, p_4 = \frac{t_1t_2t_3}{\pi_1}.$$

where

$$\pi_1 = t_1t_2 + (t_1+t_3)t_4 + t_1t_2t_3t_4, \pi_2 = t_1^2t_2 + (t_1+t_3)^2t_4 + t_1t_2t_3t_4(t_1+t_3)$$

We introduce a new class of planar networks: *A mirror symmetric circular planar electrical network* is a circular planar electrical network, whose underlying graph is mirror symmetric to itself with respect to some straight line, and the weight of each edge is the same as its mirror image (vertices and edges are allowed to lie on the mirror line).

We consider the following mirror reduction:

MR(1) Mirror symmetrically move a vertex of degree one and its incident edge, or mirror symmetrically move a self loop.

MR(2) If a vertex has degree 2, then use series reduction R(3) of circular planar electrical network on it and its mirror image.

MR(3) If there is a pair of parallel edges between two vertices, then use the parallel reduction R(4) of circular planar electrical network on them and their mirror images.

And the mirror moves:

MM1 (Mirror symmetric star-triangle move). If a vertex has exactly three incident edges or we have a triangle, use the move M of circular planar electrical network on this shape and its mirror image.

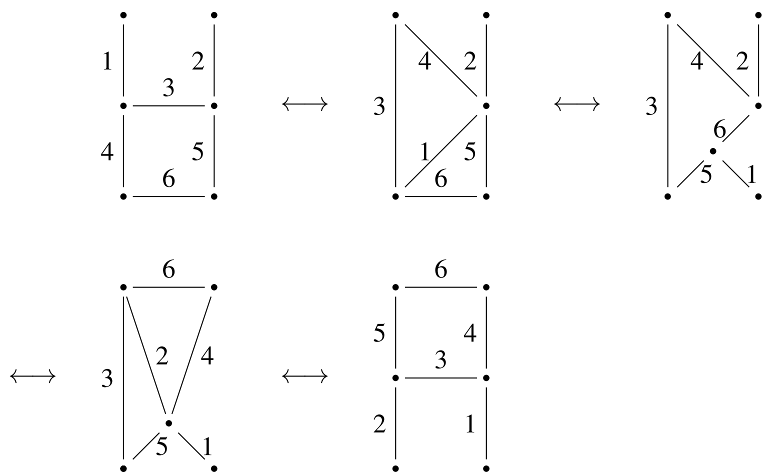

Fig. 5: Decomposition of mirror symmetric square move

MM2 (Mirror symmetric square move) We will demonstrate this move in Figure 4, and the weights are given as by the transformation $\phi : (a, b, c, d) \to (A, B, C, D)$.

$$\begin{aligned}
A &= \frac{ad + bc + cd + 2bd}{b} \\
B &= \frac{ad + bc + cd + 2bd}{(a + c)^2 d + b(c^2 + 2ad + 2cd)} \\
C &= \frac{ac(ad + bc + cd + 2bd)}{(a + c)^2 d + b(c^2 + 2ad + 2cd)} \\
D &= \frac{a^2 bd}{(a + c)^2 d + b(c^2 + 2ad + 2cd)}
\end{aligned}$$

We also have $\phi(A, B, C, D) = (a, b, c, d)$, so ϕ is an involution.

Proposition 18 *MR(1)-MR(3), MM(1)-(2) do not change the electrical properties of the mirror symmetric circular planar network, i.e. do not change the response matrix.*

Proof: (sketch)
By R(1)-R(4), and M, we can see MR(1)-MR(3) and MM(1) have no effect on the response matrix. As for MM(2), it can be decomposed as the sequence of M(star-triangle) moves (Figure 5), and the change of weights also follows this sequence of transformations. □

For now let's focus on $e_{B_{2n}}$. Let N be a mirror symmetric circular planar network with $2n$ boundary vertices, indexed clockwisely as $1, 2, \ldots, n, n', \ldots, 2', 1'$, where i and i' are mirror symmetric image of each other. Now define actions $v_i(t)$ on N (Figure 6):

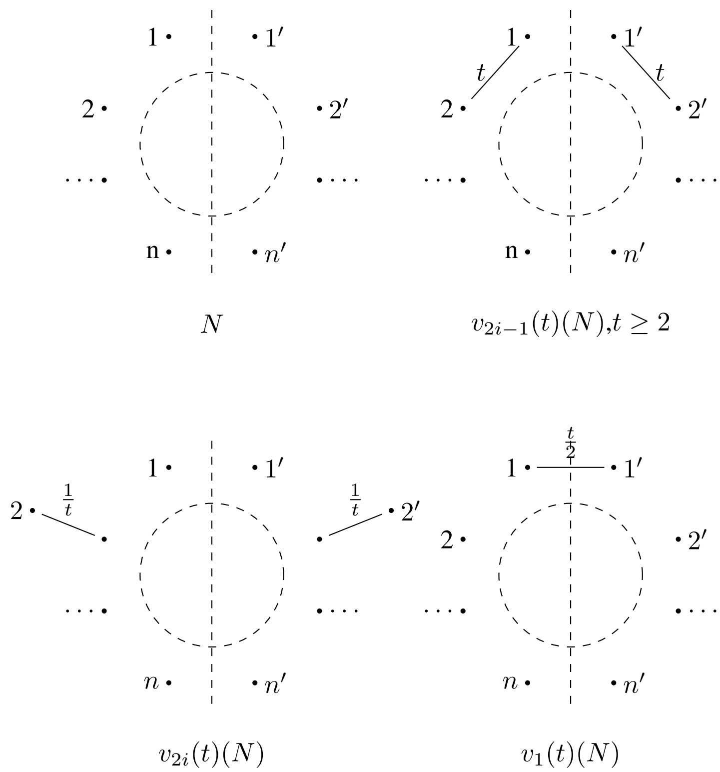

Fig. 6: $(E_{B_{2n}})_{\geq 0}$ action

Mirror symmetrically adding a boundary spike For all $i \in [n]$, define $v_{2i}(t) \cdot N$ to be the action of adding boundary spikes with weights $1/t$ on both vertices i and i', and treating the new vertices as new boundary vertices 1 and $1'$, and old boundary vertices 1 and $1'$ as interior vertices.

Mirror symmetrically adding a boundary edge For $i \in [n] \setminus \{1\}$, define $v_{2i-1}(t) \cdot N$ to be the action of adding boundary edges between vertices $i-1$ and i, and between $i'-1$ and i', both with weight t.

Adding an edge between vertices 1 and $1'$ Define $v_1(t)$ be the action of adding an boundary edge between vertices 1 and $1'$ with weight $t/2$.

Theorem 19 *The actions $\{v_i(t)\}_{i=1}^{2n}$ satisfy the relations of $\{u_i(t)\}_{i=1}^{2n}$ in Theorem 17 when acting on the mirror symmetric network N.*

Proof: (sketch) The first three relations are a direct consequence of Theorem 7. The last relation is actually a consequence of Figure 4, if we set $a = \frac{1}{t_1}, b = \frac{t_2}{2}, c = \frac{1}{t_3}, d = \frac{t_4}{2}$. $\square$

For $\omega \in B_n$, pick a reduced word $\mathbf{i} = i_1 i_2 \dots i_l$ of ω, and for $a_i \geq 0$, where $i = 1, 2, \dots, l$, define a map $\psi_{\mathbf{i}}(a_1, a_2, \dots, a_l) = u_{i_1}(a_1)u_{i_2}(a_2)\dots u_{i_l}(a_l)$. Denote the image of $\psi_{\mathbf{i}}$ to be E_ω. According to Theorem 17, the image of this map is independent of the choice of reduced words of ω. Thus again $E(\omega)$ is well-defined. We would like to prove the following result in order to construct an analogous theory as in Type A.

Conjecture 20 *E_{B_n} is a disjoint union of $E(\omega)$. Each $\psi_{\mathbf{i}}$ is a homeomorphism between $\mathbb{R}^l_{>0}$ and $E(\omega)$*

If we can prove this is true, then we get an action of $E_{B_{2n}}$ on the mirror symmetric circular planar electrical network. And we may be able to develop the same theory as in type A.

6 Possible Work in the Future

The class of mirror symmetric circular planar electrical networks is still not well studied. Similar to circular planar networks, we can ask the following two questions:

(a) What is the set $\mathcal{MP}$ of response matrices of all mirror symmetric circular planar electrical networks?

(b) Under what conditions, do two mirror symmetric networks have the same response matrix?

The second question will become more interesting since the mirror symmetric square move comes into play. However, we hope the similar theorem would also hold as in type A case. Also one can look at the images of the $(E_{B_{2n}})_{\geq 0}$ acting on the empty network, and see the relations between this image and the set $\mathcal{MP}$.

As for type C, we already know the structure of the corresponding electrical Lie algebra and Lie group. We would like to find a new class of networks which $(E_{C_n})_{\geq 0}$ could possibly act on. For type D, we'd like to understand its structure for now.

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