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A topological interpretation of the cyclotomic polynomial

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Abstract. We interpret the coefficients of the cyclotomic polynomial in terms of simplicial homology.

Résumé. Nous donnons une interprétation des coefficients du polynôme cyclotomique en utilisant l'homologie simpliciale.

Keywords: Cyclotomic polynomial, higher-dimensional tree, matroid duality, oriented matroid, simplicial matroid, simplicial homology

1 Introduction

This paper studies the cyclotomic polynomial $\Phi_n(x)$, which is defined as the minimal polynomial over \mathbb{Q} for any primitive n^{th} root of unity ζ in \mathbb{C} . It is monic, irreducible, and has degree given by the Euler phi function $\phi(n)$, with formula

$$\Phi_n(x) = \prod_{j \in (\mathbb{Z}/n\mathbb{Z})^\times} (x - \zeta^j).$$

The equation $x^n - 1 = \prod_{d|n} \Phi_d(x)$ gives a recurrence showing that all coefficients of $\Phi_n(x)$ lie in \mathbb{Z} .

Although well-studied, the coefficients of $\Phi_n(x)$ are mysterious [2, 10, 11, 14, 15, 17, 29]. We offer here two interpretations for their magnitudes, as orders of cyclic groups. In the first interpretation (Corollary 5 below) this group is a quotient of the free abelian group $\mathbb{Z}[\zeta]$ by a certain full rank sublattice.

The second interpretation is topological, given by Theorem 1 below, as the torsion in the homology of a certain simplicial complex associated with a squarefree integer $n = p_1 \cdots p_d$. These simplicial complexes originally arose in the work of Bolker [6], reappeared in the work of Kalai [13] and Adin [1] on higher-dimensional matrix-tree theorems, and were shown to be connected with cyclotomic extensions in work of J. Martin and the second author [18]. We review these simplicial complexes briefly here in order to state the result; see Section 4 for more details.

Given a positive integer p , let K_p denote a 0-dimensional abstract simplicial complex having p vertices⁽ⁱ⁾, which we will label by the residues

$$\{0 \bmod p, 1 \bmod p, \dots, (p-1) \bmod p\}$$

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⁽ⁱ⁾Note that here K_p does *not* refer to a complete graph on p vertices; we hope that this causes no confusion.

for reasons that will become clear in a moment. Given primes p_1, \dots, p_d , let

$$K_{p_1, \dots, p_d} := K_{p_1} * \dots * K_{p_d}$$

be the *simplicial join*, [21, §62], of K_{p_1}, \dots, K_{p_d} . This is a pure $(d - 1)$ -dimensional abstract simplicial complex, that may be thought of as the *complete d -partite complex* on vertex sets K_{p_1} through K_{p_d} of sizes p_1, \dots, p_d . The *facets* (maximal simplices) of K_{p_1, \dots, p_d} are labelled by sequences of residues $(j_1 \bmod p_1, \dots, j_d \bmod p_d)$. Denoting the squarefree product $p_1 \cdots p_d$ by n , the Chinese Remainder Theorem isomorphism

$$\mathbb{Z}/p_1\mathbb{Z} \times \dots \times \mathbb{Z}/p_d\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/n\mathbb{Z} \tag{1}$$

allows one to label such a facet by a residue $j \bmod n$; call this facet $F_{j \bmod n}$. Then for any subset $A \subseteq \{0, 1, \dots, \phi(n)\}$, let K_A denote the subcomplex of K_{p_1, \dots, p_d} which is generated by the facets $\{F_{j \bmod n}\}$ as j runs through the following set of residues:

$$A \cup \{\phi(n) + 1, \phi(n) + 2, \dots, n - 1, n\}.$$

Our first main result interprets the magnitudes of the coefficients of $\Phi_n(x)$. Let $\tilde{H}_i(-; \mathbb{Z})$ denote reduced simplicial homology with coefficients in \mathbb{Z} .

Theorem 1 *For a squarefree positive integer $n = p_1 \cdots p_d$, with cyclotomic polynomial $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$, one has*

$$\tilde{H}_i(K_{\{j\}}; \mathbb{Z}) = \begin{cases} \mathbb{Z}/c_j\mathbb{Z} & \text{if } i = d - 2, \\ \mathbb{Z} & \text{if both } i = d - 1 \text{ and } c_j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We furthermore interpret topologically the signs of the coefficients in $\Phi_n(x)$. For this, we use oriented simplicial homology, and orient the facet $F_{j \bmod n}$ having $j \equiv j_i \bmod p_i$ for $i = 1, 2, \dots, d$ as

$$[F_j] = [F_{j \bmod n}] = [j_1 \bmod p_1, \dots, j_d \bmod p_d].$$

Theorem 2 *Fix a squarefree positive integer $n = p_1 \cdots p_d$ with cyclotomic polynomial $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$. Then for any $j \neq j'$ such that $c_j, c_{j'} \neq 0$, one has $\tilde{H}_{d-1}(K_{\{j, j'\}}; \mathbb{Z}) \cong \mathbb{Z}$, and any nonzero $(d - 1)$ -cycle $z = \sum_{\ell} b_{\ell} [F_{\ell}]$ in this homology group will have $b_j, b_{j'} \neq 0$, with*

$$\frac{c_j}{c_{j'}} = -\frac{b_{j'}}{b_j}.$$

In particular, $c_j, c_{j'}$ have the same sign if and only if $b_j, b_{j'}$ have opposite signs.

Example 3 We illustrate these theorems for $n = 15$. Here $d = 2, p_1 = 3, p_2 = 5$, and $\phi(n) = 2 \cdot 4 = 8$. The cyclotomic polynomial is

$$\begin{aligned} \Phi_{15}(x) &= 1 - x + x^3 - x^4 + x^5 - x^7 + x^8 \\ &= (+1) \cdot (x^0 + x^3 + x^5 + x^8) + (-1) \cdot (x^1 + x^4 + x^7) + 0 \cdot (x^2 + x^6). \end{aligned}$$

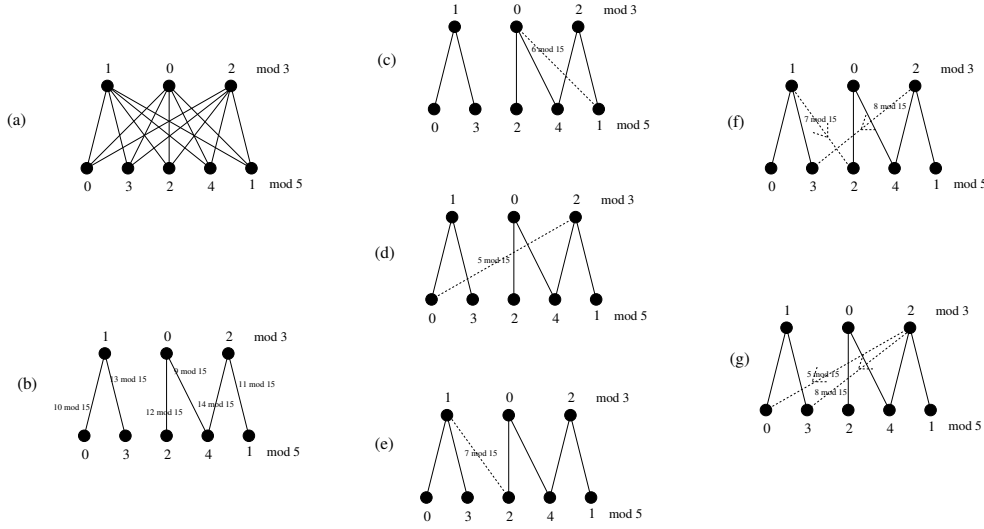


Fig. 1: The case of $\Phi_{15}(x)$

The complex $K_{p_1, p_2} = K_{3,5}$ is a complete bipartite graph with vertex sets labelled as in Figure 1(a). The subcomplex K_\emptyset generated by the edges $F_j \pmod{15}$ with $j \in \{\phi(n) + 1, \phi(n) + 2, \dots, n - 1\} = \{9, 10, 11, 12, 13, 14\}$ is the subgraph shown in Figure 1(b).

To see why the coefficient $c_6 = 0$ in $\Phi_{15}(x)$, one adds the edge $F_6 \pmod{15}$ to the graph K_\emptyset , obtaining the graph $K_{\{6\}}$, shown in Figure 1(c), which has

$$\begin{aligned} \tilde{H}_0(K_{\{6\}}; \mathbb{Z}) &= \mathbb{Z} = \mathbb{Z}/0\mathbb{Z} \\ \tilde{H}_1(K_{\{6\}}; \mathbb{Z}) &= \mathbb{Z}. \end{aligned}$$

To see why the coefficients $c_5 = +1$ or $c_7 = -1$ have magnitude 1, one adds the edge $F_5 \pmod{15}$ or $F_7 \pmod{15}$ to the graph K_\emptyset , obtaining the graphs $K_{\{5\}}$ or $K_{\{7\}}$ shown in Figures 1(d) and 1(e), which have

$$\begin{aligned} \tilde{H}_0(K_{\{5\}}; \mathbb{Z}) &= 0 = \mathbb{Z}/(+1)\mathbb{Z} \\ \tilde{H}_0(K_{\{7\}}; \mathbb{Z}) &= 0 = \mathbb{Z}/(-1)\mathbb{Z}. \end{aligned}$$

To understand the signs of the coefficients, note first that, by convention, $\Phi_{15}(x)$ is monic, so the coefficient $c_8 = c_{\phi(n)} = +1$. Therefore any other coefficient c_j should have sign

$$\text{sgn}(c_j) = \frac{\text{sgn}(c_j)}{\text{sgn}(c_8)} = -\frac{\text{sgn}(b_8)}{\text{sgn}(b_j)}$$

where $z = \sum_i b_i [F_i]$ is a nontrivial cycle in $K_{\{j,8\}}$, in which the edge $[F_j]$ is directed from the vertex $(j_1 \pmod{3})$ toward the vertex $(j_2 \pmod{5})$. As shown in Figures 1(f) and 1(g), the nontrivial cycle in $K_{\{7,8\}}$ has $[F_7], [F_8]$ oriented in the *same* direction, explaining why $c_7 = -1$, while the nontrivial cycle in $K_{\{5,8\}}$ has $[F_5], [F_8]$ oriented in the *opposite* direction, explaining why $c_5 = +1$.

The remainder of the paper is structured as follows. Section 2 describes our first interpretation for the cyclotomic polynomial, which applies much more generally to any monic polynomial in $\mathbb{Z}[x]$. Section 3 reviews some facts, underlying the main results, about duality of matroids, Plücker coordinates, and oriented matroids. Section 4 recalls the main results and establishes terminology on Kalai’s higher dimensional spanning trees in a simplicial complex. Section 5 discusses further properties of the simplicial complex K_{p_1, \dots, p_d} whose subcomplexes appear in Theorem 1 and 2. Section 6 proves these theorems. We end with Section 7, where we discuss known properties of $\Phi_n(x)$ that manifest themselves topologically.

2 Coefficients of monic polynomials in $\mathbb{Z}[x]$

Our goal here is the first interpretation for the coefficients of $\Phi_n(x)$, which applies more generally to the coefficients of *any* monic polynomial $f(x)$ in $\mathbb{Z}[x]$. Recall that when $f(x)$ is of degree r , one has an isomorphism of \mathbb{Z} -modules

$$\begin{aligned} \mathbb{Z}^r &\longrightarrow \mathbb{Z}[x]/(f(x)) \\ (a_0, a_1, \dots, a_{r-1}) &\longmapsto \sum_{j=0}^{r-1} a_j \bar{x}^j. \end{aligned}$$

As notation, given a subset A of some abelian group, let $\mathbb{Z}A$ denote the collection of all \mathbb{Z} -linear combinations of elements of A .

Proposition 4 *For a monic polynomial $f(x) = \sum_{j=0}^r c_j x^j$ of degree r in $\mathbb{Z}[x]$, one has an isomorphism of abelian groups*

$$(\mathbb{Z}[x]/(f)) / \mathbb{Z}A \cong \mathbb{Z}/c_j \mathbb{Z}$$

where A is the subset of size r given as $\{\bar{1}, \bar{x}, \bar{x}^2, \dots, \bar{x}^r\} \setminus \{\bar{x}^j\}$.

Proof: Consider the matrix in $\mathbb{Z}^{r \times (r+1)}$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -c_0 \\ 0 & 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -c_{r-1} \end{bmatrix}$$

whose columns express the elements of $\{\bar{1}, \bar{x}, \bar{x}^2, \dots, \bar{x}^r\}$ uniquely in the \mathbb{Z} -basis $\{\bar{1}, \bar{x}, \bar{x}^2, \dots, \bar{x}^{r-1}\}$ for $\mathbb{Z}[x]/(f)$. The $r \times r$ submatrix obtained by restricting this matrix to the columns indexed by A is equivalent by row and column permutations to an upper triangular matrix with diagonal entries $(1, 1, \dots, 1, -c_j)$. Hence $(\mathbb{Z}[x]/(f)) / \mathbb{Z}A \cong \mathbb{Z}/c_j \mathbb{Z}$. \square

The special case where $f(x)$ is the cyclotomic polynomial $\Phi_n(x)$ leads to the following considerations. Fix once and for all a primitive n^{th} root of unity ζ .

Corollary 5 *The cyclotomic polynomial $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$ has*

$$\mathbb{Z}[\zeta]/\mathbb{Z}A \cong \mathbb{Z}/c_j \mathbb{Z}$$

where $A = \{1, \zeta, \zeta^2, \dots, \zeta^{\phi(n)}\} \setminus \{\zeta^j\}$.

Proof: Apply the previous proposition with $f(x) = \Phi_n(x)$ and $r = \phi(n)$, noting that the ring map $\mathbb{Z}[x] \rightarrow \mathbb{Z}[\zeta]$ sending x to ζ will also send x^j to ζ^j , and induce an isomorphism $\mathbb{Z}[x]/(\Phi_n(x)) \rightarrow \mathbb{Z}[\zeta]$. \square

For later use (see the proof of Theorem 20), we note here that the set

$$P_n := \{\zeta^m\}_{m \in (\mathbb{Z}/n\mathbb{Z})^\times}$$

of all primitive n^{th} roots of unity within $\mathbb{Z}[\zeta]$ forms a \mathbb{Z} -basis whenever n is squarefree. This is a sharpening of an observation of Johnsen [12], who noted that P_n forms a \mathbb{Q} -basis of $\mathbb{Q}[\zeta]$ in the same situation.

Proposition 6 *When n is squarefree, the collection P_n of all primitive n^{th} roots of unity forms a \mathbb{Z} -basis for $\mathbb{Z}[\zeta]$.*

Proof: The result is easy when n is prime and can be deduced from the Chinese Remainder Theorem in the general case. See [22] for details. \square

3 Duality of matroids or Plücker coordinates

We will need a version of the linear algebraic duality between Plücker coordinates for complementary Grassmannians $G(r, \mathbb{F}^n), G(n - r, \mathbb{F}^n)$, or equivalently, the duality between bases and cobases in coordinatized matroids.

Proposition 7 *Let $0 \leq r \leq n$. Let M and M^\perp be matrices in $\mathbb{F}^{r \times n}$ and $\mathbb{F}^{(n-r) \times n}$, respectively, both of maximal rank, with the following property: $\ker M$ is equal to the row space of M^\perp , or equivalently, $\ker M^\perp$ is the row space of M . Then*

- (i) *there exists a scalar α in \mathbb{F}^\times having the following property: for every $(n - r)$ -subset T of $[n]$, with complementary set T^c ,*

$$\det(M|_{T^c}) = \pm\alpha \cdot \det(M^\perp|_T)$$

where $A|_J$ denotes the restriction of a matrix A to the subset of columns indexed by J , and the \pm sign depends upon the set T .

- (ii) *if one furthermore assumes that $\mathbb{F} = \mathbb{Q}$, that M and M^\perp have entries in \mathbb{Z} , and that there exists at least one $(n - r)$ -subset T_0 for which $M|_{T_0^c}, M^\perp|_{T_0}$ are both invertible over \mathbb{Z} , then the scalar α above equals ± 1 , and one has for every $(n - r)$ -subset T ,*

$$\text{coker}(M|_{T^c}) \cong \text{coker}(M^\perp|_T).$$

Here, we are thinking of $\text{coker } M$ as signifying a map between powers of \mathbb{Z} .

Proof: Both assertions can be reduced via row and column operations to the case where M takes the form $[I_r | A]$ for some r -by- $(n - r)$ matrix A , where they are easier to verify. See [22] for details. \square

The proof of Theorem 2 will ultimately rely on the following statement about duality of oriented matroids for vectors in a vector space over an ordered field \mathbb{F} , such as $\mathbb{F} = \mathbb{Q}$.

Proposition 8 Let \mathbb{F} be an ordered field, M and let M^\perp be matrices in $\mathbb{F}^{r \times n}$ and $\mathbb{F}^{(n-r) \times n}$ as in Proposition 7, that is, both of maximal rank, with $\ker M$ perpendicular to the row space of M^\perp . Let the vectors v_ℓ in \mathbb{F}^r and v_ℓ^\perp in \mathbb{F}^{n-r} be the ℓ^{th} columns of M and M^\perp . Let A be an $(r+1)$ -subset of $\{1, 2, \dots, n\}$ such that the matrix $M|_A$ in $\mathbb{F}^{r \times (r+1)}$ has full rank r , with

$$\sum_{\ell \in A} c_\ell v_\ell = 0 \quad (2)$$

the unique dependence among its columns, up to scaling. Then for any pair of nonzero coefficients $c_j, c_{j'} \neq 0$, the matrix $M^\perp|_{A^c \cup \{j, j'\}}$ in $\mathbb{F}^{(n-r) \times (n-r+1)}$ has full rank $n-r$, and the unique dependence among its columns, up to scaling,

$$\sum_{\ell \in A^c \cup \{j, j'\}} b_\ell v_\ell^\perp = 0, \quad (3)$$

will have both $b_j, b_{j'} \neq 0$, with

$$\frac{c_j}{c_{j'}} = -\frac{b_{j'}}{b_j}.$$

In particular, $c_j, c_{j'}$ have the same sign if and only if $b_j, b_{j'}$ have opposite signs.

Proof: The main observation here is that vectors in the row space of M^\perp are *covectors* for $\{v_\ell^\perp\}$. See [22] for details. □

4 Simplicial spanning trees

For a collection of subsets S of some vertex set V , let $\langle S \rangle$ denote the (abstract) simplicial complex S on V generated by S , that is, $\langle S \rangle \subset 2^V$ consists of all subsets of V contained in at least one subset from S . We recall the notion of a simplicial spanning tree in S , following Adin [1], Duval, Klivans and Martin [8], Kalai [13], and Maxwell [19].

Definition 9 Let S be the collection of facets of a pure k -dimensional (abstract) simplicial complex. Say that $R \subset S$ is an S -spanning tree if

- (i) $\langle R \rangle$ contains the entire $(k-1)$ -skeleton of $\langle S \rangle$,
- (ii) $\tilde{H}_k(\langle R \rangle; \mathbb{Z}) = 0$, and
- (iii) $\tilde{H}_{k-1}(\langle R \rangle; \mathbb{Z})$ is finite.

We point out here three well-known features of this definition.

Proposition 10 Fix the collection of facets S of a pure k -dimensional simplicial complex.

- (i) Condition (i) in Definition 9 is equivalent to $\tilde{H}_k(\langle S \rangle, \langle R \rangle; \mathbb{Z}) = \mathbb{Z}^{|S \setminus R|}$.
- (ii) Condition (ii) in Definition 9 is equivalent to $\tilde{H}_k(\langle R \rangle; \mathbb{Q}) = 0$.

(iii) All S -spanning trees R have the same cardinality, namely

$$|R| = |S| - \text{rank}_{\mathbb{Z}} \tilde{H}_k(\langle S \rangle; \mathbb{Z}). \tag{4}$$

Proof: See [22] for details. □

The following key observation essentially goes back to work of Kalai [13, Lemma 2].

Proposition 11 Fix a vertex set V and a collection of k -dimensional simplices S . Consider a collection of $(k + 1)$ -dimensional faces T of cardinality

$$|T| := \text{rank}_{\mathbb{Z}} \tilde{H}_k(\langle S \rangle; \mathbb{Z})$$

for which $T \cup \langle S \rangle$ forms a simplicial complex K , that is, all boundaries of faces in T lie in $\langle S \rangle$.

Then the following two assertions hold for any choice of an S -spanning tree R .

(i) The $|T| \times |T|$ matrix ∂ that represents the relative simplicial boundary map

$$\begin{array}{ccc} C_{k+1}(K, \langle R \rangle; \mathbb{Z}) & \rightarrow & C_k(K, \langle R \rangle; \mathbb{Z}) \\ \parallel & & \parallel \\ \mathbb{Z}^{|T|} & & \mathbb{Z}^{|S \setminus R|} \end{array}$$

is nonsingular if and only if $\tilde{H}_{k+1}(K; \mathbb{Q}) = 0$.

(ii) When the matrix ∂ is nonsingular, then $\text{coker}(\partial) = \tilde{H}_k(K, \langle R \rangle; \mathbb{Z})$.

Proof: See [22] for details. □

Definition 12 Given a collection of k -simplices S , and an S -spanning tree R , say⁽ⁱⁱ⁾ that R is *torsion-free* if Condition (iii) in Definition 9 is strengthened to the vanishing condition

$$(iv) \quad \tilde{H}_{k-1}(\langle R \rangle; \mathbb{Z}) = 0.$$

Example 13 For example, when $\langle R \rangle$ is a contractible subcomplex of $\langle S \rangle$ then it satisfies Condition (ii) of Definition 9 as well as the vanishing condition (iv). If it furthermore satisfies Condition (i) of Definition 9, then R becomes a torsion-free S -spanning tree.

A frequent combinatorial setting where this occurs (such as in Proposition 15 below) is when S is the set of facets of a (pure) *shellable* [3] simplicial complex, and R is the subset of facets which are not fully attached along their entire boundaries during the shelling process.

Proposition 14 Using the hypotheses and notation of Proposition 11, if one assumes in addition that R is torsion-free, assertion (ii) of Proposition 11 becomes the following assertion about (non-relative) homology:

$$(ii) \quad \text{When the matrix } \partial \text{ is nonsingular, then } \text{coker}(\partial) = \tilde{H}_k(K; \mathbb{Z})$$

Proof: When R is torsion-free, the long exact sequence for the pair $(K, \langle R \rangle)$ shows that $\tilde{H}_k(K; \mathbb{Z}) \cong \tilde{H}_k(K, \langle R \rangle; \mathbb{Z})$. □

⁽ⁱⁱ⁾ This condition on an S -spanning tree also plays an important role in [9] by Duval, Klivans and Martin.

5 More on the complete d -partite complex

It is well-known and easy to see that for a positive integer n having prime factorization $n = p_1^{e_1} \cdots p_d^{e_d}$ with $e_i \geq 1$, one always has $\Phi_n(x) = \Phi_{p_1 \cdots p_d}(x^{n/p_1 \cdots p_d})$. Thus it suffices to interpret the coefficients of cyclotomic polynomials for squarefree n .

In this section, we fix such a squarefree $n = p_1 \cdots p_d$, and discuss further properties of the simplicial complexes K_{p_1, \dots, p_d} , defined in Section 1, appearing in Theorems 1 and 2.

Proposition 15 *The $(d - 2)$ -dimensional skeleton of K_{p_1, \dots, p_d} is shellable, with*

$$\tilde{H}_{d-2}(K_{p_1, \dots, p_d}; \mathbb{Z}) = \mathbb{Z}^{n-\phi(n)}.$$

Proof: To show that the $(d - 2)$ -skeleton is shellable, we note the following three facts: (i) zero-dimensional complexes are all trivially shellable, (ii) joins of shellable complexes are shellable [24, Sec. 2], and (iii) skeleta of (pure) shellable simplicial complexes are shellable [5, Corollary 10.12]. Having shown that this skeleton is shellable, it therefore has only top homology; see, for example [3, Appendix]. This homology is free abelian, with rank the absolute value of its reduced Euler characteristic, namely

$$\begin{aligned} \left| \sum_{i \geq -1} (-1)^i \text{rank}_{\mathbb{Z}}(C_i) \right| &= \left| \sum_{i \geq -1} (-1)^i \sum_{\substack{I \subseteq \{1, 2, \dots, d\} \\ |I|=i+1}} \prod_{i \in I} p_i \right| = \left| \sum_{I \subseteq \{1, 2, \dots, d\}} (-1)^{|I|-1} \prod_{i \in I} p_i \right| \\ &= |(p_1 - 1) \cdots (p_d - 1) - p_1 \cdots p_d| = |\phi(n) - n|. \end{aligned}$$

□

As noted in the introduction, the Chinese Remainder Theorem isomorphism (1) identifies elements of $\mathbb{Z}/n\mathbb{Z}$ with the $(d - 1)$ -dimensional simplices of K_{p_1, \dots, p_d} . Lower dimensional faces of K_{p_1, \dots, p_d} can also be identified as cosets of subgroups within $\mathbb{Z}/n\mathbb{Z}$, but we will use this identification sparingly in this paper. For the sake of writing down oriented simplicial boundary maps, choose the following orientation on the simplices of K_{p_1, \dots, p_d} , consistent with the orientation of facets preceding Theorem 2: choose the oriented $(\ell - 1)$ -simplex $[j_{i_1} \bmod p_{i_1}, \dots, j_{i_\ell} \bmod p_{i_\ell}]$ with $i_1 < \dots < i_\ell$ as a basis element of $C_{\ell-1}(K_{p_1, \dots, p_d}; \mathbb{Z})$. The following simple observation was the crux of the results in [18].

Proposition 16 *If one identifies the indexing set $\mathbb{Z}/n\mathbb{Z}$ for the columns of the boundary map*

$$C_{d-1}(K_{p_1, \dots, p_d}; \mathbb{Z}) \rightarrow C_{d-2}(K_{p_1, \dots, p_d}; \mathbb{Z}) \tag{5}$$

with the set $\mu_n := \{\zeta^j\}_{j \in \mathbb{Z}/n\mathbb{Z}}$ of all n^{th} roots of unity, then every row of this boundary map represents a \mathbb{Q} -linear dependence on μ_n .

Proof: A row in this boundary map is indexed by an oriented $(d - 2)$ -face, which has the form

$$[j_1 \bmod p_1, \dots, j_k \widehat{\bmod} p_k, \dots, j_d \bmod p_d]$$

for some $j_k \in \{0, 1, \dots, p_k - 1\}$ and $1 \leq k \leq d$. This row will contain mostly zeroes. Its non-zero entries are all $(-1)^{k-1}$, and lie in the columns indexed by those ζ^j having $j \equiv j_i \bmod p_i$ for $i \neq k$, and $j \bmod p_k$ arbitrary. These exponents j are exactly those lying in one coset of the subgroup $p_1 \cdots \hat{p}_k \cdots p_d \mathbb{Z}/n\mathbb{Z}$ within $\mathbb{Z}/n\mathbb{Z}$. Summing ζ^j over j in such a coset gives zero. □

Example 17 Let $n = 15$ as in Example 3, and consider the matrix for the simplicial boundary map $C_1(K_{3,5}; \mathbb{Z}) \rightarrow C_0(K_{3,5}; \mathbb{Z})$. One of its rows is indexed by the 0-face $[2 \bmod 5]$ and this row has exactly three nonzero entries, all equal to $(-1)^0 = +1$. To see these signs, we rewrite $[2 \bmod 5]$ in three ways, all of which involve deleting the first entry out of two in an oriented 1-face:

$$[2 \bmod 5] = [0 \widehat{\bmod} 3, 2 \bmod 5] = [1 \widehat{\bmod} 3, 2 \bmod 5] = [2 \widehat{\bmod} 3, 2 \bmod 5].$$

The columns corresponding to these three 1-faces are indexed by the roots of unity ζ^{12} , ζ^7 , and ζ^2 , respectively. Summing these up with coefficients of positive one, we get

$$1 \cdot \zeta^{12} + 1 \cdot \zeta^7 + 1 \cdot \zeta^2 = \zeta^2(\zeta^{10} + \zeta^5 + 1),$$

which is the sum of ζ^j over j lying in a coset of $5\mathbb{Z}/15\mathbb{Z}$, and hence is zero.

Definition 18 Assume that n is squarefree and let T denote any set of $n - \phi(n)$ columns of the boundary map (5). Identify the complementary set T^c of $\phi(n)$ columns with a subset of the n^{th} roots-of-unity μ_n . Create a subcomplex of K_{p_1, \dots, p_d} by including its entire $(d - 2)$ -skeleton and attaching the subset of $(d - 1)$ -faces indexed by T . We denote this subcomplex as $K[T]$.

With this definition in mind, we will make use of an interesting feature of this labelling of the boundary map and the set P_n of primitive n^{th} roots of unity, noted already in [18, Remark 5]. For this next result, we let P_n^c denote the $(n - \phi(n))$ -element subset of μ_n indexed by the n^{th} roots of unity which are not primitive.

Proposition 19 Let n be a squarefree integer and P_n^c be as above. Then the subcomplex $K[P_n^c]$ of K_{p_1, \dots, p_d} is contractible.

Proof: Observe that the primitive roots in $\mathbb{Z}/n\mathbb{Z}$ are exactly those elements which do not vanish modulo p_i for $i = 1, \dots, d$. Tracing through the labelling of the $(d - 1)$ -faces via Ξ , we obtain the description

$$K[P_n^c] = \bigcup_{i=0}^d \text{star}_{K_{p_1, \dots, p_d}}(0 \bmod p_i),$$

where $\text{star}_{\Delta}(v)$ denotes the *simplicial star* of the vertex v inside a simplicial complex Δ . Furthermore, each intersection of these stars is nonempty and contractible, because it is the star of another face: for $I \subset [d]$,

$$\bigcap_{i \in I} \text{star}_{K_{p_1, \dots, p_d}}(0 \bmod p_i) = \text{star}_{K_{p_1, \dots, p_d}}(\{0 \bmod p_i\}_{i \in I}).$$

A standard nerve lemma [4, Theorem 10.6] then shows that $K[P_n^c]$ itself is contractible. □

Theorem 20 Let n be a squarefree integer and T be a subset of μ_n of size $n - \phi(n)$. Let $K[T]$ be the subcomplex of K_{p_1, \dots, p_d} of Definition 18. Then

$$\tilde{H}_i(K[T]; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}[\zeta]/\mathbb{Z}T^c & \text{if } i = d - 2, \\ \mathbb{Z} & \text{if both } i = d - 1 \text{ and } \text{rank}_{\mathbb{Z}}(\mathbb{Z}T^c) < \phi(n), \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathbb{Z}T^c$ is the sublattice \mathbb{Z} -spanned by the roots-of-unity $T^c \subset \mu_n$.

Proof: Choose any \mathbb{Z} -basis for $\mathbb{Z}[\zeta]$. Let M in $\mathbb{Z}^{\phi(n) \times n}$ be the matrix that expresses the n^{th} roots of unity μ_n in this basis. We construct a particular matrix M^\perp to accompany M as in Proposition 7 part (ii). Consider the collection S of all $(d - 2)$ -faces in the complete d -partite complex K_{p_1, \dots, p_d} . The complex $\langle S \rangle$ generated by S is therefore the $(d - 2)$ -skeleton of K_{p_1, \dots, p_d} . Proposition 15 implies that $\langle S \rangle$ is shellable, and that it has $\text{rank}_{\mathbb{Z}} \tilde{H}_{d-2}(\langle S \rangle; \mathbb{Z}) = n - \phi(n)$. Therefore, we are in the situation of Example 13, implying that there exists a torsion-free S -spanning tree R , and any such R will have $|S \setminus R| = n - \phi(n)$.

Our candidate for the matrix M^\perp in $\mathbb{Z}^{(n-\phi(n)) \times n}$ is the restriction of the boundary map from (5) to its rows indexed by $S \setminus R$. Proposition 16 shows that the rows of M^\perp are all perpendicular to the rows of M . Now choose T, T^c so that T^c indexes the set P_n of primitive n^{th} roots of unity. Proposition 6 implies that the maximal minor $M|_{T^c}$ of M is invertible over \mathbb{Z} , while Proposition 19 implies that the maximal minor $M^\perp|_T$ of M^\perp is invertible over \mathbb{Z} . Thus M, M^\perp satisfy the hypotheses of Proposition 7 part (ii), and combining this with Proposition 14 gives the assertion of the theorem. \square

6 Proof of Theorems 1 and 2

We are now in a position to prove Theorems 1 and 2.

Proof of Theorem 1: Let $T^c = \{1, \zeta, \zeta^2, \dots, \zeta^{\phi(n)}\} \setminus \{\zeta^j\}$ so that we have the equality of complexes $K[T] = K[\{\zeta^{\phi(n)+1}, \zeta^{\phi(n)+2}, \dots, \zeta^{n-1}\} \cup \{j\}] = K_j$. The theorem then follows from Theorem 20 and Corollary 5. \square

Proof of Theorem 2: We prove Theorem 2 by applying Proposition 8 to the matrices M, M^\perp in the proof of Theorem 20, with $A = \{1, \zeta, \zeta^2, \dots, \zeta^{\phi(n)}\}$. The dependence (2) among the columns of $M|_A$ has the same coefficients (up to scaling) as the cyclotomic polynomial, and the dependence (3) among the columns of $M^\perp|_{A^c \cup \{j, j'\}}$ has the same coefficients (up to scaling) as a nonzero cycle $z = \sum_{\ell} b_{\ell} [F_{\ell}]$ in $\tilde{H}_{d-1}(K_{\{j, j'\}}; \mathbb{Z})$. \square

7 Concordance with known properties of $\Phi_n(x)$

Here are some results about $\Phi_n(x)$ that manifest themselves topologically. See [22] for details.

1. The two maps $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ that send \bar{j} to $-\bar{j}$ and send \bar{j} to $\bar{j} + 1$ generate a dihedral group of simplicial automorphisms of K_{p_1, p_2, \dots, p_d} . One such automorphism sends the subcomplex $K_{\{j\}}$ to $K_{\{\phi(n)-j\}}$, and the subcomplex $K_{\{j, \phi(n)\}}$ to $K_{\{0, \phi(n)-j\}}$, explaining the symmetry $c_j = c_{\phi(n)-j}$ in $\Phi_n(x)$.
2. The fact that $\Phi_{2n}(x) = \Phi_n(-x)$ when n is odd manifests itself topologically as follows: the subcomplex $K_{\{j\}}$ whose homology interprets the coefficient of x^j for $\Phi_{2n}(x)$ is homotopy-equivalent to the suspension of the corresponding complex for $\Phi_n(x)$. Furthermore, there is a similar suspension relation between the complexes that predict the coefficients' signs.

3. When $d = 2$ so $n = p_1 p_2$ is the product of only two primes, all the subcomplexes $K_{\{j\}}$ of K_{p_1, p_2} are graphs. Hence their $(d - 2)$ -dimensional homology is torsion-free. It follows that the only nonzero coefficients of $\Phi_n(x)$ are ± 1 , agreeing with a well-known old observation of Migotti [20]. The explicit expansion of $\Phi_{p_1 p_2}(x)$ is given in Elder [10], Lam and Leung [16], and Lenstra [17].
4. In contrast to above, when $d \geq 3$ and the p_i 's are odd primes, $\Phi_n(x)$ often has coefficients with absolute value ≥ 2 . For example, $\Phi_{105}(x)$ has coefficient -2 on x^7 and x^{41} . The 2-dimensional subcomplexes $K_{\{7\}}$ and $K_{\{41\}}$, whose 1-homology equals $\mathbb{Z}/2\mathbb{Z}$, turn out to be surprisingly non-trivial. For example, neither one can be collapsed down to a real projective plane.

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References

- [1] R.M. Adin, Counting colorful multi-dimensional trees. *Combinatorica* **12** (1992), 247–260.
- [2] G. Bachman, On the Coefficients of Cyclotomic Polynomials *Memoirs of the American Mathematical Society* **106**, 1993.
- [3] A. Björner, Shellable and Cohen-Macaulay partially ordered sets. *Trans. Amer. Math. Soc.* **260** (1980), no. 1, 159–183.
- [4] A. Björner, Topological methods. *Handbook of combinatorics, Vol. II*, 1819–1872. Elsevier, Amsterdam, 1995
- [5] A. Björner and M. Wachs, Shellable nonpure complexes and posets. II. *Trans. Amer. Math. Soc.* **349** (1997), no. 10, 3945–3975.
- [6] E.D. Bolker, Simplicial geometry and transportation polytopes. *Trans. Amer. Math. Soc.* **217** (1976), 121–142.
- [7] H.H. Crapo and G.-C. Rota, On the foundations of combinatorial theory: Combinatorial geometries. Preliminary edition. The M.I.T. Press, Cambridge, Mass.-London, 1970.
- [8] A.M. Duval, C.J. Klivans, and J.L. Martin, Simplicial matrix-tree theorems. *Trans. Amer. Math. Soc.* **361** (2009), no. 11, 6073–6114.
- [9] A.M. Duval, C.J. Klivans, and J.L. Martin, Critical groups of simplicial complexes. preprint, arXiv:1101.3981
- [10] S. Elder, Flat Cyclotomic Polynomials, Colorado Math Circle Talk, January 2, 2010, http://www.coloradomath.org/handouts/flat_cyclotomic.pdf

- [11] Y. Gallot and P. Moree, Neighboring ternary cyclotomic coefficients differ by at most one. *J. Ramanujan Math. Soc.* **24** (2009), no. 3, 235248.
- [12] K. Johnsen, Lineare Abhängigkeiten von Einheitswurzeln. *Elem. Math.* **40** (1985), 57–59.
- [13] G. Kalai, Enumeration of \mathbb{Q} -acyclic simplicial complexes. *Israel J. Math.* **45** (1983), 337–351.
- [14] N. Kaplan, Flat cyclotomic polynomials of order three. *J. Number Theory* **127** (2007), no. 1, 118–126.
- [15] G. S. Kazandzidis, On the cyclotomic polynomial: Coefficients. *Bull. Soc. Math. Grèce (N.S.)* **4** (1963), no. 1, 1–11.
- [16] T.Y. Lam and K.H. Leung, On the Cyclotomic Polynomial $\Phi_{pq}(X)$. *Amer. Math. Monthly* **103** (1996), 562–564.
- [17] H.W. Lenstra, Vanishing sums of roots of unity, in: Proceedings, Bicentennial Congress Wiskundig Genootschap, Vrije Univ., Amsterdam, 1978, Part II, 1979, pp. 249–268.
- [18] J. Martin and V. Reiner, Cyclotomic and simplicial matroids. *Israel J. Math.* **150** (2005), 229–240.
- [19] M. Maxwell, Enumerating bases of self-dual matroids. *J. Combin. Theory Ser. A* **116** (2009), no. 2, 351–378.
- [20] A. Migotti, Zur Theorie der Kreisteilungsgleichung. *Sitzber. Math.-Naturwiss. Classe der Kaiser. Akad. der Wiss., Wien* **87** (1883), 7–14.
- [21] J.R. Munkres, Elements of algebraic topology. Addison-Wesley Publishing Company, Menlo Park, CA, 1984.
- [22] G. Musiker and V. Reiner, The cyclotomic polynomial topologically. preprint, [arXiv:1012.1844](https://arxiv.org/abs/1012.1844)
- [23] J.G. Oxley, Matroid theory. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992.
- [24] J.S. Provan and L.J. Billera, Decompositions of simplicial complexes related to diameters of convex polyhedra. *Math. Oper. Res.* **5** (1980), no. 4, 576–594.
- [25] D.J.A. Welsh, Matroid theory. London Math. Soc. Monographs **8**. Academic Press, London-New York, 1976.
- [26] N. White, Theory of matroids. *Encyclopedia of Mathematics and its Applications* **26**. Cambridge University Press, Cambridge, 1986.
- [27] N. White, Combinatorial geometries. *Encyclopedia of Mathematics and its Applications* **29**. Cambridge University Press, Cambridge, 1987.
- [28] N. White, Matroid applications. *Encyclopedia of Mathematics and its Applications* **40**. Cambridge University Press, Cambridge, 1992.
- [29] Zhao, Jia; Zhang, Xianke, Coefficients of ternary cyclotomic polynomials. *J. Number Theory* **130** (2010), no. 10, 2223–2237.