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Abstract: We consider the problem of stability analysis and control synthesis for first-order hyperbolic linear PDEs over a bounded interval with spatially varying coefficients. We propose LMI-based conditions for the stability and for the design of boundary and distributed control for this class of systems. These LMI-based conditions involve an infinite number of LMI. Hence, we show how to overapproximate these constraints using polytopic embeddings to reduce the problem to a finite number of LMI. We show the effectiveness of the overapproximation with several examples.

Keywords: Hyperbolic PDE, Lyapunov method, LMI.

1. INTRODUCTION

The wide variety of physical applications governed by hyperbolic equations make their study a dynamic field of research. Hydraulic networks (Diagne et al. (2012)), (Bastin and Coron (2011)) or gas device (Castillo et al. (2015)) are some examples of these applications.

While for finite dimensional and time-delay systems a large number of numerical techniques for stability analysis exist, for Partial Differential Equations (PDEs) these tools are mostly lacking. In this paper, we propose some techniques to verify numerically the existence of quadratic Lyapunov function for first-order hyperbolic PDEs over a bounded interval with spatially varying coefficients. Besides this analysis aspect, we propose some techniques for the synthesis of boundary and distributed controls.

The Lyapunov analysis has been a common and powerful approach to analyse and control systems in finite dimension for several decades. Among the large literature on stability analysis and stabilization of finite dimensional system let us mention (Boyd et al. (1994)) covering a large number of results on the resolution of LMI coming from the Lyapunov analysis. In (Löfberg (2004)), a numerical toolbox to solve various problems in control is presented.

Time-delay systems are widely analysed with Lyapunov type methods such as, for instance, the use of Lyapunov-Krasovskii functionals. Numerical methods have been proposed for the construction of this type of functions, see (Briat et al. (2009)), (Peet et al. (2009)), (Peet (2014)), or (Peet and Bliman (2011)).

The use of Lyapunov function is now appearing for hyperbolic systems. Lyapunov converse results have been stated recently in (Karafyllis and Krstic (2014)). More specifically, a special attention has been made on quadratic Lyapunov functions. Indeed, this class of functions allows to express conditions for stability as LMI as in (Castillo et al. (2015)), (Diagne et al. (2012)), (Prieur et al. (2014)), and (Xu and Sallet (2002)) for the linear case. LMI conditions derived by an operators approach is used in (Fridman and Orlov (2009)) for the H^∞ boundary control of parabolic and hyperbolic systems. Quadratic control Lyapunov function has also been used for 2×2 quasilinear systems (Coron et al. (2007)) and $n\times n$ quasilinear systems (Coron et al. (2008)). LMI-based conditions derived from a quadratic Lyapunov function were stated in (Castillo et al. (2013)) for the construction of boundary observers for linear as well as for quasilinear hyperbolic systems. However, the approach by a quadratic Lyapunov function is not always effective to prove stability for hyperbolic systems. Indeed, it has been shown in (Bastin and Coron (2011)) that there exist stable 2×2 linear hyperbolic systems for which there does not exist quadratic Lyapunov function.

The results in this paper are related to the resolution of the LMIs proposed in (Prieur et al. (2014)) and to the control synthesis. The LMI-based conditions involve the spatial variable, hence the number of constraints is infinite. These LMI-based conditions are analogous to stability conditions for finite-dimensional LPV systems. Hence, an approach inspired by the LPV systems is applied to find a candidate Lyapunov function. More precisely, to reduce the numerical complexity, different approximations based on properties of the exponential functions are considered in this paper. The control synthesis relies on a combination of classical techniques coming from the stabilization for discrete and continuous time finite dimensional systems. Then, the former overapproximations techniques are used to get a finite number of LMI-based conditions.

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The paper is organized as follows. In Section 2 we present the system studied in the paper. In Section 3 we propose new LMI-based conditions for the stability of such systems. In Section 4 we address the problem of the design of static boundary and distributed controllers. In Section 5 overapproximation techniques are presented to reduce the complexity of the LMI-based conditions stated so far. Finally, in Section 6, methods are tested on academic examples.

Notation. The set $\mathbb{R}^+$ is the set of nonnegative real numbers. The set of complex number is denoted by $\mathbb{C}$. The set of square real matrices of dimension n is denoted by $\mathbb{R}^{n\times n}$. Given a matrix A, the transpose of the matrix A is denoted by $A^\top$. Given N square matrices $A_1,\dots,A_N$, of respective dimension $k_1,\dots,k_N$, the block diagonal matrix $A \in \mathbb{R}^{(k_1+\dots+k_N)\times(k_1+\dots+k_N)}$ whose block diagonal matrices are $A_1,\dots,A_N$, is denoted by $\mathtt{diag}[A_1,\dots,A_N]$. The identity matrix of dimension n is denoted by I_n. For a symmetric matrix $A\in\mathbb{R}^{n\times n}$, A being positive definite is denoted by $A>0$, while A being positive semi-definite is denoted by $A\geq 0$. The smallest eigenvalues of a matrix $A\in\mathbb{R}^{n\times n}$ is denoted by $\lambda_{\min}(A)$ while its largest eigenvalues is denoted by $\lambda_{\max}(A)$. The derivative of a matrix $A(x)$ with respect to variable x is denoted by $A'(x)$. The usual Euclidian norm in $\mathbb{R}^n$ is denoted by $|\cdot|$. The set of functions $y:[0,1]\to\mathbb{R}^n$ such that $|y|^2_{L^2((0,1);\mathbb{R}^n)} = \int_0^1 |y(x)|^2 dx < \infty$, is denoted by $L^2((0,1);\mathbb{R}^n)$.

2. LINEAR HYPERBOLIC SYSTEMS

We consider the following general system

$$y_t(t,x) + \Lambda(x) y_x(t,x) = F(x) y(t,x)\,, \tag{1}$$

where $t\in\mathbb{R}^+$ is the time variable, $x\in[0,1]$ is the spatial variable, $y:\mathbb{R}^+\times[0,1]\to\mathbb{R}^n$, F and Λ are in $C^0([0,1];\mathbb{R}^{n\times n})$. The matrix $\Lambda(x)$ is diagonal and in addition $\Lambda(x) = \mathtt{diag}[\lambda_1(x),\dots,\lambda_n(x)]$ with $\lambda_k(x)<0$ for $k\in\{1,\dots,m\}$ and $\lambda_k(x)>0$ for $k\in\{m+1,\dots,n\}$, for all $x\in[0,1]$. Let us introduce the following notations

$$\Lambda(x) = \mathtt{diag}\left[\Lambda^-(x),\Lambda^+(x)\right]\,,$$
$$y = \left[y^-(t,x)\ y^+(t,x)\right]^\top\,,$$

where $y^-:\mathbb{R}^+\times[0,1]\to\mathbb{R}^m$ and $y^+:\mathbb{R}^+\times[0,1]\to\mathbb{R}^{(n-m)}$, $\Lambda^-(x)=\mathtt{diag}[\lambda_1(x),\dots,\lambda_m(x)]$, and $\Lambda^+(x)=\mathtt{diag}[\lambda_{m+1}(x),\dots,\lambda_n(x)]$. We consider the following boundary conditions

$$\begin{bmatrix} y^-(t,1) \\ y^+(t,0)\end{bmatrix} = G\begin{bmatrix} y^-(t,0) \\ y^+(t,1)\end{bmatrix}\,,\quad t\in\mathbb{R}^+\,, \tag{2}$$

where G is a matrix in $\mathbb{R}^{n\times n}$. The initial condition is

$$y(0,x) = y^0(x)\,,\quad x\in(0,1)\,, \tag{3}$$

where $y^0\in L^2((0,1);\mathbb{R}^n)$.

It can be shown that the following result holds, see (Diagne et al. (2012)) and the references therein.

Proposition 2.1. There exists a unique (weak) solution $y\in C^0\left(\mathbb{R}^+;L^2(0,1)\right)$ to the Cauchy problem (1)–(3).

Let us define the concept of *Globally Exponentially Stable* (GES) solution for system (1)–(3).

Definition 1. The system (1)–(3) is said *Globally Exponentially Stable* if there exist $\nu>0$ and $C>0$ such that, for every initial condition $y^0\in L^2((0,1);\mathbb{R}^n)$, the solution to the Cauchy problem (1)–(3) satisfies

$$|y(t,\cdot)|_{L^2((0,1);\mathbb{R}^n)} \leq Ce^{-\nu t}\left|y^0\right|_{L^2((0,1);\mathbb{R}^n)}\,,\ \forall t\in\mathbb{R}^+\,. \tag{4}$$

Let us denote $|\Lambda(x)|$ the matrix whose elements are the absolute value of the elements of the matrix $\Lambda(x)$, that is

$$|\Lambda(x)| = \mathtt{diag}\left[-\Lambda^-(x),\Lambda^+(x)\right]\,, \tag{5}$$

and let us denote by $\tilde{I}_{n,m}$ the matrix

$$\tilde{I}_{n,m} = \begin{bmatrix} -I_m & 0_{m,n-m} \\ 0_{n-m,m} & I_{n-m}\end{bmatrix}\,. \tag{6}$$

For a matrix A in $\mathbb{R}^{n\times n}$, we decompose it in four block matrices A_{--} in $\mathbb{R}^{m\times m}$, A_{-+} $\mathbb{R}^{m\times(n-m)}$, A_{+-} in $\mathbb{R}^{(n-m)\times m}$ and A_{++} in $\mathbb{R}^{(n-m)\times(n-m)}$ such that $A = \begin{bmatrix} A_{--} & A_{-+} \\ A_{+-} & A_{++}\end{bmatrix}$.

3. LYAPUNOV FUNCTIONS AND LMI-BASED STABILITY ANALYSIS

In this section, we propose a Lyapunov function, and derive some LMI-based conditions for the solution of system (1)–(3) to satisfy (4). In (Prieur et al. (2014)) sufficient conditions have been given for the stability of (1)–(3) with $\Lambda(x)$ and $F(x)$ constant. We consider a slightly different Lyapunov function

$$V(y) = \int_0^1 y^\top(t,x)\,|\Lambda(x)|^{-1}\,\mathcal{Q}(x) y(t,x) dx\,, \tag{7}$$

where

$$\mathcal{Q}(x) = \mathtt{diag}\left[e^{2\mu x}Q^-, e^{-2\mu x}Q^+\right]\,, \tag{8}$$

with Q^- in $\mathbb{R}^{m\times m}$, Q^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ are two symmetric positive definite matrices.

Remark 1. We could use the word "functional" instead of "function" since the state-space is a function space, but we have decided to keep the terminology of (Coron et al. (2007)).

Proposition 3.1. If there exist $\nu>0$, μ in $\mathbb{R}$, and symmetric positive definite matrices Q^- in $\mathbb{R}^{m\times m}$ and Q^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that the following conditions hold, for all $x\in[0,1]$,

$$\mathcal{Q}(x)\Lambda(x) = \Lambda(x)\mathcal{Q}(x)\,, \tag{9}$$

$$-2\mu\mathcal{Q}(x) + F^\top(x)\,|\Lambda(x)|^{-1}\,\mathcal{Q}(x) + \mathcal{Q}(x)\,|\Lambda(x)|^{-1}\,F(x) \leq -2\nu\,|\Lambda(x)|^{-1}\,\mathcal{Q}(x)\,, \tag{10}$$

together with

$$\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++}\end{bmatrix}^\top \tilde{I}_{n,m}\mathcal{Q}(0)\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++}\end{bmatrix} \leq \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m}\end{bmatrix}^\top \tilde{I}_{n,m}\mathcal{Q}(1)\begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m}\end{bmatrix}\,, \tag{11}$$

with $\mathcal{Q}(x)$ defined by (8), then system (1)–(3) is GES.

Proof. For the stability analysis we consider the candidate Lyapunov function (7), where $\mathcal{Q}(x)$ is given by (8). Let us introduce the constants

$$\underline{\lambda} = \min_{x\in[0,1]} \lambda_{\min}\left(|\Lambda(x)|^{-1}\mathcal{Q}(x)\right)\,, \tag{12}$$

$$\overline{\lambda} = \max_{x\in[0,1]} \lambda_{\max}\left(|\Lambda(x)|^{-1}\mathcal{Q}(x)\right)\,. \tag{13}$$

The matrix $|\Lambda(x)|^{-1}\mathcal{Q}(x)$ being positive definite, one can conclude that $\overline{\lambda},\underline{\lambda}>0$ and for all $y\in L^2((0,1);\mathbb{R}^n)$

$$\underline{\lambda}\,|y|^2_{L^2((0,1);\mathbb{R}^n)} \leq V(y) \leq \overline{\lambda}\,|y|^2_{L^2((0,1);\mathbb{R}^n)}\,. \tag{14}$$

Let us compute the time-derivative of the candidate Lyapunov function (7) along the solutions of system (1), (2). Using the commutativity condition (9), we have

$$\dot V = 2\int_0^1 y_t^\top(t,x)\left|\Lambda(x)\right|^{-1}\mathcal{Q}(x)y(t,x)dx$$
$$= -2\int_0^1 y^\top(t,x)\tilde I_{n,m}\mathcal{Q}(x)y_x(t,x)dx$$
$$+ 2\int_0^1 y^\top(t,x)\mathcal{Q}(x)\left|\Lambda(x)\right|^{-1}F(x)y(t,x)dx\,. \quad (15)$$

Noting that $-2y^\top\tilde I_{n,m}\mathcal{Q}y_x = -\left(y^\top\tilde I_{n,m}\mathcal{Q}y\right)_x + y^\top\tilde I_{n,m}\mathcal{Q}'y$ and $\tilde I_{n,m}\mathcal{Q}' = -2\mu\mathcal{Q}$, one has

$$\dot V = \begin{bmatrix} y^-(t,0)\\ y^+(t,1)\end{bmatrix}^\top \left[\begin{bmatrix} I_m & 0_{m,n-m}\\ G_{+-} & G_{++}\end{bmatrix}^\top \tilde I_{n,m}\mathcal{Q}(0)\begin{bmatrix} I_m & 0_{m,n-m}\\ G_{+-} & G_{++}\end{bmatrix}\right.$$
$$\left. - \begin{bmatrix} G_{--} & G_{-+}\\ 0_{n-m,m} & I_{n-m}\end{bmatrix}^\top \tilde I_{n,m}\mathcal{Q}(1)\begin{bmatrix} G_{--} & G_{-+}\\ 0_{n-m,m} & I_{n-m}\end{bmatrix}\right]\begin{bmatrix} y^-(t,0)\\ y^+(t,1)\end{bmatrix}$$
$$+ 2\int_0^1 y^\top(t,x)\mathcal{Q}(x)\left|\Lambda(x)\right|^{-1}F(x)y(t,x)dx$$
$$- 2\mu\int_0^1 y^\top(t,x)\mathcal{Q}(x)y(t,x)dx\,. \quad (16)$$

Then, (10) and (11) imply that $\dot V \le -2\nu V$. Hence for all $t\in\mathbb{R}^+$ one has $V(y) \le e^{-2\nu t}V\left(y^0\right)$. Combining this relation with (14), the proof is complete. ■

Remark 2. In the proof of Proposition 3.1, for the computation of the time derivative of V, we have proceeded as if the solutions were in C^1. Nonetheless, the calculus remains valid with L^2-solutions. It is due to the density of C^1-solutions in the set of L^2-solutions.

Henceforth, for a given μ in $\mathbb{R}$ let us denote $I_{n,m}(x)$ the matrix

$$I_{n,m}(x) = \mathtt{diag}\left[e^{2\mu x}I_m, e^{-2\mu x}I_{n-m}\right]\,. \quad (17)$$

Proposition 3.2. Let μ in $\mathbb{R}$. Conditions (10) and (11) are satisfied if and only if the continuous time LPV system

$$\dot p(t) = I_{n,m}(x)\left(\left|\Lambda(x)\right|^{-1}F(x) - \mu I_n\right)p(t)\,,\ x\in[0,1]\,, \quad (18)$$

and the discrete time system

$$h(t{+}1) = \mathtt{diag}\left[I_m, e^{-\mu}I_{n-m}\right]e^{\mu}G\,\mathtt{diag}\left[I_m, e^{\mu}I_{n-m}\right]h(t) \quad (19)$$

share a common block diagonal Lyapunov matrix $\mathtt{diag}\left[Q^-, Q^+\right]$, where Q^- and Q^+ are symmetric matrices in $\mathbb{R}^{m\times m}$ and $\mathbb{R}^{(n-m)\times(n-m)}$ respectively.

Proof. LMI-based condition (10) describes a condition for the stability of the continuous time LPV system (18).

LMI-based condition (11) may be developed as

$$P = \begin{bmatrix} P_{--} & P_{-+}\\ P_{+-} & P_{++}\end{bmatrix} \le 0\,, \quad (20)$$

with

$$P_{--} = e^{2\mu}G_{--}^\top Q^- G_{--} + G_{+-}^\top Q^+ G_{+-} - Q^-\,,$$
$$P_{-+} = e^{2\mu}G_{--}^\top Q^- G_{-+} + G_{+-}^\top Q^+ G_{++}\,,$$
$$P_{+-} = P_{-+}^\top\,,$$
$$P_{++} = e^{2\mu}G_{-+}^\top Q^- G_{-+} + G_{++}^\top Q^+ G_{++} - e^{-2\mu}Q^+\,.$$

The matrix P in (20) may be rewritten as

$$P = \left(e^{\mu}G\right)^\top \mathtt{diag}\left[Q^-, e^{-2\mu}Q^+\right]e^{\mu}G$$
$$- \mathtt{diag}\left[Q^-, e^{-2\mu}Q^+\right]\,. \quad (21)$$

Thus, with (21) inequality (20) leads to establish

$$P\le 0 \Leftrightarrow \mathtt{diag}\left[I_m, e^{\mu}I_{n-m}\right]e^{\mu}G^\top\mathtt{diag}\left[I_m, e^{-\mu}I_{n-m}\right]$$
$$\times\,\mathtt{diag}\left[Q^-, Q^+\right]\mathtt{diag}\left[I_m, e^{-\mu}I_{n-m}\right]$$
$$e^{\mu}G\,\mathtt{diag}\left[I_m, e^{\mu}I_{n-m}\right] \le \mathtt{diag}\left[Q^-, Q^+\right]\,.$$

Hence, condition (11) implies that the discrete time system (19) shares a common Lyapunov matrix with the continuous time LPV system (18). It concludes the proof of Proposition 3.2. ■

Remark 3. A consequence of Proposition 3.2 is a trade-off in the choice of μ between the satisfaction of (10) and (11).

4. CONTROLLER DESIGN

4.1 Boundary Control Design

We consider next the problem of boundary control design, when boundary condition (2) is given by

$$G = T + LK_B\,, \quad (22)$$

where matrices T in $\mathbb{R}^{n\times n}$, L in $\mathbb{R}^{n\times q}$ ($n>q$) are given and the matrix K_B in $\mathbb{R}^{q\times n}$ has to be designed such that system (1)–(3) with the boundary conditions (22) is GES.

Theorem 1. If there exist $\nu>0$, μ in $\mathbb{R}$, a matrix U in $\mathbb{R}^{q\times n}$, and symmetric matrices S^- in $\mathbb{R}^{m\times m}$, S^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that $S(x) = \mathtt{diag}\left[e^{-2\mu x}S^-, e^{2\mu x}S^+\right]$, and such that the following conditions hold, for all $x\in[0,1]$,

$$S(x)\Lambda(x) = \Lambda(x)S(x)\,, \quad (23)$$

$$\begin{bmatrix}\mathtt{diag}\left[S^-, e^{-2\mu}S^+\right] & \left(TS(0)+LU\right)^\top\\ TS(0)+LU & \mathtt{diag}\left[e^{-2\mu}S^-, S^+\right]\end{bmatrix} \ge 0\,, \quad (24)$$

$$-2\mu S(x) + S(x)F^\top(x)\left|\Lambda(x)\right|^{-1}$$
$$+ \left|\Lambda(x)\right|^{-1}F(x)S(x) \le -2\nu S(x)\left|\Lambda(x)\right|^{-1}\,, \quad (25)$$

then the boundary control given by (22) with

$$K_B = US(0)^{-1}\,, \quad (26)$$

makes system (1)–(3) GES.

Proof. Replacing U by $K_BS(0)$ and applying the Schur complement formula in (24) one gets

$$\mathtt{diag}\left[S^-, e^{-2\mu}S^+\right] - S(0)\left(T+LK_B\right)^\top$$
$$\times\,\mathtt{diag}\left[e^{-2\mu}S^-, S^+\right]^{-1}\left(T+LK_B\right)S(0)\ge 0\,. \quad (27)$$

Rassembling the term in one matrix and multiplying from the left and right with $S(0)^{-1}$ we get a matrix

$$M = \begin{bmatrix} M_{--} & M_{-+}\\ M_{+-} & M_{++}\end{bmatrix}\ge 0\,, \quad (28)$$

with

$$M_{--} = \left(S^-\right)^{-1} - e^{2\mu}\left(T+LK_B\right)_{--}^\top\left(S^-\right)^{-1}$$
$$\times\left(T+LK_B\right)_{--}$$
$$-\left(T+LK_B\right)_{+-}^\top\left(S^+\right)^{-1}\left(T+LK_B\right)_{+-}\,, \quad (29)$$

$$M_{-+} = -e^{2\mu}\left(T+LK_B\right)_{--}^\top\left(S^-\right)^{-1}\left(T+LK_B\right)_{-+}$$
$$-\left(T+LK_B\right)_{+-}^\top\left(S^+\right)^{-1}\left(T+LK_B\right)_{++}\,, \quad (30)$$

$$M_{+-} = M_{-+}^\top\,, \quad (31)$$

$$M_{++} = e^{-2\mu}\left(S^+\right)^{-1} - e^{2\mu}\left(T+LK_B\right)_{-+}^\top\left(S^-\right)^{-1}$$
$$\times\left(T+LK_B\right)_{-+}$$
$$-\left(T+LK_B\right)_{++}^\top\left(S^+\right)^{-1}\left(T+LK_B\right)_{++}\,. \quad (32)$$

Letting $Q^- = (S^-)^{-1}$, $Q^+ = (S^+)^{-1}$ we get condition (9) from (23), LMI-based conditions (10) from (25), (11) from the matrix M in (28). Indeed, M is equivalent to the matrix $-P$ in (20), this latter is the derivation of the inequality (11). It concludes the proof of Theorem 1. ■

4.2 Distributed Control Design

We consider that the right-hand side of (1) is of the form

$$F(x) = H(x) + B(x)K_D(x)\,, \quad x \in [0,1]\,, \tag{33}$$

where matrices $H(x)$ in $\mathbb{R}^{n\times n}$ and $B(x)$ in $\mathbb{R}^{n\times p}$ ($n > p$) are given and matrix $K_D(x)$ in $\mathbb{R}^{p\times n}$ has to be designed such that system (1)–(3) is GES with the distributed control (33). In the next we assume that $K_D(x)$ is given by

$$K_D(x) = \sum_{i=1}^{\ell} \alpha_i(x) K_i\,, \tag{34}$$

where α_i, $i = 1, \ldots, \ell$, are some continuous real functions.

Remark 4. Examples of suitable functions α_i, in (34) are the Bézier functions basis and spline basis functions of degree 1.

Theorem 2. Let an integer $\ell > 0$ be given. If there exist $\nu > 0$, μ in $\mathbb{R}$, matrices U_i in $\mathbb{R}^{p\times n}$, $i = 1, \ldots, \ell$, and positive definite symmetric matrices S^- in $\mathbb{R}^{m\times m}$, S^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that $S(x) = \mathtt{diag}\left[e^{-2\mu x}S^-, e^{2\mu x}S^+\right]$, and such that, the following conditions hold, for all $x \in [0,1]$,

$$S(x)\Lambda(x) = \Lambda(x)S(x)\,, \tag{35}$$

$$\begin{aligned}
&\left(|\Lambda(x)|^{-1} H(x) - \mu I_n\right) S(x)\\
&+ S(x)\left(|\Lambda(x)|^{-1} H(x) - \mu I_n\right)^\top\\
&+ \sum_{i=1}^{\ell} \alpha_i(x) \left(I_{n,m}(x)\right)^{-1} U_i^\top B^\top(x) |\Lambda(x)|^{-1}\\
&+ \sum_{i=1}^{\ell} \alpha_i(x) |\Lambda(x)|^{-1} B(x) U_i \left(I_{n,m}(x)\right)^{-1}\\
&\leq -2\nu |\Lambda(x)|^{-1} S(x)\,,
\end{aligned} \tag{36}$$

$$\begin{bmatrix} \mathtt{diag}\left[S^-, e^{-2\mu}S^+\right] & (GS(0))^\top \\ GS(0) & \mathtt{diag}\left[e^{-2\mu}S^-, S^+\right] \end{bmatrix} \geq 0\,, \tag{37}$$

with $I_{n,m}(x)$ given in (17), then the distributed control given by (33) and (34) with

$$K_i = U_i S(0)^{-1}\,, \quad i = 1, \ldots, \ell\,, \tag{38}$$

makes system (1)–(3) GES.

Proof. We know that system (1)–(3) is exponentially stable if conditions of Proposition 3.1 hold. To apply this result let us check (9), (10) and (11) successively. Using the Schur complement formula with (37), letting $Q^- = (S^-)^{-1}$ and $Q^+ = (S^+)^{-1}$ as in the proof of Theorem 1, conditions (9) and (11) are satisfied. We can rewrite (10) as

$$\begin{aligned}
&\left(|\Lambda(x)|^{-1} F(x) - \mu I_n\right)^\top \mathcal{Q}(x)\\
&+ \mathcal{Q}(x)\left(|\Lambda(x)|^{-1} F(x) - \mu I_n\right) \leq -2\nu |\Lambda(x)|^{-1} \mathcal{Q}(x)\,.
\end{aligned}$$

We use the expression of F given by (33) and (34) and get

$$\begin{aligned}
&\left(|\Lambda(x)|^{-1}\left(H(x) + B(x)\sum_{i=1}^{\ell}\alpha_i(x)K_i\right) - \mu I_n\right)^\top \mathcal{Q}(x)\\
&+ \mathcal{Q}(x)\left(|\Lambda(x)|^{-1}\left(H(x) + B(x)\sum_{i=1}^{\ell}\alpha_i(x)K_i\right) - \mu I_n\right)\\
&\leq -2\nu |\Lambda(x)|^{-1} \mathcal{Q}(x)\,.
\end{aligned} \tag{39}$$

This last inequality is not jointly convex in K_i and $\mathcal{Q}(x)$. To overcome this issue we multiply (39) at the left and right by $S(x)$ and we let $K_i = U_i S(0)^{-1}$, $i = 1, \ldots, \ell$. The obtained inequality is (36). This concludes the proof of Theorem 2. ■

Remark 5. The simultaneous design of a boundary control and of a distributed control is possible with the computation of matrices S^-, S^+, U and U_i , $i = 1, \ldots, \ell$, such that S^-, S^+ satisfy (23), (24), (25), (36) and (37) and U satisfies (24) and U_i, $i = 1, \ldots, \ell$ satisfy (36).

5. OVERAPPROXIMATION OF THE LMI-BASED STABILITY CONDITIONS

In this section we present practical techniques for the stability analysis. The same approach can be used for the design.

5.1 Non-Spatially Varying Case

Let us suppose $F(x) = F$, $\Lambda(x) = \Lambda$. The main goal of this section is to provide a way to numerically verify conditions of Proposition 3.1 and of Theorems 1 and 2.

For fixed μ in $\mathbb{R}$, Q^- in $\mathbb{R}^{m\times m}$ and Q^+ in $\mathbb{R}^{(n-m)\times(n-m)}$, we write

$$Q_{ij} = \mathtt{diag}\left[e^{2\mu i}Q^-, e^{-2\mu j}Q^+\right]\,, \quad i,j = 0,1\,. \tag{40}$$

Lemma 1. For all $x \in [0,1]$, $\mathcal{Q}(x)$ lies in the convex hull formed by Q_{00}, Q_{01}, and Q_{11} if $\mu > 0$ and by Q_{00}, Q_{10}, and Q_{11} if $\mu < 0$.

Proof. Without loss of generality we assume that $\mu > 0$. $\mathcal{Q} : x \mapsto \mathcal{Q}(x)$ is a parameterized curve in the (Q^-, Q^+) plane. We can express it as an explicit curve. We have $e^{2\mu x}e^{-2\mu x} = 1$, thus the expression of the explicit curve is given by

$$h(X) = \frac{1}{X}\,, \quad X \in \rho = \left[\min\left(e^{-2\mu}, 1\right), \max\left(e^{-2\mu}, 1\right)\right]\,.$$

This curve is convex on this interval. Then,

$$\frac{1}{\alpha e^{-2\mu} + (1-\alpha)} \leq \alpha g\left(e^{-2\mu}\right) + (1-\alpha) g(1)\,, \quad \alpha \in (0,1)\,,$$

where $g(X) = \left(\frac{\min\left(1,e^{2\mu}\right) - \max\left(1,e^{2\mu}\right)}{\max\left(1,e^{-2\mu}\right) - \min\left(1,e^{-2\mu}\right)}\right) X + e^{2\mu} + 1$. When X lies in ρ, $g(X)$ describes the straight line between Q_{00} and Q_{11}. Hence, for $X \in \rho$ one has $h(X) \leq g(X)$. Thus, $\mathcal{Q}(x)$ lies in the convex hull formed by Q_{00}, Q_{01}, and Q_{11}. ■

Proposition 5.1. If there exist $\nu > 0$, μ in $\mathbb{R}$, and symmetric positive definite matrices Q^- in $\mathbb{R}^{m\times m}$ and Q^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that

$$Q_{ij}\Lambda = \Lambda Q_{ij}\,, \tag{41}$$

$$-2\mu Q_{ij} + F^\top |\Lambda|^{-1} Q_{ij} + Q_{ij} |\Lambda|^{-1} F \leq -2\nu |\Lambda|^{-1} Q_{ij}\,, \tag{42}$$

hold for all $(i,j) \in \{(0,0),(0,1),(1,1)\}$ if $\mu > 0$ and for all $(i,j) \in \{(0,0),(1,0),(1,1)\}$ if $\mu < 0$, together with

$$\begin{aligned}
&\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix}^\top Q_{00} \tilde{I}_{n,m} \begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix}\\
&\leq \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix}^\top Q_{11} \tilde{I}_{n,m} \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix}\,,
\end{aligned} \tag{43}$$

then conditions (9), (10), and (11) are satisfied for all $x \in [0,1]$.

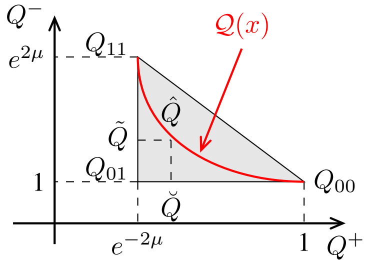

Fig. 1. Representation of the gridding method in the plane (Q^-, Q^+) with $\mu > 0$.

Proof. The inequality (43) corresponds to (11). By Lemma 1 the constraint of equality (9) and LMI (10) are embedded in the polytope formed by the points Q_{00}, Q_{01} and Q_{11} if $\mu > 0$ and by the polytope formed by the points Q_{00}, Q_{10} and Q_{11} if $\mu < 0$. Thus, conditions (9), (10), and (11) are satisfied. It concludes the proof of Proposition 5.1. ∎

Remark 6. The approximation with the exponential kernel (8) can be made thighter by increasing the number of points describing the polytope embbeding the constraints given by condition (9) and LMIs (10), (11) (see Fig. 1). For instance, on Fig. 1 there are 5 points: Q_{11}, $\tilde{Q}$, $\hat{Q}$, $\breve{Q}$ and Q_{00}. The impact of the number of points is explored numerically in the next section.

5.2 Spatially-Varying Case

We may generalize the previous results when Λ and F are both spatially varying and lie in a convex hull. Let us do it in the context of Proposition 3.1 only.

We assume that the parametrized matrix

$$W(x) = |\Lambda(x)|^{-1} F(x)\,, \tag{44}$$

lies for all $x \in [0,1]$ in the convex hull

$$\mathcal{W} := \left\{ W : W = \sum_{i=1}^{N} \alpha_i W_i,\ \sum_{i=1}^{N} \alpha_i = 1 \right\}, \tag{45}$$

for given matrices W_i, $i = 1, \dots, N$.

Proposition 5.2. If there exist $\nu > 0$, μ in $\mathbb{R}$, and diagonal positive definite matrices Q^- in $\mathbb{R}^{m\times m}$ and Q^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that

$$-2\mu Q_{jk} + W_i^\top Q_{jk} + Q_{jk} W_i \leq -2\nu \left|\bar{\Lambda}\right|^{-1} Q_{jk}\,, \tag{46}$$

$$\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix}^\top \tilde{I}_{n,m} Q_{00} \begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix} \leq \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix}^\top \tilde{I}_{n,m} Q_{11} \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix}, \tag{47}$$

where

$$\bar{\Lambda} = \mathtt{diag}\left[\min_{x\in[0,1]} \lambda_1(x), \dots, \min_{x\in[0,1]} \lambda_m(x), \max_{x\in[0,1]} \lambda_{m+1}(x), \dots, \max_{x\in[0,1]} \lambda_n(x) \right], \tag{48}$$

for all $i = 1, \dots, N$, and $(j,k) \in \{(0,0),(0,1),(1,1)\}$ if $\mu > 0$, $(j,k) \in \{(0,0),(1,0),(1,1)\}$ if $\mu < 0$, then conditions (9), (10), and (11) are satisfied for all $x \in [0,1]$.

Proof. Multiplying (46) by α_i and making the sum for $i = 1, \dots, N$, we get

$$-2\mu Q_{jk} + W(x)^\top Q_{jk} + Q_{jk} W(x) \leq -2\nu \left|\bar{\Lambda}\right|^{-1} Q_{jk}\,, \tag{49}$$

for all $x \in [0,1]$, $(j,k) = \{(0,0),(0,1),(1,1)\}$ if $\mu > 0$ and $(j,k) = \{(0,0),(1,0),(1,1)\}$ if $\mu < 0$. Using Lemma 1 and the definition of $\bar{\Lambda}$ in (48), one gets (10). Condition (9) is automatically satisfied because of the diagonal form of Q^- and Q^+. It concludes the proof of Proposition 5.2. ∎

6. NUMERICAL EXPERIMENTS

In this section, several examples are presented to illustrate the results of the paper. All the solutions of the LMIs have been computed with the Multi-Parametric Toolbox (MPT) (Herceg et al. (2013)).

6.1 Stability Analysis, illustrating Proposition 5.1

Example 1. Let us consider the following matrices

$$\Lambda = \mathtt{diag}\,[-3, 1]\,, \tag{50}$$

$$F = \begin{bmatrix} -1 & 0.2 \\ 1 & 0.2 \end{bmatrix}, \tag{51}$$

$$G = \begin{bmatrix} 0.2 & -0.3 \\ 0.6 & 0.1 \end{bmatrix}. \tag{52}$$

The matrix F in (51) is non-Hurwitz and the matrix G in (52) is such that $\rho(G) < 1$. This last property is classical for the stability analysis of linear and quasilinear hyperbolic system (Coron et al. (2008)), (Diagne et al. (2012)).

Fig. 2 shows that the result obtained with only three points for the polytope is optimal. Indeed, the numerical ν obtained with three points is the same than with higher number of points. This result might be expected because all the constraints of the LMI are enclosed by the overapproximation with the polytope described by three points. The lower curve corresponds to the result of the algorithm when the objective is to maximize ν. In order to make this objective tractable, a relaxation on the right-hand side of the inequality (10) is made. The upper curve is the result of the algorithm when the objective is to minimize the trace of $\mathcal{Q}(0)$. Unexpectedly, the second algorithm gives a better ν than the first one.

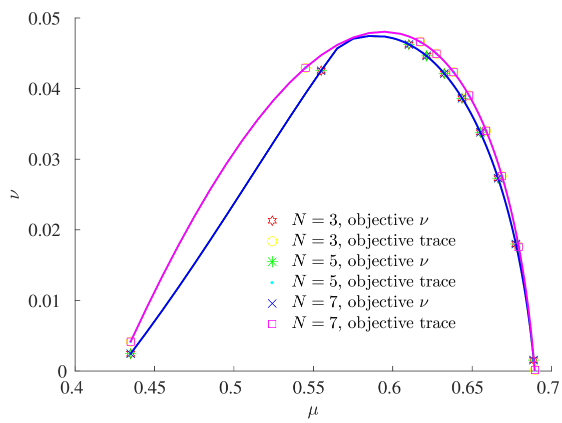

Fig. 2. Evolution of ν as a function of μ for Example 1 for the gridding method depending on the number of points.

Example 2. Let us consider the following matrices

$$\Lambda = \mathtt{diag}\,[-1, 1]\,, \tag{53}$$

$$F = \begin{bmatrix} -0.3 & 0.1 \\ 0.1 & -0.3 \end{bmatrix}, \tag{54}$$

$$G = \begin{bmatrix} 0.1 & -0.8 \\ 0.6 & -0.4 \end{bmatrix}. \tag{55}$$

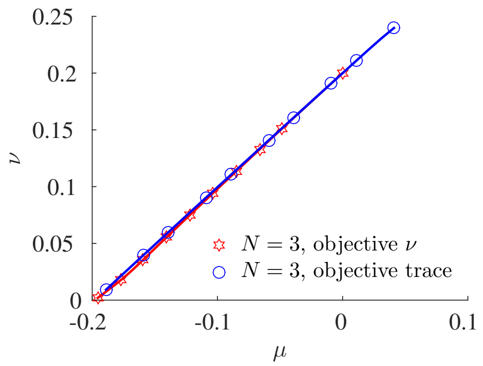

Fig. 3. Evolution of ν as a function of μ for Example 2.

In this example the matrix F in (54) is Hurwitz and the matrix G in (55) is contractive that is $\rho(G) < 1$.

The shape of the curve obtained in this example is not the same as the one presented in Example 1. This comes from the fact that the matrix F is Hurwitz, hence increase μ moves the eigenvalues of $|\Lambda|^{-1}F - \mu I_2$ in the left half-plane of $\mathbb{C}$, so it will increase the parameter ν. The algorithm stops due to LMI (11) which is no more solvable for large μ. Thus, this example illustrates also Proposition 3.2.

6.2 Controller Design, illustrating Theorems 1 and 2

Example 3. Let us consider system (1)–(3) with

$$\Lambda = \mathtt{diag}[-1, 2], \quad F = \begin{bmatrix} -0.1 & 0.1 \\ 0.5 & -0.8 \end{bmatrix},$$

under the boundary control (22) where

$$T = \begin{bmatrix} -0.5 & 1 \\ 0.5 & 1 \end{bmatrix}, \quad L^\top = [0.5 \ -1].$$

Let us choose $\nu = 0.1$. The design algorithm gives

$$\mu = 0.1580, \quad K_B = [0.5596 \ 0.7910],$$

which leads to the following boundary control

$$G = \begin{bmatrix} -0.2202 & 1.3955 \\ -0.0596 & 0.2090 \end{bmatrix}. \tag{56}$$

Example 4. Let us consider system (1)–(3) with

$$\Lambda = \mathtt{diag}[-2, 1], \quad G = \begin{bmatrix} 0.5 & -0.4 \\ 0.2 & 0.8 \end{bmatrix},$$

under the distributed control (33) where

$$H = \begin{bmatrix} -0.5 & 0.2 \\ 0.2 & 0.5 \end{bmatrix}, \quad B^\top = [0.5 \ 1], \quad (\ell, \alpha) = (1, 1).$$

Numerically, ν is fixed to 0.3. The design algorithm gives

$$\mu = 0.15, \quad K_D = [-0.3130 \ -1.1485],$$

which leads to

$$F = \begin{bmatrix} -0.6565 & -0.3743 \\ -0.1130 & -0.6485 \end{bmatrix}. \tag{57}$$

7. CONCLUSION

In this work, we have proposed a Lyapunov function and stability conditions based on it. These conditions are expressed as LMIs. We also design boundary and distributed controllers by the same techniques. All these conditions correspond to an infinite number of LMIs. Then, we proved that this numerical complexity can be relaxed. We show the effectiveness of the method with academic examples.

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