Globally Stable Implicit Euler Time-Discretization of a Nonlinear Single-Input Sliding-Mode Control System

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***Abstract*— In this note we study the effect of an implicit Euler time-discretization method on the stability of the discretization of a globally fixed-time stable, scalar differential inclusion representing a simple nonlinear system with a set-valued signum controller. The controller nonlinearity is a cubic term and it is shown that the fully-implicit method preserves the global Lyapunov stability property of the continuous-time system, contrarily the explicit discretization which does not. It allows to obtain finite-time convergence to the origin when the plant is undisturbed, while the cubic term provides the hyper-exponential convergence rate.**

Keywords: Implicit Euler discretization; sliding mode; fixed-time stability; hyper-exponential convergence.

I. Introduction

Sliding mode (SM) method is the oldest approach to robust control design introduced almost 50 years ago (see, for example, [22] and references therein). It utilizes ”relay” (discontinuous) feedback in order to force a closed-loop system to slide on a given surface in a state space.

The main theoretical advantage of sliding mode algorithms is their insensitivity to the so-called matched disturbances and uncertainties, see [7], [21]. In practice, the discontinuous feedback application yields the so-called chattering phenomenon [5], [14] expressed in high-frequency destructive oscillations of the closed-loop system. Several approaches have been proposed in order to overcome this drawback. The classical way of chattering reduction is the ”linearization” of the relay feedback law close to a sliding (switching) surface. The high-order SM control principles were introduced in [12] as a desirable ”chattering-free” alternative of the convectional technique. However, an improper implementation of SM control laws in a digital devices anyway implies tangible destructive chattering. Recently, the method of implicit discretization has been developed [1], [2] for effective digital realization of sliding mode algorithms. It has been tested on an experimental setup [23], [11], providing drastic reduction of both input and output chattering for first and second order sliding mode algorithms.

One more feature of sliding mode algorithm is finite-time reaching of the sliding surface by trajectories of the closed-loop system. Finite-time control is the subject of intensive research (see, for example, [4], [18]) motivated by control problems, which need terminations of all transition processes in a finite time. Finite-time analysis of different sliding mode systems has been done in [13], [17], [20], [16]. Finite-time stable system with a bounded settling time on a whole attraction domain has been discovered in [3]. In [19] this property was called fixed-time stability and utilized for sliding mode control design with fixed-time reaching of the sliding surface and the origin of the closed-loop system, providing a possible theoretical background of fast control system. Application of this idea to the first order sliding mode control design is presented in [6].

The present paper aims at extending the implicit discretization method introduced in [1], [2] to the nonlinear case. It shows that contrary to the explicit discretization (which is shown in [15] to lack of global stability), the implicit method preserves the continuous-time closed-loop properties like global finite-time Lyapunov stability. Section II presents the problem statement and the basic assumptions. Then the discretization scheme is studied in the ideal undisturbed case in section III. The case when a perturbation acts on the system is treated in section IV.

Mathematical Preliminaries: Let $K \in I\!R^n$ be a non empty convex closed set. The normal cone to K at $x \in K$ is defined as $N_K(x) = \{z \in I\!R^n | z^T(s-x) \leqslant 0 \text{ for all } s \in K\}$, while $N_K(x) = \emptyset$ if $x \notin K$. In particular let $K = [a,b]$, $a < b$. Then $N_{[a,b]}(x) = \{0\}$ if $a < x < b$, $I\!R^+$ if $x = b$, $I\!R^-$ if $x = a$. The sign set-valued function is $\text{sgn}(x) = 1$ if $x > 0$, -1 if $x < 0$, and $\text{sgn}(0) = [-1, 1]$. It follows from convex

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analysis that for any reals x and y: $x \in N_{[-1,1]}(y) \Leftrightarrow y \in \mathrm{sgn}(x)$.

II. Problem Statement

A. Plant and control design

Let us consider a nonlinear single-input control system

$$\dot{z}(t) = f(z(t), q(t)) + g(z(t), p(t))w + r(t), \quad t > 0, \tag{1}$$

where $z(t) \in I\!R^n$ is the state vector, $f(\cdot)$ and $g(\cdot)$ are vector fields, $w(\cdot) \in I\!R$ is the control input and $q, p, r : I\!R^{n+2} \to I\!R^n$ are single-valued vector functions representing uncertainties and noise. Let us follow the sliding mode control methodology that recommends, initially, to select an appropriate sliding surface $x(z) = 0$ in the state space, where $x : I\!R^n \to I\!R$ is a smooth function. Let us assume that after a possible suitable pre-feedback action $w(z, v)$, the system (1) admits a surface with the sliding variable equation of the form

$$\dot{x}(t) = v(t) + d(t, z) \tag{2}$$

where $x(t) = x(z(t))$ is the sliding variable, $v(\cdot)$ the new input, and the uncertain function $d(t, z)$ is an equivalent disturbance, which is assumed to satisfy $|d(z, t)| \leqslant \delta < 1$ for all t and z[1]. Next, let us design the fixed-time sliding mode control

$$v(t) = -x^3(t) - \mathrm{sgn}(x(t)). \tag{3}$$

The closed-loop system (2), (3) is globally fixed-time stable, with the Lyapunov function $V(x) = x^2$. Indeed, $\frac{d}{dt}(V \circ x)(t) = \frac{\partial V}{\partial x}\dot{x}(t) = -x(t)\mathrm{sgn}(x(t)) - x(t)^4 + x(t)d(t) \leqslant -(1-\delta)|x(t)| - x(t)^4 \leqslant -(1-\delta)V^{1/2}(x(t)) - V^2(x(t))$, that is negative definite and $V(x(t)) = 0$ for all $t > 2/(1-\delta) + 1$ (see [19] for details).

In [15] it has been shown that digital implementation of the control law (3) is difficult due to instability of the explicit Euler discretization of the closed-loop system (2) (3). This paper presents and alternative way for the realization of the fixed-time algorithm, which is based on the analysis of the implicit Euler discretization. The obtained (so-called implicit) control algorithm can be easily realized in a digital device.

[1] Quite often in practice the equivalent disturbance $d(t, z)$ in (2) is the result of not only parameter uncertainty and measurement noise, but also of the error made by the discretization of the pre-feedback.

B. Discretization of fixed-time sliding mode algorithm

In view of the above, let us consider the following sliding mode dynamics (without disturbance):

$$\begin{cases} \dot{x}(t) = v(t) \\ v(t) \in -x(t)^3 - \mathrm{sgn}(x(t)) \\ x(0) = x_0 \end{cases} \tag{4}$$

where $x(t) \in I\!R$. This system is well-posed in the sense of Fillipov, and enjoys the uniqueness of solutions property for any x_0. In the following we shall work with the Euler discretization of (4), namely:

$$\begin{cases} x_{k+1} = x_k + hv_k \\ v_k \in -x_i^3 - \mathrm{sgn}(x_j) \end{cases} \tag{5}$$

where $h > 0$ is the step size, $t_k = hk$, $x_k \triangleq x(t_k)$. From a general point of view, several discretization schemes may be designed:

- $i = k + 1$ and $j = k + 1$ (fully-implicit discretization)
- $i = k$ and $j = k + 1$, or $i = k + 1$ and $j = k$ (half-implicit discretizations)
- $i = k$ and $j = k$ (fully-explicit discretization)

In this note, we focus on the fully-implicit method and show that the results in [1], [2] can be extended to the considered nonlinear case. Let us note that the fully-explicit discretization has already been proven to be unstable in [15]. Equilibria of both continuous-time and discrete-time systems are given as the solutions of the generalized equation:

$$\begin{cases} 0 = v_* \\ v_* \in -x_*^3 - \mathrm{sgn}(x_*) \end{cases} \tag{6}$$

One infers that both (4) and (5) have the origin $x_* = 0$ as a unique fixed point, with input $v_* = 0$.

III. Analysis of the fully-implicit discretization

We first focus on the case without disturbance, and we prove that the implicit method preserves the continuous-time stability properties, while an explicit discretization does not as proved in [15]. Let us denote the set-valued part of the control input v_k as u_{k+1}, so that $u_{k+1} \in -\mathrm{sgn}(x_{k+1})$. Thus the discrete-time system reads as:

$$\begin{cases} x_{k+1} = x_k - hx_{k+1}^3 + hu_{k+1} \\ u_{k+1} \in -\mathrm{sgn}(x_{k+1}) \end{cases} \tag{7}$$

From (7) one obtains the following two generalized equations with unknown x_{k+1} and u_{k+1}, respectively:

$$\begin{cases} (1 + hx_{k+1}^2)x_{k+1} - x_k + h\zeta_{k+1} = 0 \\ \zeta_{k+1} \in \mathrm{sgn}(x_{k+1}) \end{cases} \tag{8}$$

and

$$\begin{cases} \xi_{k+1} + h\xi_{k+1}^3 - x_k - hu_{k+1} = 0 \\ \xi_{k+1} \in -N_{[-1,1]}(u_{k+1}) \end{cases} \tag{9}$$

where $N_{[-1,1]}(u_{k+1})$ is the normal cone to the interval $[-1,1]$ computed at u_{k+1}. One has (see the Mathematical Preliminaries in section I): $x_{k+1} \in -N_{[-1,1]}(u_{k+1}) \Leftrightarrow u_{k+1} \in -\mathrm{sgn}(x_{k+1})$. Thus, in a sense, the two generalized equations in (8) and (9) are dual one to each other. We adopt the notation ξ_{k+1} instead of x_{k+1} in (9) to emphasize the fact that for this generalized equation, x_{k+1} is a dummy variable. The same holds for ζ_{k+1} in (8) instead of u_{k+1}. If (x_{k+1}^*, u_{k+1}^*) is a solution (8) and (9), then the control v_k in (5) with the fully-implicit method is defined as follows

$$v_k = -(x_{k+1}^*)^3 - u_{k+1}^*. \tag{10}$$

Three comments arise at this stage:

- *As the next two propositions show, such a control input at time t_k is nonanticipative (*i.e. *it is causal), because both x_{k+1}^* and u_{k+1}^* are functions of x_k and h, but not of the future state x_{k+1}.*
- The ability to solve generalized equations as in (8) and (9) is at the core of the proposed implicit discretization method. For analytical purpose the solutions should be characterized. For implementation purpose, it may be sufficient that they are numerically solved with a specific solver.
- A complete study should start with the exact ZOH discretization of the plant (1), then the application of a discrete-time pre-feedback to obtain the sliding-mode dynamics with its equivalent disturbance in (2), and finally the analysis of the closed-loop system when a Euler discretization of the set-valued nonlinear robust controller in (3) is applied. However the Euler discretization of the controller is a control design choice.

This is quite in agreement with the results in [1], [2], [23], [11] which deal with similar generalized equations to calculate the controller, with no nonlinear term however.

Proposition 1: The generalized equation (8) has a unique solution x_{k+1}^* for any $h > 0$ and any x_k. If $x_k > h$ then $x_{k+1}^* > 0$, if $x_k < -h$ then $x_{k+1}^* < 0$, and if $|x_k| \leqslant h$ then $x_{k+1}^* = 0$.

Proof:

- 1) let $x_k > h$. Suppose first that $x_{k+1} > 0$. We obtain the nonlinear equation $(1+hx_{k+1}^2)x_{k+1} = x_k - h > 0$. A simple graphical reasoning proves that this 3rd order equation has a unique solution $x_{k+1}^* > 0$. Suppose now that $x_{k+1} < 0$, then we get $(1+hx_{k+1}^2)x_{k+1} = x_k + h > 0$, however this nonlinear equation has no negative solution, thus one infers that x_{k+1} is necessarily non negative, a contradiction. Let now $x_{k+1} = 0$, thus we obtain $0 \in -x_k + h\zeta_{k+1}$ and $\zeta_{k+1} \in [-1,1]$. Since $x_k > h$ this is not possible. We infer that the first solution satisfying $x_{k+1}^* > 0$ is the unique solution.
- 2) The case $x_k < -h$ may be analysed similarly and we conclude that the solution is the unique $x_{k+1}^* < 0$ solution of the nonlinear equation $(1+hx_{k+1}^2)x_{k+1} = x_k + h < 0$.
- 3) Let $|x_k| < h$. Suppose that $x_{k+1} > 0$, this yields $(1 + hx_{k+1}^2)x_{k+1} = x_k - h < 0$ which is impossible. Similalry if $x_{k+1} < 0$ we obtain $(1+hx_{k+1}^2)x_{k+1} = x_k + h > 0$ which is impossible. Let now $x_{k+1}^* = 0$, then $\zeta_{k+1}^* = \frac{-x_k}{h}$ solves the generalized equation (8) and is thus the unique solution.
- 4) Let now $x_k = h$. Thus the generalized equation is $0 = (1 + hx_{k+1}^2)x_{k+1} - h + h\zeta_{k+1}$, $\zeta_{k+1} \in \mathrm{sgn}(x_{k+1})$. Let us choose $x_{k+1} = 0$, so that $\zeta_{k+1} \in [-1,1]$. We obtain $0 = -h + h\zeta_{k+1}$ so that $\zeta_{k+1} = 1$ is a solution. Take now $x_{k+1} > 0$, then we get $0 = (1 + hx_{k+1}^2)x_{k+1}$, a contradiction. If we choose $x_{k+1} < 0$ we get $(1+hx_{k+1}^2)x_{k+1} = 2h$, a contradiction. We infer that $x_{k+1}^* = 0$ and $\zeta_{k+1}^* = 1$ is the unique solution.
- 5) The last case $x_k = -h$ yields the unique solution $x_{k+1}^* = 0$ and $\zeta_{k+1}^* = -1$.

■

Proposition 1 yields in particular that $|x_k| \leqslant h$ implies $x_{k+1} = 0$: if the state reaches the ball of radius h centered at the origin, then the next states stay at the origin. Most importantly the control input has to be calculated uniquely at each time step. This is the object of the next proposition.

Proposition 2: The generalized equation (9) has a unique solution u_{k+1}^* for any $h > 0$ and any x_k. If $x_k \geqslant h$ then $u_{k+1}^* = -1$, if $x_k \leqslant -h$ then $u_{k+1}^* = 1$, and if $|x_k| < h$ then $u_{k+1}^* = -\frac{x_k}{h} \in [-1,1]$. Moreover x_{k+1}^* is a function of x_k and h only.

Proof: The proof is led in a similar way as the proof of Proposition 1, step by step in a constructive way.

- 1) let $x_k > h$, and take $u_{k+1} = -1$ so that $\xi_{k+1} \in I\!R^+$. Then we obtain $0 = \xi_{k+1} + h\xi_{k+1}^3 - x_k + h \Rightarrow \xi_{k+1} + h\xi_{k+1}^3 = x_k - h > 0$. This equation has a unique solution $\xi_{k+1}^* > 0$ as may be checked graphically since the function $x \mapsto x + hx^3$ is strictly increasing. Now take

$u_{k+1}=1$. One obtains $\xi_{k+1}+h\xi_{k+1}^3=h+x_k$ while $\xi_{k+1}\in I\!R^-$ and $h+x_k>0$: this is impossible. Now take $|u_{k+1}|<1$, from which it follows that $\xi_{k+1}=0$. Then $0=h-x_k$ which is impossible. Hence there is a unique solution $u_{k+1}^*=-1$.
- 2) The case $x_k<-h$ may be analysed similarly and the unique solution is found to be $u_{k+1}^*=1$.
- 3) Take now $|x_k|<h$. Let $u_{k+1}=-\frac{x_k}{h}$, hence one obtains $0=\xi_{k+1}+h\xi_{k+1}^3$, with $\xi_{k+1}\in -N_{[-1,1]}(u_{k+1})$. Since $|u_{k+1}|=\frac{|x_k|}{h}<1$ one has $\xi_{k+1}=0$ and we conclude that $u_{k+1}=-\frac{x_k}{h}$ is a solution. Take now $u_{k+1}=1$. We obtain $\xi_{k+1}+h\xi_{k+1}^3=x_k+h$ with $\xi_{k+1}\in I\!R^-$. Since $x_k+h>0$ this is impossible. The same reasoning may be done for $u_{k+1}=-1$, and one infers that $u_{k+1}^*=-\frac{x_k}{h}$ is the only solution.
- 4) Finally consider $x_k=h$. Then $0=\xi_{k+1}+h\xi_{k+1}^3-h-hu_{k+1}$. Consider $u_{k+1}=-\frac{x_k}{h}=-1$. Then $\xi_{k+1}\in I\!R^+$ and $0=\xi_{k+1}(1+h\xi_{k+1}^2)$ which has the unique solution $\xi_{k+1}=0$. Consider now $u_{k+1}=1$, hence the generalized equation is $\xi_{k+1}+h\xi_{k+1}^3=2h$ with $\xi_{k+1}\in I\!R^-$: this is impossible. Finally let $|u_{k+1}|<1\Rightarrow\xi_{k+1}=0$. The generalized equation becomes $h=hu_{k+1}$, a contradiction. Thus $u_{k+1}^*=-1$.
- 5) A similar reasoning holds for $x_k=-h$ to infer $u_{k+1}^*=1$.
- The last point follows from the fact that in cases 1) and 2), x_{k+1}^* is obtained as the root of a third order polynomial whose solutions are in (13) (14) below.

■

It is noteworthy that the framework of the fully-implicit method is that of difference inclusions, since the signum function may take values in the interior of $[-1,1]$. However, similarly to what happens in continuous-time (within Filippov's mathematical framework), the input u_{k+1}, which may be viewed as a selection of the set-valued right-hand side of the system, is uniquely defined as a function of the state. For a real x, $\lfloor x\rfloor$ is the integer n such that $n\leqslant x<n+1$.

Proposition 3: Let $h>0$. The discrete-time system (5) with the fully-implicit method, has a globally Lyapunov finite-time stable equilibrium $x^*=0$ and the equilibrium is reached after a finite number of steps N. Moreover $x_k=0$ for all $k\geqslant N+1$ and $u_{k+1}=0$ for all $k\geqslant N+2$, while $N\leqslant\lfloor\frac{x_0}{h}\rfloor$ for any $x_0\in I\!R$.

Proof: let us analyse the quantities $|x_{k+1}|-|x_k|$ when $x_k>h$, $x_k<-h$ and $|x_k|\leqslant h$.

- 1) Let $x_k>h$, then from Proposition 2 $u_{k+1}=-1$ so from (5) we get: $x_{k+1}=x_k-h-hx_{k+1}^3$. From Proposition 1 one has $x_{k+1}>0$, consequently $x_{k+1}<x_k-h\Rightarrow|x_{k+1}|<|x_k|-h$.
- 2) Let $x_k<-h$, then from Proposition 2 $u_{k+1}=1$ so from (5) we get: $x_{k+1}=x_k+h-hx_{k+1}^3$. From Proposition 1 one has $x_{k+1}<0$, consequently $x_{k+1}>x_k+h\Rightarrow|x_{k+1}|<|x_k|-h$.
- 3) Let $|x_k|\leqslant h$, from Proposition 1 one has $x_{k+1}=0$ while from Proposition 2 $u_{k+1}=-\frac{x_k}{h}$. So one infers that $x_{k+n}=0$ for all $n\geqslant 1$ and $u_{k+1}=0$ for all $n\geqslant 2$.

Let us now prove that $V(x_k)=|x_k|$ is a Lyapunov function for (5) with the fully-implicit method, for any $h>0$ and any x_0. Outside the ball $|x_k|\leqslant h$, it strictly decreases as $V(x_{k+1})=V(x_k)-h$ and $h>0$. Suppose that the initial condition is $x(t_0)=x_0$ for some finite $x_0=$. Then $V(x_k)=|x_0|-kh$, from which it follows that there exists a finite integer N such that $V(x_N)=|x_N|=0$. From item 3) the result follows. For the last part of the proposition, suppose that $x_0>h$. Let us show that $N\leqslant\lfloor\frac{x_0}{h}\rfloor$ for any $x_0\in I\!R$. One has $x_k<x_0-kh$ for any $k\geqslant 1$. We are seeking the least integer N such that $x_0-kh\leqslant h\Leftrightarrow k+1\geqslant\frac{x_0}{h}$. Thus for all $k+1\geqslant\lfloor\frac{x_0}{h}\rfloor+1$ one obtains $x_{k+1}=0$. Similar calculations hold for the case $x_0<-h$. ■

Remark 1: The Lyapunov function for the continuous-time case shown in section II, is $V(x)=x^2$. It is not too difficult to show that $V(x_k)=x_k^2$ is also a Lyapunov function for the discrete-time case. Consider for instance case 1) in the proof of Proposition 3. One obtains $V(x_{k+1})-V(x_k)=(x_{k+1}-x_k)(x_{k+1}+x_k)=-h(1+x_{k+1}^3)(x_{k+1}+x_k)$, which is negative because both $x_k>0$ and $x_{k+1}>0$ in this case. The other cases are treated similarly.

Remark 2: Proposition 3 proves that after a finite number of steps, the discrete-time state and the discrete-time input have no numerical chattering, *i.e.* no spurious oscillations around the sliding surface and no bang-bang behaviour of the input occur due solely to the discretization process, contrarily to what happens with an explicit discretization as shown analytically in [8], [9], [10] and experimentally in [23], [11]. This is why it is worth analysing the disturbance-free case.

Notice that the presented calculation of N does not incorporate the decrease due to the term $-x_{k+1}^3$, and is therefore an overestimation of the real number of steps needed to attain the origin. In fact one may deduce a more accurate estimation for N. Let $[\cdot]_+$ be the projector to $\mathbb{R}_+$, *i.e.* $[x]_+ = x$ for $x > 0$, $[x]_+ = 0$ for $x \leqslant 0$.

Corollary 1: The fully-implicit method implies that x_k attains the equilibrium after a finite number of steps N for any $h > 0$ and for any bounded initial condition x_0, with:

$$N \leqslant \left\lfloor 2 + \left[\log_3 \frac{[\ln(\sqrt{h}|x_0|)]_+}{\ln(2+h^{3/2})}\right]_+ \right\rfloor \quad \text{for } h \geqslant 1 \tag{11}$$

and

$$N \leqslant \left\lfloor \frac{\min\{1,|x_0|\}}{h} + \frac{[\ln\min\{1/\sqrt{h},|x_0|\}]_+}{\ln(1+2h)} + \left[\log_3 \frac{[\ln(\sqrt{h}|x_0|)]_+}{\ln(2+h^{3/2})}\right]_+ \right\rfloor \quad \text{for } h < 1. \tag{12}$$

Proof: In order to take into account the effect of the cubic term, at least for large initial conditions, let us define $\varphi_k := \sqrt{h}x_k$, and suppose that $\varphi_0 > 1$ (*i.e.* $x_0 > 1/\sqrt{h}$). Multiplying both sides of the equality $hx_{k+1}^3 + x_{k+1} = x_k - h$ by $\sqrt{h}$ we derive $\varphi_{k+1}^3 + \varphi_{k+1} = \varphi_k - h^{3/2}$. On the one hand, if $\varphi_k \leqslant 2 + h^{3/2}$ then $\varphi_{k+1} \leqslant 1$. On the other hand, $\varphi_{k+1}^3 \leqslant \varphi_k \Rightarrow \ln\varphi_k \leqslant \frac{\ln\varphi_{k-1}}{3}$ and $\ln\varphi_k \leqslant \frac{\ln\varphi_0}{3^k} \leqslant \ln(2 + h^{3/2})$ for $k \geqslant \lfloor \log_3 \log_{2+h^{3/2}}(\varphi_0)\rfloor_+$. Therefore, $\sqrt{h}x_k \leqslant 1$ for $k \geqslant 1 + \lfloor \log_3 \log_{2+h^{3/2}}(\varphi_0)\rfloor_+$. Hence, due to Proposition 3 the inequality (11) holds. Since, $2hx_{k+1} + x_{k+1} \leqslant h(x_{k+1}^3 + 1) + x_{k+1} = x_k$ for any $x_{k+1} \geqslant 1$, then $x_{k+n} \leqslant 1$ for $n \geqslant \frac{\ln(|x_k|)}{\ln(2h+1)}$. And again due to Proposition 3 the inequality (12) also holds. ■

The proven corollary implies that, in addition to finite-time stability, the fully implicit system has hyper exponential convergence rate. In contrast to the continuous-time case, we are not able to prove that the settling time $T = hN$ is globally bounded. However, the property of fast convergence is preserved in the discrete-time system. Moreover if $|x_0| \leqslant 1$ then the presented estimate coincides with the estimation given by Proposition 3, and if $|x_0| > 1$ then $N < \frac{|x_0|}{h}$.

Remark 3: In practice, the fully-implicit method may be implemented at time $t_k = kh$ by solving the generalized equations (8) and (9) with a suitable numerical solver, as implemented for instance in the open-source SICONOS software[2]. For the particular case of this paper one notices that if $x_k > h$ then x_{k+1}^* is the unique root of $x_{k+1}^3 + \frac{1}{h}x_{k+1} - \frac{x_k - h}{h} = 0$. Equivalently:

$$\begin{aligned} x_{k+1}^* = {} & 2^{-\frac{1}{3}}\left(\tfrac{x_k-h}{h} + \left(\left(\tfrac{x_k-h}{h}\right)^2 + 4\tfrac{1}{27h^3}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} \\ & + 2^{-\frac{1}{3}}\left(\tfrac{x_k-h}{h} - \left(\left(\tfrac{x_k-h}{h}\right)^2 + 4\tfrac{1}{27h^3}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} \end{aligned} \tag{13}$$

and similarly for $x_k < -h$:

$$\begin{aligned} x_{k+1}^* = {} & 2^{-\frac{1}{3}}\left(\tfrac{x_k+h}{h} + \left(\left(\tfrac{x_k+h}{h}\right)^2 + 4\tfrac{1}{27h^3}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} \\ & + 2^{-\frac{1}{3}}\left(\tfrac{x_k+h}{h} - \left(\left(\tfrac{x_k+h}{h}\right)^2 + 4\tfrac{1}{27h^3}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} \end{aligned} \tag{14}$$

If $|x_k| \leqslant h$ then $x_{k+1}^* = 0$. One way to interpret the above results, is that they allow us to analyze such a switching difference equation. Having the explicit representation of x_{k+1}^* the discrete control law (10) can be easily calculated using Proposition 2.

IV. The case with a disturbance

Let us nevertheless consider now that a disturbance $d(z,t)$ with $|d(z,t)| \leqslant \delta < 1$ acts on the system in (4): $\dot{x}(t) = u - x(t)^3 + d(z,t)$. The continuous-time system is still globally finite-time Lyapunov stable. Similar results as in Propositions 1, 2, 3 and Corollary 1 can be obtained. However the system's behaviour near the origin is slightly different. The plant discrete-time dynamics is given by:

$$x_{k+1} = x_k + hu_{k+1} + hd_k \tag{15}$$

where $d_k = d(x_k, t_k)$. Since $d(x,t)$ is unknown, the controller is calculated from a nominal system as follows:

$$\begin{cases} \tilde{x}_{k+1} = x_k + hu_{k+1} \\ u_{k+1} \in -\tilde{x}_{k+1}^3 - \mathrm{sgn}(\tilde{x}_{k+1}) \end{cases} \tag{16}$$

where $\tilde{x}_{k+1}$ replaces ξ_{k+1}. The same generalized equations as in (8) and (9) can be constructed from (16) which is the counterpart of (7). Mimicking Propositions 1 2 and 3 we get:

Proposition 4: The generalized equation (16) has a unique solution $\tilde{x}_{k+1}^*$ for any x_k and $h > 0$: if $x_k > h$ then $\tilde{x}_{k+1}^* < 0$ (and $u_{k+1}^* = -1$), if $x_k < -h$ then $\tilde{x}_{k+1}^* > 0$ (and $u_{k+1}^* = 1$), and if $|x_k| \leqslant h$ then $\tilde{x}_{k+1}^* = 0$ (and $u_{k+1}^* = -\frac{x_k}{h}$). Moreover the surface $\tilde{x}_k = 0$ is attained after a finite number of steps.

The next corollary states what happens on the "sliding surface" $\tilde{x}_k = 0$ in terms of the set-valued control input and disturbance compensation.

[2] http://siconos.gforge.inria.fr/

Corollary 2: Let $\tilde{x}_k = 0$ for all $k \geqslant n$ and some $n \geqslant 0$. Then $u_k = -hd_{k-1}$ for all $k \geqslant n+1$.

Proof: From (16) it follows that $\tilde{x}_k = 0$ implies $u_{k+1} = -\frac{x_k}{h}$. Suppose that the nominal system satisfies $\tilde{x}_{k+1} = 0$, then obviously, the discretized plant state satisfies $x_{k+1} = hd_k$. Thus we infer that $x_k = hd_{k-1}$. The result follows. ∎

Remark 4: The presence of a cubic term in the control input, implies that $v(\cdot)$ takes large values during the transient period if the initial conditions are large. This certainly limits the practical usefulness of such controllers. The goal of this article is however merely to prove that the implicit discretization method introduced in [1], [2] and successfully experimentally tested in [23], [11], also applies in a nonlinear setting, and has the very nice feature to preserve the continuous-time closed-loop properties like global finite-time Lyapunov stability. This is in big contrast with the widely used explicit discretization which does not share such properties.

V. Conclusions

In this note we have shown that the implicit time-discretization method introduced for linear sliding-mode dynamics in [1], [2], extends nicely to the scalar, nonlinear case. It preserves the continuous-time closed-loop system properties like the global finite-time Lyapunov stability of the unique equilibrium point. Moreover it compensates the disturbances by a factor h, and suppresses the input and ouput numerical chattering. This papers treats only the case of a single-input first order sliding mode control system. It demonstrates the successful applicability of the implicit control design method for fixed-time algorithms. The extension of this concept to the case of the second order sliding mode systems is an important subject for future research.

References

[1] V. Acary and B. Brogliato. Implicit Euler numerical scheme and chattering-free implementation of sliding mode systems. *Systems and Control Letters*, 59:284–293, 2010.

[2] V. Acary, B. Brogliato, and Y. Orlov. Chattering-free digital sliding-mode control with state observer and disturbance rejection. *IEEE Transactions on Automatic Control*, 57(5):1087–1101, 2012.

[3] V. Andrieu, L. Praly, and A. Astolfi. Homogeneous approximation, recursive obsertver and output feedback. *SIAM Journal of Control and Optimization*, 47(4):1814–1850, 2008.

[4] S.P. Bhat and D.S. Bernstein. Finite-time stability of continuous autonomous systems. *SIAM Journal of Control and Optimization*, 38(3):751–766, 2000.

[5] I. Boiko, L. Fridman, A. Pisano, and E. Usai. Analysis of chattering in systems with second-order sliding modes. *IEEE Transactions on Automatic Control*, 52(11):2085–2102, 2007.

[6] E. Cruz-Zavala, J. A. Moreno, and L. Fridman. Uniform sliding mode controllers and uniform sliding surfaces. *IMA Journal of Mathematical Control and Information*, 29(4):491–505, 2012.

[7] C. Edwards and S. Spurgeon. *Sliding Mode Control: Theory And Applications*. CRC Press., 1998.

[8] Z. Galias and X. Yu. Complex discretization behaviours of a simple sliding-mode control system. *IEEE Transactions on Circuits and Systems–II: Express Briefs*, 53(8):652–656, August 2006.

[9] Z. Galias and X. Yu. Euler's discretization of single input sliding-mode control systems. *IEEE Transactions on Automatic Control*, 52(9):1726–1730, September 2007.

[10] Z. Galias and X. Yu. Analysis of zero-order holder discretization of two-dimensional sliding-mode control systems. *IEEE Transactions on Circuits and Systems–II: Express Briefs*, 55(12):1269–1273, December 2008.

[11] O. Huber, V. Acary, B. Brogliato, and F. Plestan. Discrete-time twisting controller without numerical chattering: Analysis and experimental results with an implicit method. In *53rd IEEE Annual Conference on Decision and Control*, pages 4373–4378, Los Angeles, USA, December 2014. DOI: 10.1109/CDC.2014.7040071.

[12] A. Levant. Sliding order and sliding accuracy in sliding mode control. *International Journal of Control*, 58(6):1247–1263, 1993.

[13] A. Levant. Homogeneity approach to high-order sliding mode design. *Automatica*, 41:823–830, 2005.

[14] A. Levant. Chattering analysis. *IEEE Transactions on Automatic Control*, 55(6):1380 – 1389, 2010.

[15] A. Levant. On fixed and finite time stability in sliding mode control. In *52th IEEE Conference on Decision and Control*, pages 4260–4265, Firenze, Italy, 2013.

[16] J Moreno. Strict Lyapunov functions for the super-twisting algorithm. *IEEE Transactions on Automatic Control*, 57(4):1035 – 1040, 2012.

[17] Y. Orlov. Finite time stability and robust control synthesis of uncertain switched systems. *SIAM Journal of Control and Optimization*, 43(4):1253–1271, 2005.

[18] Y. Orlov. *Discontinous Systems: Lyapunov Analysis and Robust Synthesis under Uncertainty Conditions*. Springer-Verlag, 2009.

[19] A. Polyakov. Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Transactions on Automatic Control*, 57(8):2106–2110, 2012.

[20] A. Polyakov and A. Poznyak. Lyapunov function design for finite-time convergence analysis: "twisting" controller for second-order sliding mode realization. *Automatica*, 45:444–448, 2009.

[21] Y. Shtessel, C. Edwards, L. Fridman, and A. Levant. *Sliding Mode Control and Observation*. Birkhauser, 2014.

[22] V.I. Utkin. Variable structure systems with sliding modes. *IEEE Transactions on Automatic Control*, 22(2):212 – 222, 1977.

[23] B. Wang, B. Brogliato, V. Acary, A. Boubakir, and F. Plestan. Experimental comparisons between implicit and explicit implementations of discrete-time sliding mode controllers: Toward input and output chattering suppression. *IEEE Transactions on Control Systems Technology*, 2015. DOI: 10.1109/TCST.2015.2396473.