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Abstract. Any continuous map of an N-dimensional simplex Δ_N with colored vertices to a d-dimensional manifold M must map r points from disjoint rainbow faces of Δ_N to the same point in M, assuming that $N \geq (r-1)(d+1)$, no r vertices of Δ_N get the same color, and our proof needs that r is a prime. A face of Δ_N is called a *rainbow face* if all vertices have different colors.

This result is an extension of our recent "new colored Tverberg theorem", the special case of $M = \mathbb{R}^d$. It is also a generalization of Volovikov's 1996 topological Tverberg theorem for maps to manifolds, which arises when all color classes have size 1 (i.e., without color constraints); for this special case Volovikov's proofs, as well as ours, work when r is a prime power.

Résumé. Étant donné un simplex Δ_N de dimension N ayant les sommets colorés, une face de Δ_N est dite *arc-en-ciel*, si tous les sommets de cette face ont des couleurs différentes. Toute fonction continue d'un simplex Δ_N de dimension N aux sommets colorés vers une variété d-dimensionnelle M doit envoyer r points provenant de faces arc-en-ciel disjointes de Δ_N au mêmes points dans M; en supposant que $N \geq (r-1)(d+1)$, un ensemble de r sommets de Δ_N doit être coloré à l'aide d'au moins deux couleurs. Notre démonstration requiert que r soit un nombre premier.

Ce résultat est une extension de notre "nouveau théorème de Tverberg coloré", le cas particulier où $M = \mathbb{R}^d$. Il est également une généralisation du théorème de Tverberg topologique de Volovikov datant de 1996, pour les fonctions vers une variété, dont les classes de couleurs sont de taille 1 (c'est-à-dire sans contraintes de couleur). Dans ce cas particulier, la démonstration de Volovikov et la nôtre fonctionnent lorsque r est une puissance d'un premier.

Keywords: equivariant algebraic topology, convex geometry, colored Tverberg problem, configuration space/test map scheme, group cohomology

†The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement no. 247029-SDModels. Also supported by the grant ON 174008 of the Serbian Ministry of Science and Environment. `pavleb@mi.sanu.ac.rs`

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§The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement no. 247029-SDModels. `ziegler@math.fu-berlin.de`

1 Introduction

More than 50 years ago, the Cambridge undergraduate Bryan Birch [5] showed that "$3N$ points in a plane" can be split into N triples that span triangles with a non-empty intersection. He also conjectured a sharp, higher-dimensional version of this, which was proved by Helge Tverberg [15] in 1964.

In a 1988 Computational Geometry paper [2], Bárány, Füredi & Lovász noted that they needed a "colored version of Tverberg's theorem". Soon after this Bárány & Larman [3] proved such a theorem for rN colored points in a plane where the number of overlapping faces r is 2 or 3. Moreover, they conjectured a general version for any higher dimension d and any number of overlaps $r \geq 2$, offering a proof by Lovász for the case $r = 2$ and any dimension d. A 1992 paper [17] by Živaljević & Vrećica obtained this in a slightly weaker version, though not with a tight bound on the number of points. The proof relied on equivariant topology and beautiful combinatorics of "chessboard complexes".

Recently we proposed a new "colored Tverberg theorem", which is tight, generalizes Tverberg's original theorem in the case of primes and gives the best known answers for the Bárány–Larman conjecture.

Theorem 1.1 (Tight colored Tverberg theorem [7]) *For $d \geq 1$ and a prime $r \geq 2$, set $N := (d+1)(r-1)$, and let the $N+1$ vertices of an N-dimensional simplex Δ_N be colored such that all color classes are of size at most $r-1$.*

Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$ there are r disjoint faces $F_1, \ldots, F_r$ of Δ_N such that the vertices of each face F_i have all different colors and the images under f have a point in common: $f(F_1) \cap \ldots \cap f(F_r) \neq \emptyset$.

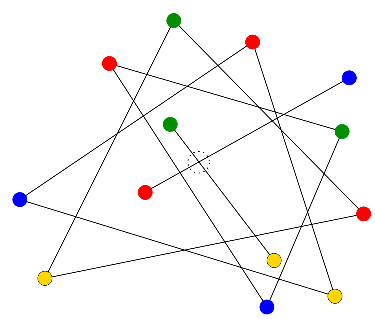

Fig. 1: Example of Theorem 1.1 for $d = 2$, $r = 5$, $N + 1 = 13$.

Here a *coloring* of the vertices of the simplex Δ_N is a partition of the vertex set into color classes, $C_1 \uplus \ldots \uplus C_m$. The condition $|C_i| \leq r - 1$ implies that there are at least $d + 2$ different color classes. In the following, a face whose all vertices have different colors, $|F_j \cap C_i| \leq i$ for all 1, will be called a *rainbow face*. Figure 1 shows an example for Theorem 1.1.

Theorem 1.1 is tight in the sense that it fails for maps of a simplex of smaller dimension, or if r vertices have the same color. It implies an optimal result for the Bárány–Larman conjecture in the case where $r+1$ is a prime, and an asymptotically-optimal bound in general; see [7, Corollaries 2.4, 2.5]. The special case where all vertices of Δ_N have different colors, $|C_i| = 1$, is the prime case of the topological Tverberg theorem, as proved by Bárány, Shlosman & Szűcs [4].

In this talk we present an extension of Theorem 1.1 that treats continuous maps $R \to M$ from the a subcomplex R of the N-simplex to an arbitrary d-dimensional manifold M with boundary in place of $\mathbb{R}^d$. Here, R is the *rainbow subcomplex* Δ_N, which consists of all rainbow faces.

Theorem 1.2 (Tight colored Tverberg theorem for M) *For $d \geq 1$ and a prime $r \geq 2$, set $N := (d+1)(r-1)$, and let the $N+1$ vertices of an N-dimensional simplex Δ_N be colored such that all color classes are of size at most $r-1$. Let R be the corresponding rainbow subcomplex.*

Then for every continuous map $f : R \to M$ to a d-dimensional manifold, the rainbow subcomplex R has r disjoint rainbow faces whose images under f have a point in common.

Theorem 1.2 without color constraints (that is, when all color classes are of size 1, and thus all faces are rainbow faces and $R = \Delta_N$) was previously obtained by Volovikov [16], using different methods. His proof (as well as ours in the case without color constraints) works for prime powers r.

An extension of Theorem 1.2 to a prime power that is not a prime seems out of reach at this point, even in the case $M = \mathbb{R}^d$. Similarly, for the case when r is not a prime power there currently does not seem to be a viable approach to the case without color constraints, even for $M = \mathbb{R}^d$. This is the remaining open case of the topological Tverberg conjecture [4].

Finally we remark that the restriction of the domain to a proper subcomplex of Δ_N, as given by Theorem 1.2, appears to be a non-trivial strengthening, even though any partition can use only faces in $R \subset \Delta_N$ of dimension at most $N - r + 1$. Let us give an example to illustrate that. Let $d = r = 2$ and let M be the 2-dimensional sphere. Then $N = 3$ and we give the vertices of the tetrahedra Δ_N all different colors. Since the N-dimensional face of Δ_N is never part of a Tverberg partition, we might guess that the conclusion of Theorem 1.2 should hold true also for any map $f : \partial\Delta_3 \to M$. However this is wrong: any homeomorphism f gives a counter-example!

2 Proof

In this extended abstract we only consider the case when f extends to a map $\Delta^N \to M$ on the whole simplex. If the given number of colors used to color the vertices is at least $d + 3 + \left\lfloor \frac{d}{r-1} \right\rfloor$ then the same proof will also work for non-extendable maps $f : R \to M$. Our proof of the general case of Theorem 1.2 needs some additional machinery due to Volovikov [16].

We prove Theorem 1.2 in this case in two steps:

- First, a geometric reduction lemma implies that it suffices to consider only manifolds M that are of the form $M = \widetilde{M} \times I^g$, where $I = [0, 1]$ and $\widetilde{M}$ is another manifold. More precisely we will need for the second step that
$$(r-1)\dim(M) > r \cdot \text{cohdim}(M), \tag{1}$$
where cohdim(M) is the cohomology dimension of M. This is done in Section 2.1.
- In the second step, we can assume (1) and prove Theorem 1.2 for maps $\Delta_N \to \widetilde{M}$ via the configuration space/test map scheme and Fadell–Husseini index theory, see Sections 2.2 and 2.4. The basic idea is the following: Assuming that Theorem 1.2 has a counter-example, construct an equivariant map from it. Then we show using equivariant topology that such a map cannot exist.

In the second step we rely on the computation of the Fadell–Husseini index of joins of chessboard complexes that we obtained in [8, Corollary 2.6].

2.1 A geometric reduction lemma

In the proof of Theorem 1.2 we may assume that M satisfy the above inequality (1) by using the following reduction lemma repeatedly.

Lemma 2.1 *Theorem 1.2 for parameters (d, r, M, f) can be derived from the case with parameters $(d', r', M', f') = (d + 1, r, M \times I, f')$, where the continuous map f' is defined in the following.*

Proof: Suppose we have to prove the theorem for the parameters (d, r, M, f). Let $d' = d + 1$, $r' = r$, and $M' = M \times I$. Then $N' := (d' + 1)(r - 1) = N + r - 1$. Let $v_0, \ldots, v_N, v_{N+1}, \ldots, v_{N'}$ denote the vertices of $\Delta_{N'}$. We regard Δ_N as the front face of $\Delta_{N'}$ with vertices $v_0, \ldots, v_N$. We give the new vertices $v_{N+1}, \ldots, v_{N'}$ a new color. Define a new map $f' : \Delta_{N'} \to M'$ by

$$\lambda_0 v_0 + \ldots + \lambda_{N'} v_{N'} \longmapsto \left(f(\lambda_0 v_0 + \ldots + \lambda_{N-1} v_{N-1} + (\lambda_N + \ldots + \lambda_{N'}) v_n), \lambda_{N+1} + \ldots + \lambda_{N'}\right).$$

Suppose we can show Theorem 1.2 for the parameters (d', r', M', f'). That is, we found a Tverberg partition $F'_1, \ldots, F'_r$ for these parameters. Put $F_i := F'_i \cap \Delta_N$. Since f' maps the front face Δ_N to $M \times \{0\}$ and since $\Delta_{N'}$ has only $r - 1 < r$ vertices more than Δ_N, already the F_i will intersect in $M \times \{0\}$. Hence the r faces $F_1, \ldots, F_r$ form a solution for the original parameters (d, r, M, f). This reduction is sketched in Figure 2. □

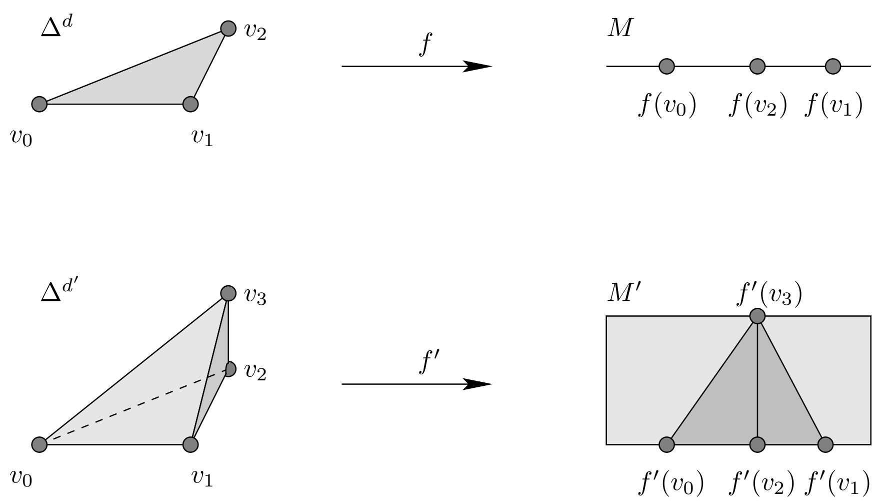

Fig. 2: Exemplary reduction in the case $d = 1$, $r = 2$, $N = 2$.

If the reduction lemma is applied $g = 1 + \left\lfloor \frac{d}{r-1} \right\rfloor$ times, the problem is reduced from the arbitrary parameters (d, r, M, f) to parameters (d'', r'', M'', f'') where $M'' = M \times I^g$. Thus M'' has vanishing cohomology in its g top dimensions. Therefore $(r - 1)\dim(M'') > r \cdot \mathrm{cohdim}(M'')$.

Having this reduction in mind, in what follows we may simply assume that the manifold M already satisfies inequality (1).

2.2 The configuration space/test map scheme

Now we reduce Theorem 1.2 to a problem in equivariant topology. Suppose we are given a continuous map

$$f : \Delta_N \longrightarrow M,$$

and a coloring of the vertex set vert$(\Delta_N) = [N+1] = C_0 \uplus \ldots \uplus C_m$ such that the color classes C_i are of size $|C_i| \leq r-1$. We want to find a colored Tverberg partition, that is, pairwise disjoint rainbow faces $F_1, \ldots, F_r$ of Δ_N, $|F_j \cap C_i| \leq 1$, whose images under f intersect.

The test map F is constructed using f in the following way. Let $f^{*r} : (\Delta_N)^{*r} \longrightarrow_{\mathbb{Z}_r} M^{*r}$ be the r-fold join of f. Since we are interested in pairwise disjoint faces $F_1, \ldots, F_r$, we restrict the domain of f^{*r} to the simplicial r-fold 2-wise deleted join of Δ_N, $(\Delta_N)^{*r}_{\Delta(2)} = [r]^{*(N+1)}$. This is the subcomplex of $(\Delta_N)^{*r}$ consisting of all joins $F_1 * \ldots * F_r$ of pairwise disjoint faces. (See [13, Chapter 5.5] for an introduction to these notions.) Since we are interested in colored faces F_j, we restrict the domain further to the subcomplex

$$R^{*r}_{\Delta(2)} = (C_0 * \ldots * C_m)^{*r}_{\Delta(2)} = [r]^{*|C_0|}_{\Delta(2)} * \ldots * [r]^{*|C_m|}_{\Delta(2)}.$$

This is the subcomplex of $(\Delta_N)^{*r}$ consisting of all joins $F_1 * \ldots * F_r$ of pairwise disjoint rainbow faces. The space $[r]^{*k}_{\Delta(2)}$ is known as the *chessboard complex* $\Delta_{r,k}$ [13, p. 163]. We write

$$K := (\Delta_{r,|C_0|}) * \ldots * (\Delta_{r,|C_m|}). \tag{2}$$

Hence we get a *test map*

$$F' : K \longrightarrow_{\mathbb{Z}_r} M^{*r}.$$

Let $T_{M^{*r}} := \{\sum_{i=1}^{r} \frac{1}{r} \cdot x \ : \ x \in M\}$ be the thin diagonal of M^{*r}. Its complement $M^{*r} \backslash T_{M^{*r}}$ is called the topological r-fold r-wise deleted join of M and it is denoted by $M^{*r}_{\Delta(r)}$.

The preimages $(F')^{-1}(T_{M^{*r}})$ of the thin diagonal correspond exactly to the colored Tverberg partitions. Hence the image of F' intersects the diagonal if and only if f admits a colored Tverberg partition.

Suppose that f admits *no* colored Tverberg partition, then the test map F' induces a $\mathbb{Z}_r$-equivariant map that avoids $T_{M^{*r}}$, that is,

$$F : K \longrightarrow_{\mathbb{Z}_r} M^{*r}_{\Delta(r)}. \tag{3}$$

We will derive a contradiction to the existence of such an equivariant map using the Fadell–Husseini index theory.

2.3 The Fadell–Husseini index

In this section we review equivariant cohomology of G-spaces via the Borel construction. This will provide the right tool to prove the non-existence of the test-map (3). We refer the reader to [1, Chap. V] and [10, Chap. III] for more details.

In the following H^* denotes singular or Čech cohomology with $\mathbb{F}_r$-coefficients, where r is a prime. Let G a finite group and let EG be a contractible free G-CW complex, for example the infinite join $G*G*\cdots$, suitably topologized. The quotient $BG := EG/G$ is called the *classifying space of* G. To every G-space X we can associate the *Borel construction* $EG \times_G X := (EG \times X)/G$, which is the total space of the fibration $X \hookrightarrow EG \times_G X \xrightarrow{pr_1} BG$.

The *equivariant cohomology* of a G-space X is defined as the ordinary cohomology of the Borel construction,

$$H^*_G(X) \ := \ H^*(EG \times_G X).$$

If X is a G-space, we define the *cohomological index* of X, also called the *Fadell–Husseini index* [11], [12], to be the kernel of the map in cohomology induced by the projection from X to a point,

$$\mathrm{Ind}_G(X) \ := \ \ker \left(H^*_G(\mathrm{pt}) \xrightarrow{p^*} H^*_G(X) \right) \ \subseteq \ H^*_G(\mathrm{pt}).$$

The cohomological index is monotone in the sense that if there is a G-map $X \longrightarrow_G Y$ then

$$\mathrm{Ind}_G(X) \supseteq \mathrm{Ind}_G(Y). \tag{4}$$

If r is odd then the cohomology of $\mathbb{Z}_r$ with $\mathbb{F}_r$-coefficients as an $\mathbb{F}_r$-algebra is

$$H^*(\mathbb{Z}_r) = H^*(B\mathbb{Z}_r) \cong \mathbb{F}_r[x,y]/(y^2),$$

where $\deg(x) = 2$ and $\deg(y) = 1$. If $r = 2$ then $H^*(\mathbb{Z}_r) \cong \mathbb{F}_2[t]$, $\deg t = 1$.

The index of the configuration space K, defined in (2), was computed in [8, Corollary 2.6]:

Theorem 2.2 $\mathrm{Ind}_{\mathbb{Z}_r}(K) = H^{*\geq N+1}(B\mathbb{Z}_r)$.

Therefore in the proof of Theorem 1.2 it remains to show that $\mathrm{Ind}_{\mathbb{Z}_r}(M^{*r}_{\Delta(r)})$ contains a non-zero element in dimension less or equal to N. Indeed, the monotonicity of the index (4) then implies the non-existence of a test map (3), which in turn implies the existence of a colored Tverberg partition.

Let us remark that the index of K becomes larger with respect to inclusion than in Theorem 2.2 if just one color class C_i has more than $r-1$ elements. That is, in this case our proof of Theorem 1.2 does not work anymore. In fact, for any r and d there exist $N+1$ colored points in $\mathbb{R}^d$ such that one color class is of size r and all other color classes are singletons that admit no colored Tverberg partition.

2.4 The index of the deleted join of the manifold

In this section we prove that $\mathrm{Ind}_{\mathbb{Z}_r} M^{*r}_{\Delta(r)}$ contains a non-zero element in degree N. Together with Theorem 2.2 we deduce that $\mathrm{Ind}_{\mathbb{Z}_r} M^{*r}_{\Delta(r)}$ is not contained in $\mathrm{Ind}_{\mathbb{Z}_r}(K)$, hence by the monotonicity of the index, the test-map (3) does not exist, which finishes the proof.

We have inclusions

$$T_{M^{*r}} \ \hookrightarrow \ \left\{ \sum \lambda_i x \in M^{*r} \ : \ \lambda_i > 0, \sum \lambda_i = 1, x \in M \right\} \cong M \times \Delta^\circ_{r-1} \ \hookrightarrow \ M^{*r},$$

where Δ°_{r-1} denotes the open $(r-1)$-simplex. Since M is a smooth $\mathbb{Z}_r$-invariant manifold, $T_{M^{*r}}$ has a $\mathbb{Z}_r$-equivariant tubular neighborhood in M^{*r}; see [6, Section VI.2]. Its closure can be described as the disk bundle $D(\xi)$ of an equivariant vector bundle ξ over M. We denote its sphere bundle by $S(\xi)$. The fiber F of ξ is as a $\mathbb{Z}_r$-representation the $(d+1)$-fold sum of W_r, where $W_r = \{x \in \mathbb{R}[\mathbb{Z}_r] \ : \ x_1 + \ldots + x_r = 0\}$ is the augmentation ideal of $\mathbb{R}[\mathbb{Z}_r]$.

The representation sphere $S(F)$ is of dimension $N-1$. It is a free $\mathbb{Z}_r$-space, hence its index is

$$\mathrm{Ind}_{\mathbb{Z}_r}(S(F)) = H^{*\geq N}(B\mathbb{Z}_r). \tag{5}$$

This can be directly deduced from the Leray–Serre spectral sequence associated to the Borel construction $E\mathbb{Z}_r \times_{\mathbb{Z}_r} S(F) \to B\mathbb{Z}_r$, noting that the images of the differentials to the bottom row give precisely the index of $S(F)$. The latter can be seen from the edge-homomorphism. For background on Leray–Serre spectral sequences we refer to [14, Chapters 5, 6].

The Leray–Serre spectral sequence associated to the fibration $S(\xi) \to M$ collapses at E_2, since $N = (r-1)(d+1) \geq d+1$ and hence there is no differential between non-zero entries. Thus the map $i^* : H^{N-1}(S(\xi)) \to H^{N-1}(S(F))$ induced by inclusion is surjective.

The Mayer–Vietoris sequence associated to the triple $(D(\xi), M^{*r}_{\Delta(r)}, M^{*r})$ contains the subsequence

$$H^{N-1}(M^{*r}_{\Delta(r)}) \oplus H^{N-1}(D(\xi)) \xrightarrow{j^*+k^*} H^{N-1}(S(\xi)) \xrightarrow{\delta} H^N(M^{*r}).$$

We see that $H^N(M^{*r})$ is zero: This follows from the formula

$$\widetilde{H}^{*+(r-1)}(M^{*r}) \cong \widetilde{H}^*(M)^{\otimes r},$$

as long as $N-(r-1) > re$, where e is the cohomological dimension of M. This inequality is equivalent to $d > \frac{r}{r-1}e$, which can be assumed by applying the reduction from Section 2.1 at least $\lfloor 1 + \frac{e}{r-1} \rfloor$ times. Hence we can assume that $H^N(M^{*r}) = 0$.

Furthermore inequality (1) implies that $N-1 \geq d > \text{cohdim}(M)$. Hence the term $H^{N-1}(D(\xi)) = H^{N-1}(M)$ of the sequence is zero as well.

Thus the map $j^* : H^{N-1}(M^{*r}_{\Delta(r)}) \to H^{N-1}(S(\xi))$ is surjective. Therefore the composition $(j \circ i)^* : H^{N-1}(M^{*r}_{\Delta(r)}) \to H^{N-1}(S(F))$ is surjective as well. We apply the Borel construction functor $E\mathbb{Z}_r \times_{\mathbb{Z}_r} (\_) \to B\mathbb{Z}_r$ to this map and apply Leray–Serre spectral sequences; see Figure 3.

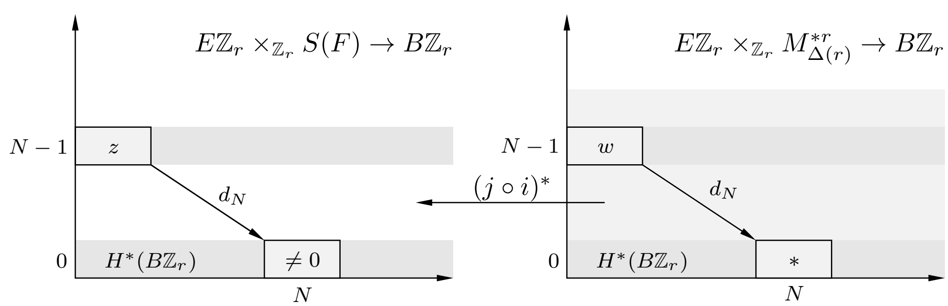

Fig. 3: We associate to the map $S(F) \xrightarrow{j \circ i} M^{*r}_{\Delta(r)}$ the Borel constructions and spectral sequences to deduce that $M^{*r}_{\Delta(r)}$ contains a non-zero element in dimension N.

At the E_2-pages, the generator z of $H^{N-1}(S(F))$ has a preimage w since $(j \circ i)^*$ is surjective. At the E_N-pages $(j \circ i)^*(d_N(w)) = d_N(z)$, which is non-zero by (5). Hence $d_N(w) \neq 0$, which is an element in the kernel of the edge-homomorphism $H^*(B\mathbb{Z}_r) \to H^*_{\mathbb{Z}_r}(M^*_{\Delta(r)})$.

Therefore, the index of $M^{*r}_{\Delta(r)}$ contains a non-zero element in dimension N. This completes the proof of Theorem 1.2 if f can be extended to Δ^N. $\square$

References

[1] A. Adem, J. Milgram. *Cohomology of Finite Groups*, Grundlehren der mathematischen Wissenschaften, Vol. 309, Springer 1994.

[2] I. Bárány, Z. Füredi, L. Lovász. *On the number of halving planes*, Combinatorica 10, 1990, 175-183.

[3] I. Bárány, D. G. Larman. *A colored version of Tverberg's theorem*, J. London Math. Soc. 45, 1992, 314-320.

[4] I. Bárány, S. B. Shlosman, A. Szűcs. *On a topological generalization of a theorem of Tverberg*, J. London Math. Soc. 23, 1981, 158-164.

[5] B. J. Birch. *On 3N points in the plane*, Math. Proc. Cambridge Phil. Soc. 55, 1959, 289-293.

[6] G. E. Bredon. *Introduction to Compact Transformation Groups*, New York, Academic Press, 1972.

[7] P. V. M. Blagojević, B. Matschke, G. M. Ziegler. *Optimal bounds for the colored Tverberg problem*, arxiv:0910.4987v2, 2009, 11 pages.

[8] P. V. M. Blagojević, B. Matschke, G. M. Ziegler. *Optimal bounds for a colorful Tverberg–Vrećica type problem*, Adv. Math. 226 (6), 2011, 5198-5215.

[9] P. V. M. Blagojević, B. Matschke, G. M. Ziegler. *A tight colored Tverberg theorem for maps to manifolds*, preprint, 2010, 7 pages.

[10] T. tom Dieck. *Transformation Groups*, de Gruyter Studies in Math., Vol. 8, Berlin, 1987.

[11] E. Fadell, S. Husseini. *Relative cohomological index theories*, Adv. Math. 64, 1987, 1-31.

[12] E. Fadell, S. Husseini. *An ideal-valued cohomological index theory with applications to Borsuk–Ulam and Bourgin–Yang theorems*, Ergodic Theory and Dynamical Systems 8^*, 1988, 73-85.

[13] J. Matoušek. *Using the Borsuk–Ulam Theorem. Lectures on Topological Methods in Combinatorics and Geometry*, Universitext, Springer 2003.

[14] J. McCleary. *A User's Guide to Spectral Sequences*, Cambridge University Press, second edition, 2001.

[15] H. Tverberg. *A generalization of Radon's theorem*, J. London Math. Soc. 41, 1966, 123-128.

[16] A. Yu. Volovikov. *On a topological generalization of the Tverberg theorem*, Math. Notes 59, No. 3, 1996, 324-326.

[17] S. Vrećica, R. Živaljević. *The colored Tverberg's problem and complex of injective functions*, J. Combin. Theory Ser. A, 61, 1992, 309-318.