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# *A tight colored Tverberg theorem for maps to manifolds (extended abstract)*

Pavle V. M. Blagojević<sup>1†</sup>, Benjamin Matschke<sup>2‡</sup> and Günter M. Ziegler<sup>2§</sup>

<sup>1</sup>*Mathematički Institut SANU, Beograd, Serbia*

<sup>2</sup>*Institut für Mathematik und Informatik, FU Berlin, Berlin, Germany*

**Abstract.** Any continuous map of an  $N$ -dimensional simplex  $\Delta_N$  with colored vertices to a  $d$ -dimensional manifold  $M$  must map  $r$  points from disjoint rainbow faces of  $\Delta_N$  to the same point in  $M$ , assuming that  $N \geq (r-1)(d+1)$ , no  $r$  vertices of  $\Delta_N$  get the same color, and our proof needs that  $r$  is a prime. A face of  $\Delta_N$  is called a *rainbow face* if all vertices have different colors.

This result is an extension of our recent “new colored Tverberg theorem”, the special case of  $M = \mathbb{R}^d$ . It is also a generalization of Volovikov’s 1996 topological Tverberg theorem for maps to manifolds, which arises when all color classes have size 1 (i.e., without color constraints); for this special case Volovikov’s proofs, as well as ours, work when  $r$  is a prime power.

**Résumé.** Étant donné un simplexe  $\Delta_N$  de dimension  $N$  ayant les sommets colorés, une face de  $\Delta_N$  est dite *arc-en-ciel*, si tous les sommets de cette face ont des couleurs différentes. Toute fonction continue d’un simplexe  $\Delta_N$  de dimension  $N$  aux sommets colorés vers une variété  $d$ -dimensionnelle  $M$  doit envoyer  $r$  points provenant de faces arc-en-ciel disjointes de  $\Delta_N$  au mêmes points dans  $M$ ; en supposant que  $N \geq (r-1)(d+1)$ , un ensemble de  $r$  sommets de  $\Delta_N$  doit être coloré à l’aide d’au moins deux couleurs. Notre démonstration requiert que  $r$  soit un nombre premier.

Ce résultat est une extension de notre “nouveau théorème de Tverberg coloré”, le cas particulier où  $M = \mathbb{R}^d$ . Il est également une généralisation du théorème de Tverberg topologique de Volovikov datant de 1996, pour les fonctions vers une variété, dont les classes de couleurs sont de taille 1 (c’est-à-dire sans contraintes de couleur). Dans ce cas particulier, la démonstration de Volovikov et la nôtre fonctionnent lorsque  $r$  est une puissance d’un premier.

**Keywords:** equivariant algebraic topology, convex geometry, colored Tverberg problem, configuration space/test map scheme, group cohomology

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<sup>‡</sup>Supported by Deutsche Telekom Stiftung. [matschke@math.fu-berlin.de](mailto:matschke@math.fu-berlin.de)

<sup>§</sup>The research leading to these results has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement no. 247029-SDModels. [ziegler@math.fu-berlin.de](mailto:ziegler@math.fu-berlin.de)

## 1 Introduction

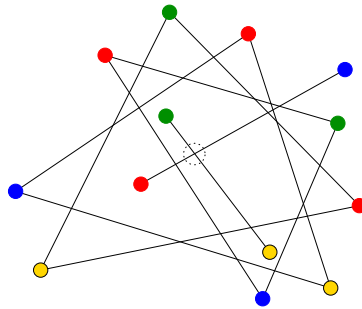
More than 50 years ago, the Cambridge undergraduate Bryan Birch [5] showed that “ $3N$  points in a plane” can be split into  $N$  triples that span triangles with a non-empty intersection. He also conjectured a sharp, higher-dimensional version of this, which was proved by Helge Tverberg [15] in 1964.

In a 1988 Computational Geometry paper [2], Bárány, Füredi & Lovász noted that they needed a “colored version of Tverberg’s theorem”. Soon after this Bárány & Larman [3] proved such a theorem for  $rN$  colored points in a plane where the number of overlapping faces  $r$  is 2 or 3. Moreover, they conjectured a general version for any higher dimension  $d$  and any number of overlaps  $r \geq 2$ , offering a proof by Lovász for the case  $r = 2$  and any dimension  $d$ . A 1992 paper [17] by Živaljević & Vrećica obtained this in a slightly weaker version, though not with a tight bound on the number of points. The proof relied on equivariant topology and beautiful combinatorics of “chessboard complexes”.

Recently we proposed a new “colored Tverberg theorem”, which is tight, generalizes Tverberg’s original theorem in the case of primes and gives the best known answers for the Bárány–Larman conjecture.

**Theorem 1.1 (Tight colored Tverberg theorem [7])** *For  $d \geq 1$  and a prime  $r \geq 2$ , set  $N := (d + 1)(r - 1)$ , and let the  $N + 1$  vertices of an  $N$ -dimensional simplex  $\Delta_N$  be colored such that all color classes are of size at most  $r - 1$ .*

*Then for every continuous map  $f : \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$  disjoint faces  $F_1, \dots, F_r$  of  $\Delta_N$  such that the vertices of each face  $F_i$  have all different colors and the images under  $f$  have a point in common:  $f(F_1) \cap \dots \cap f(F_r) \neq \emptyset$ .*



**Fig. 1:** Example of Theorem 1.1 for  $d = 2, r = 5, N + 1 = 13$ .

Here a *coloring* of the vertices of the simplex  $\Delta_N$  is a partition of the vertex set into color classes,  $C_1 \uplus \dots \uplus C_m$ . The condition  $|C_i| \leq r - 1$  implies that there are at least  $d + 2$  different color classes. In the following, a face whose all vertices have different colors,  $|F_j \cap C_i| \leq 1$  for all  $i$ , will be called a *rainbow face*. Figure 1 shows an example for Theorem 1.1.

Theorem 1.1 is tight in the sense that it fails for maps of a simplex of smaller dimension, or if  $r$  vertices have the same color. It implies an optimal result for the Bárány–Larman conjecture in the case where  $r + 1$  is a prime, and an asymptotically-optimal bound in general; see [7, Corollaries 2.4, 2.5]. The special case where all vertices of  $\Delta_N$  have different colors,  $|C_i| = 1$ , is the prime case of the topological Tverberg theorem, as proved by Bárány, Shlosman & Szűcs [4].

In this talk we present an extension of Theorem 1.1 that treats continuous maps  $R \rightarrow M$  from the a subcomplex  $R$  of the  $N$ -simplex to an arbitrary  $d$ -dimensional manifold  $M$  with boundary in place of  $\mathbb{R}^d$ . Here,  $R$  is the *rainbow subcomplex*  $\Delta_N$ , which consists of all rainbow faces.

**Theorem 1.2 (Tight colored Tverberg theorem for  $M$ )** For  $d \geq 1$  and a prime  $r \geq 2$ , set  $N := (d + 1)(r - 1)$ , and let the  $N + 1$  vertices of an  $N$ -dimensional simplex  $\Delta_N$  be colored such that all color classes are of size at most  $r - 1$ . Let  $R$  be the corresponding rainbow subcomplex.

Then for every continuous map  $f : R \rightarrow M$  to a  $d$ -dimensional manifold, the rainbow subcomplex  $R$  has  $r$  disjoint rainbow faces whose images under  $f$  have a point in common.

Theorem 1.2 without color constraints (that is, when all color classes are of size 1, and thus all faces are rainbow faces and  $R = \Delta_N$ ) was previously obtained by Volovikov [16], using different methods. His proof (as well as ours in the case without color constraints) works for prime powers  $r$ .

An extension of Theorem 1.2 to a prime power that is not a prime seems out of reach at this point, even in the case  $M = \mathbb{R}^d$ . Similarly, for the case when  $r$  is not a prime power there currently does not seem to be a viable approach to the case without color constraints, even for  $M = \mathbb{R}^d$ . This is the remaining open case of the topological Tverberg conjecture [4].

Finally we remark that the restriction of the domain to a proper subcomplex of  $\Delta_N$ , as given by Theorem 1.2, appears to be a non-trivial strengthening, even though any partition can use only faces in  $R \subset \Delta_N$  of dimension at most  $N - r + 1$ . Let us give an example to illustrate that. Let  $d = r = 2$  and let  $M$  be the 2-dimensional sphere. Then  $N = 3$  and we give the vertices of the tetrahedra  $\Delta_N$  all different colors. Since the  $N$ -dimensional face of  $\Delta_N$  is never part of a Tverberg partition, we might guess that the conclusion of Theorem 1.2 should hold true also for any map  $f : \partial\Delta_3 \rightarrow M$ . However this is wrong: any homeomorphism  $f$  gives a counter-example!

## 2 Proof

In this extended abstract we only consider the case when  $f$  extends to a map  $\Delta^N \rightarrow M$  on the whole simplex. If the given number of colors used to color the vertices is at least  $d + 3 + \lfloor \frac{d}{r-1} \rfloor$  then the same proof will also work for non-extendable maps  $f : R \rightarrow M$ . Our proof of the general case of Theorem 1.2 needs some additional machinery due to Volovikov [16].

We prove Theorem 1.2 in this case in two steps:

- First, a geometric reduction lemma implies that it suffices to consider only manifolds  $M$  that are of the form  $M = \widetilde{M} \times I^q$ , where  $I = [0, 1]$  and  $\widetilde{M}$  is another manifold. More precisely we will need for the second step that

$$(r - 1) \dim(M) > r \cdot \text{cohdim}(M), \tag{1}$$

where  $\text{cohdim}(M)$  is the cohomology dimension of  $M$ . This is done in Section 2.1.

- In the second step, we can assume (1) and prove Theorem 1.2 for maps  $\Delta_N \rightarrow \widetilde{M}$  via the configuration space/test map scheme and Fadell–Husseini index theory, see Sections 2.2 and 2.4. The basic idea is the following: Assuming that Theorem 1.2 has a counter-example, construct an equivariant map from it. Then we show using equivariant topology that such a map cannot exist.

In the second step we rely on the computation of the Fadell–Husseini index of joins of chessboard complexes that we obtained in [8, Corollary 2.6].

### 2.1 A geometric reduction lemma

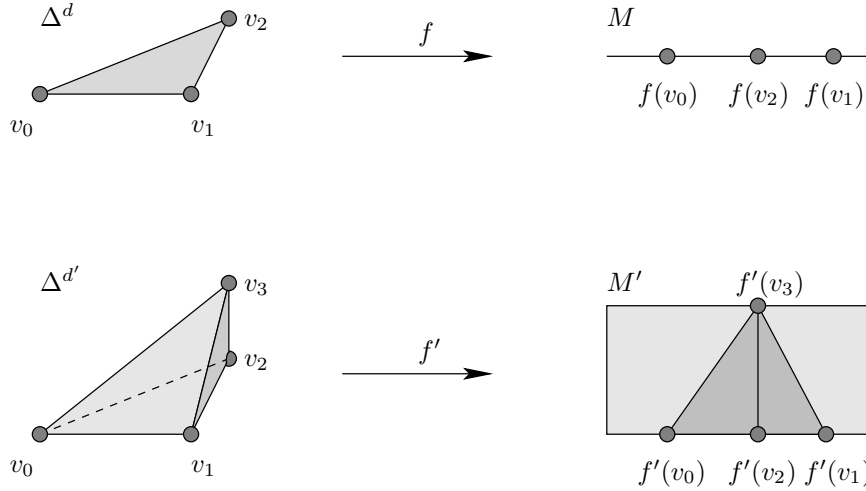
In the proof of Theorem 1.2 we may assume that  $M$  satisfy the above inequality (1) by using the following reduction lemma repeatedly.

**Lemma 2.1** *Theorem 1.2 for parameters  $(d, r, M, f)$  can be derived from the case with parameters  $(d', r', M', f') = (d + 1, r, M \times I, f')$ , where the continuous map  $f'$  is defined in the following.*

**Proof:** Suppose we have to prove the theorem for the parameters  $(d, r, M, f)$ . Let  $d' = d + 1$ ,  $r' = r$ , and  $M' = M \times I$ . Then  $N' := (d' + 1)(r - 1) = N + r - 1$ . Let  $v_0, \dots, v_N, v_{N+1}, \dots, v_{N'}$  denote the vertices of  $\Delta_{N'}$ . We regard  $\Delta_N$  as the front face of  $\Delta_{N'}$  with vertices  $v_0, \dots, v_N$ . We give the new vertices  $v_{N+1}, \dots, v_{N'}$  a new color. Define a new map  $f' : \Delta_{N'} \rightarrow M'$  by

$$\lambda_0 v_0 + \dots + \lambda_{N'} v_{N'} \mapsto (f(\lambda_0 v_0 + \dots + \lambda_{N-1} v_{N-1} + (\lambda_N + \dots + \lambda_{N'}) v_n), \lambda_{N+1} + \dots + \lambda_{N'}).$$

Suppose we can show Theorem 1.2 for the parameters  $(d', r', M', f')$ . That is, we found a Tverberg partition  $F'_1, \dots, F'_r$  for these parameters. Put  $F_i := F'_i \cap \Delta_N$ . Since  $f'$  maps the front face  $\Delta_N$  to  $M \times \{0\}$  and since  $\Delta_{N'}$  has only  $r - 1 < r$  vertices more than  $\Delta_N$ , already the  $F_i$  will intersect in  $M \times \{0\}$ . Hence the  $r$  faces  $F_1, \dots, F_r$  form a solution for the original parameters  $(d, r, M, f)$ . This reduction is sketched in Figure 2.  $\square$



**Fig. 2:** Exemplary reduction in the case  $d = 1, r = 2, N = 2$ .

If the reduction lemma is applied  $g = 1 + \lfloor \frac{d}{r-1} \rfloor$  times, the problem is reduced from the arbitrary parameters  $(d, r, M, f)$  to parameters  $(d'', r'', M'', f'')$  where  $M'' = M \times I^g$ . Thus  $M''$  has vanishing cohomology in its  $g$  top dimensions. Therefore  $(r - 1) \dim(M'') > r \cdot \text{cohdim}(M'')$ .

Having this reduction in mind, in what follows we may simply assume that the manifold  $M$  already satisfies inequality (1).

## 2.2 The configuration space/test map scheme

Now we reduce Theorem 1.2 to a problem in equivariant topology. Suppose we are given a continuous map

$$f : \Delta_N \longrightarrow M,$$

and a coloring of the vertex set  $\text{vert}(\Delta_N) = [N + 1] = C_0 \uplus \dots \uplus C_m$  such that the color classes  $C_i$  are of size  $|C_i| \leq r - 1$ . We want to find a colored Tverberg partition, that is, pairwise disjoint rainbow faces  $F_1, \dots, F_r$  of  $\Delta_N$ ,  $|F_j \cap C_i| \leq 1$ , whose images under  $f$  intersect.

The test map  $F$  is constructed using  $f$  in the following way. Let  $f^{*r} : (\Delta_N)^{*r} \longrightarrow_{\mathbb{Z}_r} M^{*r}$  be the  $r$ -fold join of  $f$ . Since we are interested in pairwise disjoint faces  $F_1, \dots, F_r$ , we restrict the domain of  $f^{*r}$  to the simplicial  $r$ -fold 2-wise deleted join of  $\Delta_N$ ,  $(\Delta_N)_{\Delta(2)}^{*r} = [r]_{\Delta(2)}^{*(N+1)}$ . This is the subcomplex of  $(\Delta_N)^{*r}$  consisting of all joins  $F_1 * \dots * F_r$  of pairwise disjoint faces. (See [13, Chapter 5.5] for an introduction to these notions.) Since we are interested in colored faces  $F_j$ , we restrict the domain further to the subcomplex

$$R_{\Delta(2)}^{*r} = (C_0 * \dots * C_m)_{\Delta(2)}^{*r} = [r]_{\Delta(2)}^{*|C_0|} * \dots * [r]_{\Delta(2)}^{*|C_m|}.$$

This is the subcomplex of  $(\Delta_N)^{*r}$  consisting of all joins  $F_1 * \dots * F_r$  of pairwise disjoint rainbow faces. The space  $[r]_{\Delta(2)}^{*k}$  is known as the *chessboard complex*  $\Delta_{r,k}$  [13, p. 163]. We write

$$K := (\Delta_{r,|C_0|}) * \dots * (\Delta_{r,|C_m|}). \quad (2)$$

Hence we get a *test map*

$$F' : K \longrightarrow_{\mathbb{Z}_r} M^{*r}.$$

Let  $T_{M^{*r}} := \{\sum_{i=1}^r \frac{1}{r} \cdot x : x \in M\}$  be the thin diagonal of  $M^{*r}$ . Its complement  $M^{*r} \setminus T_{M^{*r}}$  is called the topological  $r$ -fold  $r$ -wise deleted join of  $M$  and it is denoted by  $M_{\Delta(r)}^{*r}$ .

The preimages  $(F')^{-1}(T_{M^{*r}})$  of the thin diagonal correspond exactly to the colored Tverberg partitions. Hence the image of  $F'$  intersects the diagonal if and only if  $f$  admits a colored Tverberg partition.

Suppose that  $f$  admits *no* colored Tverberg partition, then the test map  $F'$  induces a  $\mathbb{Z}_r$ -equivariant map that avoids  $T_{M^{*r}}$ , that is,

$$F : K \longrightarrow_{\mathbb{Z}_r} M_{\Delta(r)}^{*r}. \quad (3)$$

We will derive a contradiction to the existence of such an equivariant map using the Fadell–Husseini index theory.

## 2.3 The Fadell–Husseini index

In this section we review equivariant cohomology of  $G$ -spaces via the Borel construction. This will provide the right tool to prove the non-existence of the test-map (3). We refer the reader to [1, Chap. V] and [10, Chap. III] for more details.

In the following  $H^*$  denotes singular or Čech cohomology with  $\mathbb{F}_r$ -coefficients, where  $r$  is a prime. Let  $G$  a finite group and let  $EG$  be a contractible free  $G$ -CW complex, for example the infinite join  $G * G * \dots$ , suitably topologized. The quotient  $BG := EG/G$  is called the *classifying space of  $G$* . To every  $G$ -space  $X$  we can associate the *Borel construction*  $EG \times_G X := (EG \times X)/G$ , which is the total space of the fibration  $X \hookrightarrow EG \times_G X \xrightarrow{pr_1} BG$ .

The *equivariant cohomology* of a  $G$ -space  $X$  is defined as the ordinary cohomology of the Borel construction,

$$H_G^*(X) := H^*(EG \times_G X).$$

If  $X$  is a  $G$ -space, we define the *cohomological index* of  $X$ , also called the *Fadell–Husseini index* [11], [12], to be the kernel of the map in cohomology induced by the projection from  $X$  to a point,

$$\text{Ind}_G(X) := \ker \left( H_G^*(\text{pt}) \xrightarrow{p^*} H_G^*(X) \right) \subseteq H_G^*(\text{pt}).$$

The cohomological index is monotone in the sense that if there is a  $G$ -map  $X \rightarrow_G Y$  then

$$\text{Ind}_G(X) \supseteq \text{Ind}_G(Y). \tag{4}$$

If  $r$  is odd then the cohomology of  $\mathbb{Z}_r$  with  $\mathbb{F}_r$ -coefficients as an  $\mathbb{F}_r$ -algebra is

$$H^*(\mathbb{Z}_r) = H^*(B\mathbb{Z}_r) \cong \mathbb{F}_r[x, y]/(y^2),$$

where  $\deg(x) = 2$  and  $\deg(y) = 1$ . If  $r = 2$  then  $H^*(\mathbb{Z}_r) \cong \mathbb{F}_2[t]$ ,  $\deg t = 1$ .

The index of the configuration space  $K$ , defined in (2), was computed in [8, Corollary 2.6]:

**Theorem 2.2**  $\text{Ind}_{\mathbb{Z}_r}(K) = H^{*\geq N+1}(B\mathbb{Z}_r)$ .

Therefore in the proof of Theorem 1.2 it remains to show that  $\text{Ind}_{\mathbb{Z}_r}(M_{\Delta(r)}^{*r})$  contains a non-zero element in dimension less or equal to  $N$ . Indeed, the monotonicity of the index (4) then implies the non-existence of a test map (3), which in turn implies the existence of a colored Tverberg partition.

Let us remark that the index of  $K$  becomes larger with respect to inclusion than in Theorem 2.2 if just one color class  $C_i$  has more than  $r - 1$  elements. That is, in this case our proof of Theorem 1.2 does not work anymore. In fact, for any  $r$  and  $d$  there exist  $N + 1$  colored points in  $\mathbb{R}^d$  such that one color class is of size  $r$  and all other color classes are singletons that admit no colored Tverberg partition.

### 2.4 The index of the deleted join of the manifold

In this section we prove that  $\text{Ind}_{\mathbb{Z}_r} M_{\Delta(r)}^{*r}$  contains a non-zero element in degree  $N$ . Together with Theorem 2.2 we deduce that  $\text{Ind}_{\mathbb{Z}_r} M_{\Delta(r)}^{*r}$  is not contained in  $\text{Ind}_{\mathbb{Z}_r}(K)$ , hence by the monotonicity of the index, the test-map (3) does not exist, which finishes the proof.

We have inclusions

$$T_{M^{*r}} \hookrightarrow \left\{ \sum \lambda_i x \in M^{*r} : \lambda_i > 0, \sum \lambda_i = 1, x \in M \right\} \cong M \times \Delta_{r-1}^\circ \hookrightarrow M^{*r},$$

where  $\Delta_{r-1}^\circ$  denotes the open  $(r - 1)$ -simplex. Since  $M$  is a smooth  $\mathbb{Z}_r$ -invariant manifold,  $T_{M^{*r}}$  has a  $\mathbb{Z}_r$ -equivariant tubular neighborhood in  $M^{*r}$ ; see [6, Section VI.2]. Its closure can be described as the disk bundle  $D(\xi)$  of an equivariant vector bundle  $\xi$  over  $M$ . We denote its sphere bundle by  $S(\xi)$ . The fiber  $F$  of  $\xi$  is as a  $\mathbb{Z}_r$ -representation the  $(d + 1)$ -fold sum of  $W_r$ , where  $W_r = \{x \in \mathbb{R}[\mathbb{Z}_r] : x_1 + \dots + x_r = 0\}$  is the augmentation ideal of  $\mathbb{R}[\mathbb{Z}_r]$ .

The representation sphere  $S(F)$  is of dimension  $N - 1$ . It is a free  $\mathbb{Z}_r$ -space, hence its index is

$$\text{Ind}_{\mathbb{Z}_r}(S(F)) = H^{*\geq N}(B\mathbb{Z}_r). \tag{5}$$

This can be directly deduced from the Leray–Serre spectral sequence associated to the Borel construction  $E\mathbb{Z}_r \times_{\mathbb{Z}_r} S(F) \rightarrow B\mathbb{Z}_r$ , noting that the images of the differentials to the bottom row give precisely the index of  $S(F)$ . The latter can be seen from the edge-homomorphism. For background on Leray–Serre spectral sequences we refer to [14, Chapters 5, 6].

The Leray–Serre spectral sequence associated to the fibration  $S(\xi) \rightarrow M$  collapses at  $E_2$ , since  $N = (r - 1)(d + 1) \geq d + 1$  and hence there is no differential between non-zero entries. Thus the map  $i^* : H^{N-1}(S(\xi)) \rightarrow H^{N-1}(S(F))$  induced by inclusion is surjective.

The Mayer–Vietoris sequence associated to the triple  $(D(\xi), M_{\Delta(r)}^{*r}, M^{*r})$  contains the subsequence

$$H^{N-1}(M_{\Delta(r)}^{*r}) \oplus H^{N-1}(D(\xi)) \xrightarrow{j^*+k^*} H^{N-1}(S(\xi)) \xrightarrow{\delta} H^N(M^{*r}).$$

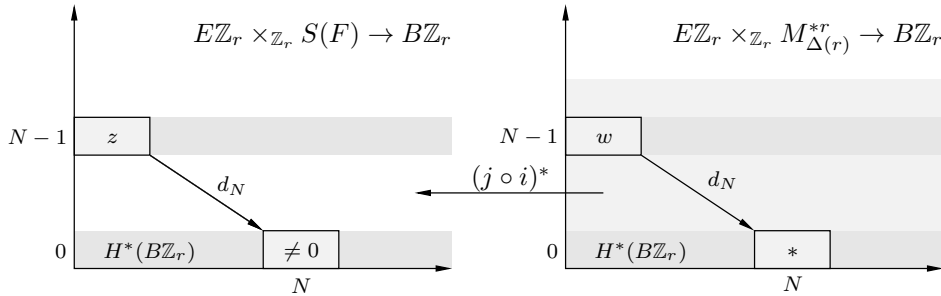
We see that  $H^N(M^{*r})$  is zero: This follows from the formula

$$\tilde{H}^{*+(r-1)}(M^{*r}) \cong \tilde{H}^*(M)^{\otimes r},$$

as long as  $N - (r - 1) > re$ , where  $e$  is the cohomological dimension of  $M$ . This inequality is equivalent to  $d > \frac{r}{r-1}e$ , which can be assumed by applying the reduction from Section 2.1 at least  $\lceil 1 + \frac{e}{r-1} \rceil$  times. Hence we can assume that  $H^N(M^{*r}) = 0$ .

Furthermore inequality (1) implies that  $N - 1 \geq d > \text{cohdim}(M)$ . Hence the term  $H^{N-1}(D(\xi)) = H^{N-1}(M)$  of the sequence is zero as well.

Thus the map  $j^* : H^{N-1}(M_{\Delta(r)}^{*r}) \rightarrow H^{N-1}(S(\xi))$  is surjective. Therefore the composition  $(j \circ i)^* : H^{N-1}(M_{\Delta(r)}^{*r}) \rightarrow H^{N-1}(S(F))$  is surjective as well. We apply the Borel construction functor  $E\mathbb{Z}_r \times_{\mathbb{Z}_r} (-) \rightarrow B\mathbb{Z}_r$  to this map and apply Leray–Serre spectral sequences; see Figure 3.



**Fig. 3:** We associate to the map  $S(F) \xrightarrow{j \circ i} M_{\Delta(r)}^{*r}$  the Borel constructions and spectral sequences to deduce that  $M_{\Delta(r)}^{*r}$  contains a non-zero element in dimension  $N$ .

At the  $E_2$ -pages, the generator  $z$  of  $H^{N-1}(S(F))$  has a preimage  $w$  since  $(j \circ i)^*$  is surjective. At the  $E_N$ -pages  $(j \circ i)^*(d_N(w)) = d_N(z)$ , which is non-zero by (5). Hence  $d_N(w) \neq 0$ , which is an element in the kernel of the edge-homomorphism  $H^*(B\mathbb{Z}_r) \rightarrow H_{\mathbb{Z}_r}^*(M_{\Delta(r)}^*)$ .

Therefore, the index of  $M_{\Delta(r)}^{*r}$  contains a non-zero element in dimension  $N$ . This completes the proof of Theorem 1.2 if  $f$  can be extended to  $\Delta^N$ .  $\square$



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