Robust Finite-time Stabilization and Observation of a Planar System Revisited

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***Abstract*— The second order planar nonlinear affine control problem is studied. A homogeneous robust finite-time stabilizing control is developed for the most general case of matched and, the more challenging, mismatched nonlinear perturbations. A homogeneous observer is designed for the planar system. Explicit restrictions on the observer gains and nonlinearities are presented. The main contribution lies in the proposed combination of the explicit and implicit Lyapunov function methods as well as weighted homogeneity while providing finite-time stability analysis. Theoretical results are supported by numerical simulations.**

I. INTRODUCTION

The problem of finite-time stabilization goes back to the classical results of optimal control design [10]. Indeed, the time-optimal bang-bang algorithm represented in state feedback form is perhaps the most famous example of a finite-time stabilizing control. The first theoretical investigations of finite-time stability and stabilization appear in [26], where, in particular, the definition of finite-time stability as well as the corresponding Lyapunov theorems are developed for general dynamical systems. Subsequently, finite-time controllers are presented in [11]. Currently this topic is popular in the context of sliding mode [17], [18], [24], nonlinear [5], [6], [21], [23], robust [3], [12] and optimal [9], [25] control systems. The problem of finite-time stability also appears in mechanical systems with dry friction [1], [7].

Finite-time stability analysis is frequently related to the so-called homogeneity property of a dynamical system. Homogeneity theory has an established history going back to Euler and his homogeneous function theorem. Currently the so-called weighted homogeneity [5], [14], [18] has been introduced in the paper of V.I. Zubov [27]. It is well-known [5], [14], [18] that asymptotically stable systems with negative homogeneity degree are finite-time stable. It is noteworthy that the problem of computing an upper bound on the finite settling time of homogeneous finite time systems without the help of finite time Lyapunov functions is an interesting problem. Recent results for the planar case can be found in [21] which makes use of a general result provided in [18].

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Second order nonlinear models cover rather a large class of the physical (in particular, mechanical) and biological systems. This paper deals with finite-time control and estimation of planar systems. It revises results of [21] for the case of both matched and mismatched uncertainties. The principal contribution of this analysis is two fold. Firstly, a combination of the explicit and (homogeneity-based) implicit Lyapunov function analysis is presented for asymptotic and finite-time stability, respectively. It allows the control parameters and system nonlinearities to be quantitatively explored as well settling time estimates to be specified. Secondly, the upper bound on the unmatched disturbances for the double integrator system covers a broader class of disturbances with a non-Lipschitz upper bound. The paper is organized as follows. Section II presents the problem formulation followed by Sections III and IV which outline the controller and the observer respectively. Numerical examples are given in Section V. Section VI concludes the paper.

II. PROBLEM STATEMENT

Consider the system

$$\begin{cases} \dot{x}_1(t) = x_2(t) + \omega_1(t, x_1(t), x_2(t)), \\ \dot{x}_2(t) = u(t) + \omega_2(t, x_1(t), x_2(t)), \end{cases} \quad t \in \mathbb{R}, \quad (1)$$

where $x_1, x_2 \in \mathbb{R}$ are states, u is the control input, $\omega_1 : \mathbb{R}^3 \to \mathbb{R}$ and $\omega_2 : \mathbb{R}^3 \to \mathbb{R}$ are non-linear functions.

In this paper we deal with two problems:

- *finite-time stabilization* of the system (1) under the assumption that the nonlinear functions ω_1 and ω_2 are **unknown** but appropriately bounded;
- *finite-time observation* of the system (1) with the output

$$y(t) = x_1(t) \quad (2)$$

under the assumption that the nonlinear functions ω_1 and ω_2 are **known**.

Recent Lyapunov based results on finite-time control and observation for the perturbed double integrator can be found in [16], [17], [19]. The most recent robust finite time stabilization result when $\omega_1 = 0$ and $|\omega_2| \leq M|x_2|^{\alpha}$, for some a priori known constant $M > 0$, can be found in the reference [20]. The results in [2], [21] present finite time stabilization for the case of matched non-Lipscitz perturbations while the result in [13] studies robustness to $\mathcal{C}^1$ matched disturbances, i.e. $\omega_1 = 0$ and $\omega_2 \neq 0$. The paper [15] presents output feedback finite time stabilization for mismatched perturbations. It is noted in this reference that the output and not the whole state is finite time stabilized. In comparison, the proposed full state feedback based results in this paper are superior for the planar case as finite time stabilization of both the states $x_1(t)$ and $x_2(t)$ is achieved via the implicit Lyapunov function approach. Existing homogeneous[1] controllers [4], [19] and homogeneous observers [22] are utilised in this paper. The main challenge is to achieve finite-time stabilization as well as observation of the system (1) in the presence of the perturbations $\omega_1 \neq 0$ and $\omega_2 \neq 0$.

III. Finite-Time Controller

Define the homogeneous feedback law [4], [19] as follows

$$u(t){=}-\mu_1|x_1(t)|^{\frac{\alpha}{2-\alpha}}\mathrm{sign}[x_1(t)]{-}\mu_2|x_2(t)|^{\alpha}\mathrm{sign}[x_2(t)], \quad (3)$$

where $\alpha \in [0,1)$, $\mu_1 > 0$, $\mu_2 > 0$ are control parameters. It is well-known [4, Th. 2], [18, Th. 3.1], that in the unperturbed case with $\omega_1 = \omega_2 = 0$, the closed-loop system (1)-(3) is homogeneous of negative degree. Hence, asymptotic stability of the origin implies finite-time stability, i.e. the origin will be reached after some finite instant of time. The theorem below characterizes (possibly uncertain) functions ω_1 and ω_2, which do not destroy the finite-time stability property.

Theorem 1: If for some $\alpha \in [0,1)$ and $p_{ij} \geq 0$, $i,j = 1,2$ the nonlinear functions ω_1 and ω_2 are bounded as follows

$$\begin{array}{l} |\omega_1(t,x_1,x_2)| \leq p_{11}|x_1|^{\frac{1}{2-\alpha}} + p_{12}|x_2|, \\ |\omega_2(t,x_1,x_2)| \leq p_{21}|x_1|^{\frac{\alpha}{2-\alpha}} + p_{22}|x_2|^{\alpha}, \end{array} \quad (4)$$

for all $(t,x_1,x_2) \in \mathbb{R}^3$ and the control parameters $\mu_1 > 0, \mu_2 > 0$ satisfies the restrictions

$$\begin{array}{c} \frac{2\mu_1-3p_{11}-(\mu_2+p_{22})(2-\alpha)}{2\xi_1+\xi_3(\alpha+1)} > \frac{\left(\frac{2}{3-\alpha}\right)^{\frac{3-\alpha}{2}}}{\mu_1^{\frac{2-\alpha}{2}}}, \\ \frac{2\mu_1-3p_{11}-(\mu_2+p_{22})(2-\alpha)}{2\xi_1+\xi_3(\alpha+1)} > \frac{2+p_{11}+2p_{21}+\alpha(\mu_2+p_{22})}{2\xi_2-\xi_3(1-\alpha)} > 0, \\ \xi_1 = \frac{(3-\alpha)(\mu_1p_{11}+\alpha(\mu_1p_{12}+p_{21})/(\alpha+1))}{2^{\frac{3-\alpha}{2}}(\mu_1(2-\alpha))^{\frac{\alpha-1}{2}}}, \\ \xi_2 = \frac{(3-\alpha)(\mu_2-p_{22}-(\mu_1p_{12}+p_{21})/(\alpha+1))}{2^{\frac{3-\alpha}{2}}} \\ \xi_3 = (\mu_1(2-\alpha))^{\frac{\alpha-1}{2}}\xi_1, \end{array} \quad (5)$$

then the feedback (3) stabilizes the system (1) to the origin in finite time with the settling time estimate (18).

Proofs of all the theorems are given in the Appendix, where settling-time estimates are also presented.

The nonlinear structure of the closed-loop system (1)-(3) implies a nonlinear restriction (5) to the control gains μ_1 and μ_2. Note that the system of nonlinear inequalities (5) is feasible for sufficiently small $p_{11} \geq 0$ and $p_{12} \geq 0$.

Consider some particular cases.

- If $p_{11} = p_{12} = p_{21} = 0$ then the system (5) is feasible for any $p_{22} \geq 0$, since it is equivalent to $\mu_1 > (\mu_2 + p_{22})(2-\alpha)/2$ and $\mu_2 > p_{22}$. This coincides with the conditions derived in [20].
- In the case of matched perturbations ($p_{11} = p_{12} = 0$), it can be shown that
$$\begin{array}{c} \mu_2 > p_{22} + \frac{2-\alpha}{2}p_{21}, \\ \mu_1 > \frac{2-\alpha}{2}(\mu_2{+}p_{22}){+}\frac{\alpha p_{21}}{(3-\alpha)^{\frac{1-\alpha}{2}}}\left(\frac{(2-\alpha)^{\frac{1-\alpha}{2}}}{(\alpha+1)\sqrt{\mu_1}}{+}\frac{0.5}{\mu_1^{\frac{2-\alpha}{2}}}\right), \\ \frac{\mu_1-\frac{2-\alpha}{2}(\mu_2+p_{22})}{(\mu_1(2-\alpha))^{0.5(1-\alpha)}+0.5(\alpha+1)} > \frac{\alpha p_{21}(1+p_{21}+0.5\alpha(\mu_2+p_{22}))}{(\alpha+1)(\mu_2-p_{22}-0.5p_{21}(2-\alpha))}. \end{array}$$
The obtained system of inequalities is feasible with respect to μ_1, μ_2 for any $p_{21} \geq 0$ and $p_{22} \geq 0$.
- In the sliding mode case ($\alpha = 0$), the inequalities (4) become
$$\begin{array}{l} |\omega_1(t,x_1,x_2)| \leq p_{11}|x_1|^{\frac{1}{2}} + p_{12}|x_2|, \\ |\omega_2(t,x_1,x_2)| \leq p_{21} + p_{22}, \end{array}$$
and the inequalities (5) can be rewritten as follows
$$\mu_1 > \mu_2 + \left(3/2 + 0.5/\sqrt{3}\right)p_{11} + p_{22} + p_{11}\frac{\sqrt{\mu_1}}{\sqrt{6}}$$
$$\mu_2 > p_{21} + p_{22} + \mu_1\left(p_{12} + p_{11}\right)$$
$$\frac{\mu_1-\mu_2-\frac{3}{2}p_{11}-p_{22}}{\mu_1p_{11}\left(1+2^{\frac{3}{2}}\sqrt{\mu_1}\right)} > \frac{1+\frac{1}{2}p_{11}+p_{21}}{\mu_2-\mu_1(p_{12}+p_{11})-p_{22}-p_{21}}.$$
Evidently, the last system of inequalities is feasible for $p_{12} < 1$ and sufficiently small $p_{11} < 1 - p_{12}$.

IV. Finite-Time Observer

Consider now the problem of finite-time observation of the system (1) with the measured output (2). Consider the so-called homogeneous observer [22] of the form

$$\begin{array}{l} \frac{d\hat{x}_1(t)}{dt}{=}-\nu_1\,|e_1(t)|^{\frac{1+\beta}{2}}\,\mathrm{sign}[e_1(t)]{+}\hat{x}_2(t){+}\omega_1(t{,}y(t){,}\hat{x}_2(t)) \\ \frac{d\hat{x}_2(t)}{dt}{=}-\nu_2\,|e_1(t)|^{\beta}\,\mathrm{sign}[e_1(t)] + \omega_2(t,y(t),\hat{x}_2(t)) \end{array} \quad (6)$$

where $\nu_1 > 0$, $\nu_2 > 0$, $\beta \in [0,1)$ and $e_1(t) = \hat{x}_1(t) - y(t)$. Denote $e_2(t) = \hat{x}_2(t){-}x_2(t)$. The observation error equation has the form

$$\begin{array}{l} \dot{e}_1(t) = -\nu_1|e_1(t)|^{\frac{1+\beta}{2}}\mathrm{sign}[e_1(t)] + e_2(t) + \Delta\omega_1, \\ \dot{e}_2(t) = -\nu_2|e_1(t)|^{\beta}\mathrm{sign}[e_1(t)] + \Delta\omega_2, \end{array} \quad (7)$$

where $\Delta\omega_1 = \omega_1(t,y(t),\hat{x}_2(t)) - \omega_1(t,y(t),x_2(t))$ and $\Delta\omega_2 = \omega_2(t,y(t),\hat{x}_2(t)) - \omega_2(t,y(t),x_2(t))$.

Theorem 2: If for some $\beta \in [0,1)$, $q_1 \geq 0$ and $q_2 \geq 0$ the inequalities

$$|\Delta\omega_1| \leq q_1|e_2| \quad \text{and} \quad |\Delta\omega_2| \leq q_2|e_2|^{\frac{2\beta}{1+\beta}} \quad (8)$$

hold and the observer parameters $\nu_1 > 0$ and $\nu_2 > 0$ satisfy the restrictions

$$\begin{array}{c} \frac{\frac{1}{2}-q_1-\frac{q_2\beta}{(1+\beta)\sqrt{\nu_2}}}{\sqrt{\nu_2}\left(\eta_2+\frac{1+3\beta}{2(1+\beta)}\eta_3\right)} > \left(\frac{2(1+\beta)}{3+\beta}\right)^{\frac{3+\beta}{2(1+\beta)}}, \\ \frac{\frac{1}{2}-q_1-\frac{q_2\beta}{(1+\beta)\sqrt{\nu_2}}}{\sqrt{\nu_2}\left(\eta_2+\frac{1+3\beta}{2(1+\beta)}\eta_3\right)} > \frac{\left(\sqrt{\nu_2}+\frac{\nu_1^2}{2\sqrt{\nu_2}}+\frac{q_2}{1+\beta}\right)}{\left(\eta_1-\frac{1-\beta}{2(1+\beta)}\eta_3\right)} > 0, \\ \eta_1 = \frac{(3+\beta)}{2(1+\beta)^{\frac{3+\beta}{2(1+\beta)}}}\left(\nu_1-\frac{2q_1\beta}{1+3\beta}\right), \\ \eta_2 = \frac{(3+\beta)}{2(1+\beta)(2\nu_2)^{\frac{1-\beta}{2(1+\beta)}}}\left(\frac{q_1(1+\beta)}{1+3\beta}+\frac{q_2}{\nu_2}\right), \\ \eta_3 = \frac{(3+\beta)}{2(1+\beta)^{\frac{3+\beta}{2(1+\beta)}}}\left(\frac{q_1(1+\beta)}{1+3\beta}+\frac{q_2}{\nu_2}\right). \end{array} \quad (9)$$

[1]A function $g : \mathbb{R}^n \to \mathbb{R}$ (resp. a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$) is said to be r-homogeneous of degree m iff for all $\lambda > 0$ and for all $x \in \mathbb{R}^n$ we have $g(D(\lambda)x) = \lambda^m g(x)$ (resp. $f(D(\lambda)x) = \lambda^m D(\lambda)f(x)$), where $D(\lambda) = \mathrm{diag}\{\lambda^{r_i}\}_{i=1}^n$ and $r = (r_1, \ldots, r_n)^T \in \mathbb{R}_+^n$.

then the system (7) is finite-time stable.

The system of inequalities (9) is feasible for sufficiently small $q_1 \geq 0$ and $q_2 \geq 0$. To estimate the upper bounds of the parameters q_1 and q_2, consider particular cases.

- If $q_1 = q_2 = 0$, the inequalities (9) imply that the system (7) is finite time stable for any positive $\nu_1 > 0$ and $\nu_2 > 0$.
- If $q_2 = 0$ then the system of inequalities (9) is feasible for any $q_1 \in [0, 0.5)$.
- For $q_1 = 0$ the system (9) is feasible $\forall\, q_2 \geq 0$.

V. Numerical Examples

A. Finite-time stabilization of the uncertain planar system

Consider the nonlinear system of the form

$$\begin{cases} \dot{x}_1(t) = \frac{x_1}{4+4|x_1|} + \frac{x_2|x_2|}{8\sqrt{x_1^2+x_2^2}} + x_2, \\ \dot{x}_2(t) = u(t), \end{cases} \tag{10}$$

where $x_1, x_2 \in \mathbb{R}$. It is straightforward to see that the restriction (4) is satisfied for $\alpha = 0$ with $p_{11} = p_{12} = 1/8$ and $p_{21} = p_{22} = 0$. Selecting $\mu_1 = 32$ and $\mu_2 = 24$, the conditions (5) hold. The simulation results are given in Fig. 1. The settling time obtained from numerical simulations is

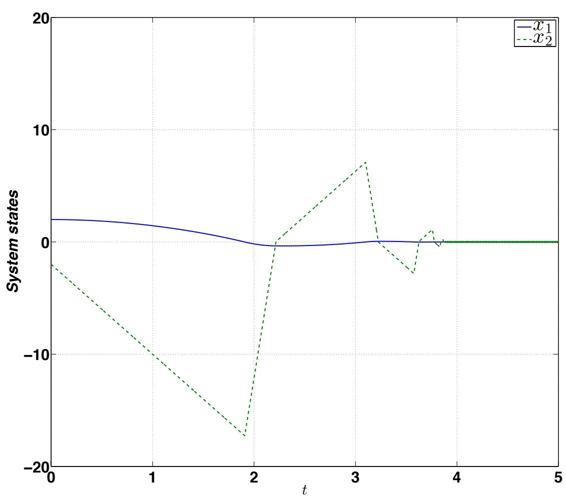

Fig. 1. Simulation results for finite-time stabilization

around 4. The analytical estimate of the settling time (see, the proof of Theorem 1, formula (18)) gives 8.4 that is conservative, although, less conservative than the latest result [21].

B. Finite-time observation of the planar system

Consider the non-linear positive system

$$\begin{cases} \dot{x}_1(t) = -2x_1(t) + 1 - \frac{1}{2}x_1(t)x_2(t), \\ \dot{x}_2(t) = \left(\frac{1}{4} - x_1(t)\right) x_2^\rho(t), \end{cases} \tag{11}$$

where $x_1 \geq 0$ and $x_2 \geq 0$ are state variables, $\rho \in [0, 1)$. The paper [8] utilizes this system with $\rho = 1$ as the tumour-immune model for the so-called "steady control of tumour growth". In this case, the variable x_1 is related to the density of the lymphocyte population and x_2 is proportional to the density of the tumour cells. Due to physical restrictions, the variables are assumed to be bounded as follows $0 < x_1^{\min} \leq x_1(t) \leq x_1^{\max}$ and $0 \leq x_2(t) \leq x_2^{\max}$ for all $t \in \mathbb{R}$. It is assumed that the variable x_1 can only be measured and the problem is to reconstruct the state variable x_2 by means of the model (11).

Introduce a state observer of the form

$$\frac{d\hat{x}_1(t)}{dt} = 1 - 2x_1(t) - \frac{x_1(t)}{2}\hat{x}_2(t) - \tilde{\nu}_1(t)|e_1(t)|^{\frac{1+\beta}{2}}\mathrm{sign}[e_1(t)],$$
$$\frac{d\hat{x}_2(t)}{dt} = \frac{1}{4}\hat{x}_2^\rho(t) - x_1(t)\hat{x}_2^\rho(t) + \tilde{\nu}_2(t)|e_1(t)|^\beta \mathrm{sign}[e_1(t)],$$

where $e_1(t) = \hat{x}_1(t) - x_1(t)$,

$$\tilde{\nu}_1(t) = \frac{\nu_1 x_1(t)}{2} \quad \text{and} \quad \tilde{\nu}_2(t) = \frac{\nu_2 x_1(t)}{2},$$

with $\nu_1 > 0$ and $\nu_2 > 0$. The change of time $\tau = \int_0^t \frac{2}{x_1(s)} ds$ allows the error equation to be expressed in the form

$$\dot{e}_1(\tau) = e_2(\tau) - \nu_1|e_1(\tau)|^{\frac{1+\beta}{2}}\mathrm{sign}[e_1(\tau)],$$
$$\dot{e}_2(\tau) = \frac{(4x_1(\tau)-1)(\hat{x}_2^\rho(t) - x_2^\rho(t))}{2x_1(\tau)} - \nu_2|e_1(\tau)|^\beta \mathrm{sign}[e_1(\tau)],$$

where $e_2 = x_2(t) - \hat{x}_2(t)$ Therefore, the obtained error equation is equivalent to (7) with $\Delta\omega_1 = 0$ and $|\Delta\omega_2| \leq q_2 \cdot |e_2|^\rho$, where $q_2 = \max\limits_{x_1 \in \{x_1^{\min}, x_1^{\max}\}} \left|2 - \frac{1}{2x_1}\right|$.

This satisfies the conditions of Theorem 2 if $\beta = \frac{\rho}{2-\rho}$. Note that for the case $\rho = 1$, the finite-time homogeneous observer $\beta \in [0, 1)$ can be constructed locally using the estimate of $\Delta\omega_2$ and a-priori boundedness of the state variables x_1 and x_2.

Fig. 2 presents simulation results for $\rho = 0.5$, $x_1^{\min} = 0.1$, $x_1^{\max} = 1$ and the parameters $\nu_1 = 10$ and $\nu_2 = 400$ satisfying (9).

VI. Conclusions

Results on robust finite time stabilization are presented. A combination of homogeneous Lyapunov functions and implicit Lyapunov functions are utilised to prove finite time stability of planar uncertain system. Both the problems of finite time stable controller design and finite time stable observer design are analysed in the presence of mismatched disturbances that admit a non-Lipschitz upper bound. This is superior to existing results in the area. An interesting open problem is to identify similar Lyapunov functions to extend these results to more general $n-$dimensional system.

VII. Appendix

Below Young's inequality is used whereby:

$$|z_1||z_2| \leq \frac{|z_1|^r}{r} + \frac{|z_2|^q}{q}, \quad \frac{1}{r} + \frac{1}{q} = 1, \quad r, q > 1, \tag{12}$$

where z_1, z_2 are real numbers.

A. Proof of Theorem 1

I. Asymptotic Stability (Explicit Lyapunov Method)

Consider the Lyapunov function candidate

$$V_0(x_1, x_2) = cU^{\frac{3-\alpha}{2}}(x_1, x_2) + \mu_1^{\frac{2-\alpha}{2}} x_1 x_2, \tag{13}$$

where $c > 0$ and

$$U(x_1, x_2) = \mu_1 \frac{2-\alpha}{2}|x_1|^{\frac{2}{2-\alpha}} + \frac{1}{2}x_2^2. \tag{14}$$

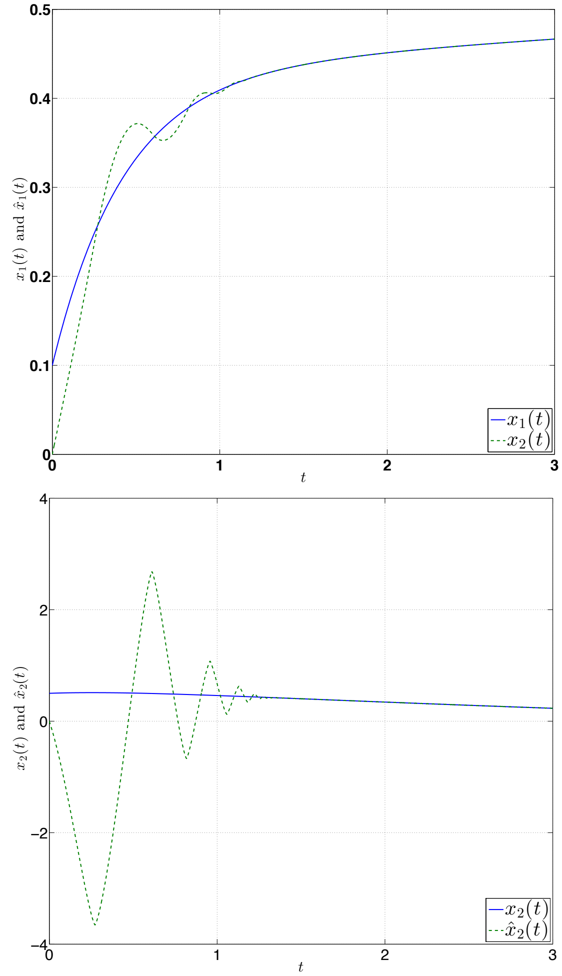

Fig. 2. Simulation results for finite-time observation

Applying Young's inequality for $q = 3-\alpha$ and $r = \frac{3-\alpha}{2-\alpha}$ produces

$$\left(\mu_1^{\frac{2-\alpha}{2}}|x_1|\right)^{\frac{2}{3-\alpha}}|x_2|^{\frac{2}{3-\alpha}} \leq \mu_1\frac{2-\alpha}{3-\alpha}|x_1|^{\frac{2}{2-\alpha}} + \frac{1}{3-\alpha}x_2^2.$$

Hence, for

$$c > \left(\frac{2}{3-\alpha}\right)^{\frac{3-\alpha}{2}} \tag{15}$$

it follows that $V_0(x_1,x_2) > 0$ outside the origin, i.e. V_0 is a positive definite function.

Since

$$\frac{d}{dt}U^{\frac{3-\alpha}{2}}(x_1,x_2) = \frac{(3\text{-}\alpha)\left(\mu_1|x_1|^{\frac{\alpha}{2-\alpha}}\operatorname{sign}[x_1]\omega_1-\mu_2|x_2|^{\alpha+1}+x_2\omega_2\right)U^{\frac{1-\alpha}{2}}}{2} \leq \frac{(3\text{-}\alpha)\left(\mu_1p_{11}|x_1|^{\frac{\alpha+1}{2\text{-}\alpha}}+(\mu_1p_{12}+p_{21})|x_1|^{\frac{\alpha}{2\text{-}\alpha}}|x_2|-(\mu_2\text{-}p_{22})|x_2|^{\alpha+1}\right)U^{\frac{1\text{-}\alpha}{2}}}{2}$$

holds true, applying Young's inequality for $q = \alpha+1$ and $r = \frac{\alpha+1}{\alpha}$ produces

$$|x_1|^{\frac{\alpha}{2-\alpha}}|x_2| \leq \alpha\frac{|x_1|^{\frac{\alpha+1}{2-\alpha}}}{\alpha+1} + \frac{|x_2|^{\alpha+1}}{\alpha+1}.$$

The following may now be derived

$$\frac{d}{dt}U^{\frac{3-\alpha}{2}}(x_1,x_2) \leq -\frac{(3\text{-}\alpha)\left(\mu_2-p_{22}-\frac{\mu_1p_{12}+p_{21}}{\alpha+1}\right)}{2}U^{\frac{1\text{-}\alpha}{2}}|x_2|^{\alpha+1} + \frac{(3-\alpha)(\mu_1p_{11}+\alpha(\mu_1p_{12}+p_{21})/(\alpha+1))}{2}U^{\frac{1-\alpha}{2}}|x_1|^{\frac{\alpha+1}{2-\alpha}} \leq$$
$$-\xi_2x_2^2+\xi_1|x_1|^{\frac{2}{2-\alpha}}+\xi_3|x_1|^{\frac{\alpha+1}{2-\alpha}}|x_2|^{1-\alpha} \leq$$
$$-(\xi_2-0.5\xi_3(1-\alpha))x_2^2+(\xi_1+0.5\xi_3(\alpha+1))|x_1|^{\frac{2}{2-\alpha}},$$

where

$$\xi_1 = \frac{(3-\alpha)(\mu_1p_{11}+\alpha(\mu_1p_{12}+p_{21})/(\alpha+1))}{2^{\frac{3-\alpha}{2}}(\mu_1(2-\alpha))^{\frac{\alpha-1}{2}}},$$

$$\xi_2 = \frac{(3-\alpha)\left(\mu_2-p_{22}-(\mu_1p_{12}+p_{21})/(\alpha+1)\right)}{2^{\frac{3-\alpha}{2}}},$$

$$\xi_3 = (\mu_1(2-\alpha))^{\frac{\alpha-1}{2}}\xi_1,$$

and Young's inequality is applied on the final step. On the other hand,

$$\frac{d}{dt}(x_1x_2) = x_2^2+x_2\omega_1\text{-}\mu_1|x_1|^{\frac{2}{2\text{-}\alpha}}\text{-}\mu_2x_1|x_2|^{\alpha}\operatorname{sign}[x_2]+x_1\omega_2 \leq$$
$$(1+p_{21})x_2^2+p_{11}|x_1|^{\frac{1}{2-\alpha}}|x_2|-(\mu_1-p_{11})|x_1|^{\frac{2}{2\text{-}\alpha}}+$$
$$(\mu_2+p_{22})|x_1||x_2|^{\alpha}$$

Applying Young's inequality for $q = \frac{2}{\alpha}$ and $p = \frac{2}{2-\alpha}$

$$|x_1||x_2|^{\alpha} \leq \frac{2-\alpha}{2}|x_1|^{\frac{2}{2-\alpha}} + \frac{\alpha}{2}|x_2|^2$$

and the inequality

$$|x_1|^{\frac{1}{2-\alpha}}|x_2| \leq \frac{|x_1|^{\frac{2}{2-\alpha}}}{2} + \frac{x_2^2}{2}$$

produce

$$\dot{V}_0(x_1,x_2) \leq$$
$$-\left(\mu_1^{\frac{2\text{-}\alpha}{2}}\left(\mu_1-\frac{3p_{11}}{2}-\frac{(\mu_2+p_{22})(2-\alpha)}{2}\right)-c\xi_1-\frac{c\xi_3(\alpha+1)}{2}\right)|x_1|^{\frac{2}{2-\alpha}}$$
$$-\left(c\xi_2-\frac{c\xi_3(1-\alpha)}{2}-\mu_1^{\frac{2-\alpha}{2}}\left(1+\frac{p_{11}}{2}+p_{21}+\frac{\alpha(\mu_2+p_{22})}{2}\right)\right)x_2^2.$$

Therefore, if the inequalities (15) and

$$\frac{\mu_1-\frac{3p_{11}}{2}-\frac{(\mu_2+p_{22})(2-\alpha)}{2}}{\xi_1+\frac{\xi_3(\alpha+1)}{2}} > \frac{c}{\mu_1^{\frac{2\text{-}\alpha}{2}}} > \frac{1+\frac{p_{11}}{2}+p_{21}+\frac{\alpha(\mu_2+p_{22})}{2}}{\xi_2-\frac{\xi_3(1-\alpha)}{2}} \tag{16}$$

hold then the function (13) is a strict Lyapunov function for the closed-loop system (1). In the sliding mode case ($\alpha = 0$) then the right-hand side of the closed-loop system is discontinuous. The Lyapunov function-based stability analysis can be provided in this case using the concept of Filippov solutions (see, for example, [18]).

II. Finite-time Stability (Implicit Lyapunov Method)

The function $V_0(x_1,x_2)$ is r-homogeneous of degree $\frac{3-\alpha}{1-\alpha}$ with the weights: $r_1 = \frac{2-\alpha}{1-\alpha}$ and $r_2 = \frac{1}{1-\alpha}$. Indeed, $V_0(\lambda^{r_1}x_1,\lambda^{r_2}x_2) = \lambda^{\frac{3-\alpha}{1-\alpha}}V_0(x_1,x_2)$, $\forall x_1,x_2 \in \mathbb{R}$ and $\forall\lambda > 0$.

Let us denote $\Omega = \left\{(z_1,z_2)\in\mathbb{R}^2 : V_0(z_1,z_2) = 1\right\}$ and $\gamma = \inf\limits_{(z_1,z_2)\in\Omega} l_1|z_1|^{\frac{2}{2-\alpha}}+l_2z_2^2$, where

$$l_1 = \mu_1^{\frac{2\text{-}\alpha}{2}}\left(\mu_1-\frac{3p_{11}}{2}-\frac{(\mu_2+p_{22})(2-\alpha)}{2}\right)-c\xi_1-\frac{c\xi_3(\alpha+1)}{2} > 0,$$

$$l_2 = c\xi_2-\frac{c\xi_3(1-\alpha)}{2}-\mu_1^{\frac{2-\alpha}{2}}\left(1+\frac{p_{11}}{2}+p_{21}+\frac{\alpha(\mu_2+p_{22})}{2}\right) > 0.$$

In this case, $\dot{V}_0(x_1,x_2) \leq -\gamma < 0$ if $(x_1,x_2) \in \Omega$.

The finite-time Lyapunov function will be designed using the implicit Lyapunov function method:

Theorem [23]

If there exists a continuous function

$$Q : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$$
$$(V,x) \to Q(V,x)$$

that is

C1) continuously differentiable in the domain $\mathbb{R}_+ \times \mathbb{R}^n$*;*

C2) for any $x \in \mathbb{R}^n \backslash \{0\}$ there exist $V^- \in \mathbb{R}_+$ and $V^+ \in \mathbb{R}_+$:

$$Q(V^-,x) < 0 < Q(V^+,x);$$

C3) for $\Omega = \left\{(V,x) \in \mathbb{R}^{n+1} : Q(V,x) = 0\right\}$ *the limits*

$$\lim_{\substack{x \to 0\\(V,x)\in\Omega}} V = 0, \quad \lim_{\substack{V \to 0^+\\(V,x)\in\Omega}} \|x\| = 0, \quad \lim_{\substack{x \to \infty\\(V,x)\in\Omega}} V = +\infty;$$

exist;

C4) for $\forall V \in \mathbb{R}_+$ *and* $\forall x \in \mathbb{R}^n \backslash \{0\}$ *the inequality*

$$-\infty < \frac{\partial Q(V,x)}{\partial V} < 0$$

holds;

C5) $\forall t \in \mathbb{R}_+$, $\forall V \in \mathbb{R}_+, \forall x \in \mathbb{R}^n \backslash \{0\} : Q(V,x) = 0$

$$\frac{\partial Q(V,x)}{\partial x} f(x) \leq cV^{1-\mu} \frac{\partial Q(V,x)}{\partial V}$$

where $c > 0$ *and* $0 < \mu \leq 1$ *are some constants, then the equilibrium point* $x = 0$ *of the system*

$$\dot{x} = f(x)$$

is globally finite time stable with the following settling time estimate

$$T(x_0) \leq \frac{V^\mu(x_0)}{c\mu},$$

where $V(x) : Q(V,x) = 0$.

In order to apply the implicit Lyapunov function theorem to the system (1), define

$$Q(V,x) = V_0\left(V^{-r_1}x_1, V^{-r_2}x_2\right) - 1, \tag{17}$$

where $x = (x_1,x_2)^T \in \mathbb{R}^2$, $r_1 = \frac{2-\alpha}{1-\alpha}, r_2 = \frac{1}{1-\alpha}$ are homogeneity weights of the system (1). It can be easily checked that the conditions C1)-C4) hold. On the one hand,

$$\tfrac{\partial Q}{\partial V} = V_0(x_1,x_2)\tfrac{\partial}{\partial V}V^{-\frac{3-\alpha}{1-\alpha}} = -\tfrac{3-\alpha}{1-\alpha}\tfrac{V_0(\frac{x_1}{V^{r_1}},\frac{x_2}{V^{r_2}})}{V} = -\tfrac{3-\alpha}{1-\alpha}\tfrac{1}{V}$$

if $Q(V,x) = 0$.

On the other hand, for the system (1)

$$\frac{\partial Q}{\partial x}f(x) = V^{-\frac{3-\alpha}{1-\alpha}}\dot{V}_0(x_1,x_2) \leq -V^{-\frac{3-\alpha}{1-\alpha}}(l_1|x_1|^{\frac{2}{2-\alpha}} + l_2x_2^2) =$$
$$-V^{-1}\Big(l_1\left|V^{-r_1}x_1\right|^{\frac{2}{2-\alpha}} + l_2\left(V^{-r_2}x_2\right)^2\Big) \leq -\gamma V^{-1} \text{ if } Q(V,x) = 0.$$

Therefore, the condition C5) also holds for $\mu = 1$ and $c = \frac{\gamma(1-\alpha)}{3-\alpha}$ and the settling time admits the following estimate:

$$T(x) \leq \tfrac{3-\alpha}{\gamma(1-\alpha)}V_0^{\frac{1-\alpha}{3-\alpha}}(x_1,x_2). \tag{18}$$

III. Estimation of γ. If $z_2 \neq 0$ then $V_0(z_1,z_2) = 1$ can be rewritten as follows

$$c\left(\tfrac{\mu_1(2-\alpha)}{2}\tfrac{|z_1|^{\frac{2}{2-\alpha}}}{z_2^2} + \tfrac{1}{2}\right)^{\frac{3-\alpha}{2}} + \left(\mu_1\tfrac{|z_1|^{\frac{2}{2-\alpha}}}{z_2^2}\right)^{\frac{2-\alpha}{2}} \text{sign}[z_1z_2] = |z_2|^{\alpha-3}$$

Hence, denoting $y_1 = \frac{|z_1|^{\frac{2}{2-\alpha}}}{z_2^2}$ and $y_2 = |z_2|^{3-\alpha}$ allows the following derivation:

$$\gamma = \inf_{y_1 \geq 0} \frac{l_1y_1 + l_2}{\left(c\left(\frac{\mu_1(2-\alpha)}{2}y_1 + \frac{1}{2}\right)^{\frac{3-\alpha}{2}} + \mu_1^{\frac{2-\alpha}{2}}y_1^{\frac{2-\alpha}{2}}\right)^{\frac{2}{3-\alpha}}}$$

In the general case, the parameter γ can be calculated numerically. However, if $\frac{l_2}{l_1} \leq \frac{1}{\mu_1(2-\alpha)}$ then the function under $\inf$ is monotone decreasing by $y_1 \in [0,+\infty)$ and

$$\gamma = \frac{2l_1}{\mu_1(2-\alpha)c^{\frac{2}{3-\alpha}}}.$$

B. Proof of Theorem 2

I. Asymptotic Stability (Explicit Lyapunov Method)

Consider the Lyapunov function candidate

$$V_0(e_1,e_2) = cU^{\frac{3+\beta}{2(1+\beta)}}(e_1,e_2) - \frac{1}{\sqrt{\nu_2}}e_1e_2, \tag{19}$$

where $c > 0$ and

$$U(e_1,e_2) = \tfrac{1}{\beta+1}|e_1|^{\beta+1} + \tfrac{1}{2\nu_2}e_2^2. \tag{20}$$

Applying Young's inequality for $r = \frac{3+\beta}{2}$ and $q = \frac{3+\beta}{1+\beta}$ it can be derived that

$$|x_1|^{\frac{2(1+\beta)}{3+\beta}}\left(\tfrac{|x_2|}{\sqrt{\nu_2}}\right)^{\frac{2(1+\beta)}{3+\beta}} \leq \tfrac{2}{3+\beta}|x_1|^{1+\beta} + \tfrac{1+\beta}{(3+\beta)\nu_2}x_2^2.$$

Therefore, for

$$c > \left(\tfrac{2(1+\beta)}{3+\beta}\right)^{\frac{3+\beta}{2(1+\beta)}} \tag{21}$$

the function V_0 is positive definite.

Since

$$\tfrac{d}{dt}U^{\frac{3+\beta}{2(1+\beta)}}(e_1,e_2) = \tfrac{3+\beta}{2(1+\beta)}U^{\frac{1-\beta}{2(1+\beta)}}\dot{U} =$$
$$\tfrac{(3+\beta)U^{\frac{1-\beta}{2(1+\beta)}}}{2(1+\beta)}\left(-\nu_1|e_1|^{\frac{1+3\beta}{2}} + \Delta\omega_1|e_1|^\beta\text{sign}[e_1] + \tfrac{e_2\Delta\omega_2}{\nu_2}\right) \leq$$
$$\tfrac{(3+\beta)U^{\frac{1-\beta}{2(1+\beta)}}}{2(1+\beta)}\left(-\nu_1|e_1|^{\frac{1+3\beta}{2}} + q_1|e_1|^\beta|e_2| + \tfrac{q_2|e_2|^{\frac{1+3\beta}{1+\beta}}}{\nu_2}\right),$$

then applying Young's inequality for $r = \frac{1+3\beta}{2\beta}$, $q = \frac{1+3\beta}{1+\beta}$

$$|e_1|^\beta|e_2| \leq \tfrac{2\beta}{1+3\beta}|e_1|^{\frac{1+3\beta}{2}} + \tfrac{1+\beta}{1+3\beta}|e_2|^{\frac{1+3\beta}{1+\beta}}$$

it follows that

$$\tfrac{d}{dt}U^{\frac{3+\beta}{2(1+\beta)}}(e_1,e_2) \leq \tfrac{(3+\beta)}{2(1+\beta)}U^{\frac{1-\beta}{2(1+\beta)}} \times$$
$$\left(-\left(\nu_1 - \tfrac{2q_1\beta}{1+3\beta}\right)|e_1|^{\frac{1+3\beta}{2}} + \left(\tfrac{q_1(1+\beta)}{1+3\beta} + \tfrac{q_2}{\nu_2}\right)|e_2|^{\frac{1+3\beta}{1+\beta}}\right) \leq$$
$$-\eta_1|e_1|^{1+\beta} + \eta_2|e_2|^2 + \eta_3|e_1|^{\frac{1-\beta}{2}}|e_2|^{\frac{1+3\beta}{1+\beta}},$$

where

$$\eta_1 = \frac{(3+\beta)\left(\nu_1 - \frac{2q_1\beta}{1+3\beta}\right)}{2(1+\beta)^{\frac{3+\beta}{2(1+\beta)}}}, \eta_3 = \frac{(3+\beta)\left(\frac{q_1(1+\beta)}{1+3\beta} + \frac{q_2}{\nu_2}\right)}{2(1+\beta)^{\frac{3+\beta}{2(1+\beta)}}},$$

$$\eta_2 = \frac{(3+\beta)}{2(1+\beta)(2\nu_2)^{\frac{1-\beta}{2(1+\beta)}}}\left(\frac{q_1(1+\beta)}{1+3\beta}+\frac{q_2}{\nu_2}\right),$$

Applying again Young's inequality for $r=\frac{2(1+\beta)}{1-\beta}$, $q=\frac{2(1+\beta)}{1+3\beta}$ it is obtained that $|e_1|^{\frac{1-\beta}{2}}|e_2|^{\frac{1+3\beta}{1+\beta}} \le \frac{1-\beta}{2(1+\beta)}|e_1|^{1+\beta}+\frac{1+3\beta}{2(1+\beta)}e_2^2$,

$$\frac{dU^{\frac{3+\beta}{2(1+\beta)}}(e_1,e_2)}{dt} \le -\left(\eta_1 - \frac{(1-\beta)\eta_3}{2(1+\beta)}\right)|e_1|^{1+\beta} + \left(\eta_2 + \frac{(1+3\beta)\eta_3}{2(1+\beta)}\right)e_2^2.$$

On the other hand,

$$\frac{d}{dt}\left(-\frac{e_1 e_2}{\sqrt{\nu_2}}\right) \le -\frac{1-q_1}{\sqrt{\nu_2}}e_2^2 + \frac{\nu_1|e_1|^{\frac{1+\beta}{2}}|e_2| + q_2|e_1||e_2|^{\frac{2\beta}{1+\beta}}}{\sqrt{\nu_2}} + \sqrt{\nu_2}|e_1|^{1+\beta}$$

and applying Young's inequality for $r = 1+\beta$, $q = \frac{1+\beta}{\beta}$ it is obtained that $|e_1|\left(\nu_2^{-1/2}|e_2|^{\frac{2\beta}{1+\beta}}\right) \le \frac{|e_1|^{1+\beta}}{1+\beta} + \frac{\beta}{1+\beta}e_2^2\nu_2^{-1}$,

$$\frac{d}{dt}\left(-\frac{e_1 e_2}{\sqrt{\nu_2}}\right) \le -\frac{\frac{1}{2}-q_1-\frac{q_2\beta}{(1+\beta)\sqrt{\nu_2}}}{\sqrt{\nu_2}}e_2^2 + \left(\sqrt{\nu_2} + \frac{\nu_1^2}{2\sqrt{\nu_2}} + \frac{q_2}{1+\beta}\right)|e_1|^{1+\beta}$$

Hence,

$$\dot V_0(e_1,e_2) \le -\left(\frac{\frac{1}{2}-q_1-\frac{q_2\beta}{(1+\beta)\sqrt{\nu_2}}}{\sqrt{\nu_2}} - c\left(\eta_2 + \frac{1+3\beta}{2(1+\beta)}\eta_3\right)\right)e_2^2$$
$$-\left(c\left(\eta_1 - \frac{1-\beta}{2(1+\beta)}\eta_3\right) - \left(\sqrt{\nu_2} + \frac{\nu_1^2}{2\sqrt{\nu_2}} + \frac{q_2}{1+\beta}\right)\right)|e_1|^{1+\beta}.$$

Hence, the inequality (21) and the inequalities

$$\frac{\frac{1}{2}-q_1-\frac{q_2\beta}{(1+\beta)\sqrt{\nu_2}}}{\sqrt{\nu_2}\left(\eta_2+\frac{1+3\beta}{2(1+\beta)}\eta_3\right)} > c > \frac{\left(\sqrt{\nu_2}+\frac{\nu_1^2}{2\sqrt{\nu_2}}+\frac{q_2}{1+\beta}\right)}{\left(\eta_1-\frac{1-\beta}{2(1+\beta)}\eta_3\right)}$$

guarantee that V_0 is a strict Lyapunov function for (7).

II. Finite-time Stability (Implicit Lyapunov Method) The function $V_0(x_1,x_2)$ is r-homogeneous with the same weights as the original system : $r_1 = \frac{2}{1-\beta}$ and $r_2 = \frac{1+\beta}{1-\beta}$. Namely, $V_0(\lambda^{r_1}x_1, \lambda^{r_2}x_2) = \lambda^{\frac{3+\beta}{1-\beta}}V_0(x_1,x_2)$ for all $x_1 \in \mathbb{R}, x_2 \in \mathbb{R}$ and $\lambda > 0$.

Denote $\Omega = \left\{(z_1,z_2)\in\mathbb{R}^2 : V_0(z_1,z_2)=1\right\}$ and $\gamma = \inf_{(z_1,z_2)\in\Omega} l_1|z_1|^{\frac{2}{2-\alpha}} + l_2 z_2^2 > 0$, where $l_1 = c\left(\eta_1 - \frac{1-\beta}{2(1+\beta)}\eta_3\right) - \left(\sqrt{\nu_2}+\frac{\nu_1^2}{2\sqrt{\nu_2}}+\frac{q_2}{1+\beta}\right) > 0$ and $l_2 = \frac{\frac{1}{2}-q_1-\frac{q_2\beta}{(1+\beta)\sqrt{\nu_2}}}{\sqrt{\nu_2}} - c\left(\eta_2+\frac{1+3\beta}{2(1+\beta)}\eta_3\right) > 0$. In this case, $\dot V_0(x_1,x_2) \le -\gamma < 0$ if $(x_1,x_2)\in\Omega$.

The proof of finite-time stability can be provided by means of the implicit Lyapunov function method similarly to Theorem 1 and the settling time estimate has the form

$$T(x) \le \frac{3+\beta}{\gamma(1-\beta)}V_0^{\frac{1-\beta}{3+\beta}}(x_1,x_2). \qquad (22)$$

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