

Shortest path poset of Bruhat intervals

Saúl A. Blanco

► **To cite this version:**

Saúl A. Blanco. Shortest path poset of Bruhat intervals. Bousquet-Mélou, Mireille and Wachs, Michelle and Hultman, Axel. 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011), 2011, Reykjavik, Iceland. Discrete Mathematics and Theoretical Computer Science, DMTCS Proceedings vol. AO, 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011), pp.191-198, 2011, DMTCS Proceedings. <hal-01215085>

HAL Id: hal-01215085

<https://hal.inria.fr/hal-01215085>

Submitted on 13 Oct 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Shortest path poset of Bruhat intervals

Saúl A. Blanco[†]

Department of Mathematics, Cornell University, Ithaca, NY, USA

Abstract. Let $[u, v]$ be a Bruhat interval and $B(u, v)$ be its corresponding Bruhat graph. The combinatorial and topological structure of the longest u - v paths of $B(u, v)$ has been extensively studied and is well-known. Nevertheless, not much is known of the remaining paths. Here we describe combinatorial properties of the shortest u - v paths of $B(u, v)$. We also derive the non-negativity of some coefficients of the complete **cd**-index of $[u, v]$.

Résumé. Soit $[u, v]$ un intervalle de Bruhat et $B(u, v)$ le graphe de Bruhat associé. La structure combinatoire et topologique des plus longs chemins de u à v dans $B(u, v)$ est bien comprise, mais on sait peu de chose des autres chemins. Nous décrivons ici les propriétés combinatoires des plus courts de chemins de u à v . Nous prouvons aussi que certains coefficients du **cd**-indice complet de $[u, v]$ sont positifs.

Keywords: Bruhat interval, shortest-path poset, complete **cd**-index

1 Introduction

While the paths of the Bruhat graph $B(u, v)$ of the Bruhat interval $[u, v]$ only depend on the isomorphism type of $[u, v]$ (see [Dye91]), all of the u - v paths of $B(u, v)$ are needed to compute the \tilde{R} -polynomial, as well as the complete **cd**-index of $[u, v]$. Unfortunately, the structure of $B(u, v)$ is not easy to understand. Thus we focus on the shortest paths of $B(u, v)$, since their combinatorial structure is more manageable. In particular, they form a Hasse diagram of a poset, which we denote by $SP(u, v)$.

The order of the paper is as follows: In Section 2 we summarize the basic properties of $SP(u, v)$, and describe their structure two specific cases: (i) if W is finite, with $u = e$ and $v = w_0^W$ (longest-length element of W) and (ii) if the number of rising chains (under a reflection order) is one. In Section 2.3 we provide an algorithm that allows us to separate the chains in $SP(u, v)$ into subposets, each of which has properties resembling properties of $[u, v]$. In Section 3 we derive consequences of the work done to the complete **cd**-index.

1.1 Basic definitions

Let (W, S) be a Coxeter system, and let $T \stackrel{\text{def}}{=} T(W) = \{wsw^{-1} : s \in S, w \in W\}$ be the set of *reflections* of (W, S) . The *Bruhat graph* of (W, S) , denoted by $B(W, S)$ or simply $B(W)$, is the directed graph with vertex set W , and a directed edge $w_1 \rightarrow w_2$ between $w_1, w_2 \in W$ if $\ell(w_1) < \ell(w_2)$ and there exists $t \in T$ with $tw_1 = w_2$. Here ℓ denotes the *length function* of (W, S) . The edges of $B(W)$ are labeled

[†]The author was supported in part by NSF grant DMS-0555268.

by reflections; for instance the edge $w_1 \rightarrow w_2$ is labeled with t . The Bruhat graph of an interval $[u, v]$, denoted by $B(u, v)$, is the subgraph of $B(W)$ obtained by only considering the elements of $[u, v]$. A *path* in the Bruhat graph $B(u, v)$, will always mean a *directed* path from u to v . As it is the custom, we will label these paths by listing the edges that are used. Furthermore, we denote the set of paths of length k in $B(u, v)$ by $B_k(u, v)$.

A *reflection order* $<_T$ is a total order of T so that $r <_T rtr <_T rtrtr <_T \dots <_T trt <_T t$ or $t <_T trt <_T trtrt <_T \dots <_T rtr <_T r$ for each Coxeter system $(\langle r, t \rangle, \{r, t\})$ where $r, t \in T$. Let $\Delta = (t_1, t_2, \dots, t_k)$ be a path in $B(u, v)$, and define the *descent set* of Δ by $D(\Delta) = \{j : t_{j+1} <_T t_j\} \subset [k-1]$. If $D(\Delta) = \emptyset$, we say that Δ is rising.

Let $w(\Delta) = x_1 x_2 \cdots x_{k-1}$, where $x_i = \mathbf{a}$ if $t_i < t_{i+1}$, and $x_i = \mathbf{b}$, otherwise. In other words, set x_i to \mathbf{a} if $i \notin D(\Delta)$ and to \mathbf{b} if $i \in D(\Delta)$. Billera and Brenti [BB] showed that $\sum_{\Delta \in B(u, v)} w(\Delta)$ becomes a polynomial in the variables \mathbf{c} and \mathbf{d} , where $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. This polynomial is called the *complete cd-index* of $[u, v]$, and it is denoted by $\tilde{\psi}_{u, v}(\mathbf{c}, \mathbf{d})$. Notice that the complete \mathbf{cd} -index of $[u, v]$ is an encoding of the distribution of the descent sets of each path Δ in the Bruhat graph of $[u, v]$, and thus seems to depend on $<_T$. However, it can be shown that this is not the case. For details on the complete \mathbf{cd} -index, see [BB].

As an example, consider S_3 with generators $s_1 = (1\ 2)$ and $s_2 = (2\ 3)$. Then $t_1 = s_1 <_T t_2 = s_1 s_2 s_1 <_T t_3 = s_2$ is a reflection ordering. The paths of length 3 are: (t_1, t_2, t_3) , (t_1, t_3, t_1) , (t_3, t_1, t_3) , and (t_3, t_2, t_1) , that encode to $\mathbf{a}^2 + \mathbf{ab} + \mathbf{ba} + \mathbf{b}^2 = \mathbf{c}^2$. There is one path of length 1, namely t_2 , which encodes simply to 1. So $\tilde{\psi}_{u, v}(\mathbf{c}, \mathbf{d}) = \mathbf{c}^2 + 1$.

Given a monomial $m \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$, we denote the coefficient of m in $\tilde{\psi}_{u, v}(\mathbf{c}, \mathbf{d})$ by $[m]_{u, v}$. Notice that $[\mathbf{c}^n]_{u, v}$ is the number of rising paths in $B_{n+1}(u, v)$.

2 Shortest path poset

We begin with some basic properties of $SP(u, v)$.

Proposition 2.1 *Let $[u, v]$ be a Bruhat interval, then the undirected edges of the shortest paths of $B(u, v)$ form the Hasse diagram of a poset.*

We point out that in general the edges of paths in $B_k(u, v)$ need not form a Hasse diagram of a poset. Indeed, it is possible to have elements $u \leq x_0 < x_1 < x_2 < x_3 \leq v$ so that $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow x_3$ and $x_0 \rightarrow x_3$ are all in $B(u, v)$.

We call the poset of Proposition 2.1, the *shortest path poset* of $[u, v]$, and we denote it by $SP(u, v)$. Furthermore, the edges of the Hasse diagram of $SP(u, v)$ inherit the labels of the corresponding edges in $B(u, v)$. In particular, we say that a maximal chain C in $SP(u, v)$ is *rising* if the path corresponding to C in $B(u, v)$ is rising.

Proposition 2.2 *$SP(u, v)$ is a graded poset, and for $x \in SP(u, v)$, the rank of x is the length of the shortest u - x path in $B(u, x)$.*

To illustrate the definition consider B_2 and $SP(e, \underline{1}\underline{2})$ as depicted in Figure 1. Notice that the rank of $SP(e, \underline{1}\underline{2})$ is 2, the length of the shortest paths in $B(B_2)$.

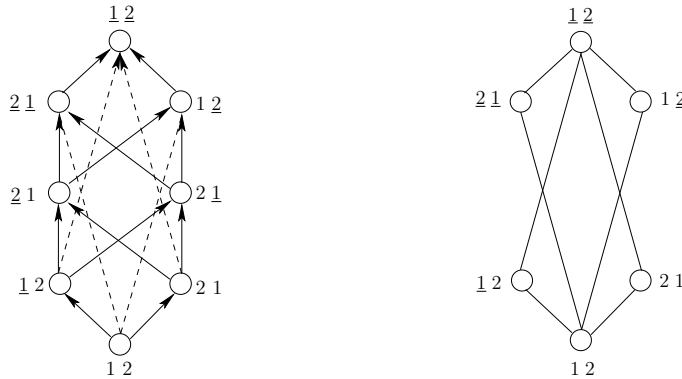


Fig. 1: $B(B_2)$ and $SP(B_2)$.

2.1 Finite Coxeter groups

For any finite Coxeter group W , there is a word w_0^W of maximal length. It is a well-known fact that $\ell(w_0^W) = |T|$. For any $w \in W$, one can write $t_1 t_2 \cdots t_n = w$ for some $t_1, t_2, \dots, t_n \in T$. If n is minimal, then we say that w is T -reduced, and that the absolute length of w is n . We write $\ell_T(w) = n$.

Notice that for $w \in W$, if $\ell_T(w) = m$, then $t_1 t_2 \cdots t_m = w$ for some reflections in T , but this does not mean that (t_1, t_2, \dots, t_m) is a (directed) path in $B(e, w)$. Nevertheless, for finite W and $w = w_0^W$, (t_1, t_2, \dots, t_m) and any of its permutations $(t_{\tau(1)}, t_{\tau(2)}, \dots, t_{\tau(m)})$, $\tau \in A_{m-1}$, are paths in $B(W)$ (see Theorem 2.3 below).

Let $SP(W)$ denote the poset $SP(e, w_0^W)$. The combinatorial structure of $SP(W)$ was described in [Bla09]. For the sake of completeness, we include the main results therein.

Theorem 2.3 *Let W be a finite Coxeter group and $\ell_0 = \ell_T(w_0^W)$, the absolute length of the longest element of W . Then $SP(W)$ is isomorphic to the union of Boolean posets of rank ℓ_0 . Each copy of $B(\ell_0)$ share at least e and w_0^W*

We summarize the number of Boolean posets that form $SP(W)$ and the rank of $SP(W)$ for each finite Coxeter group in Table 1.

where

$$b_n = 1 + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{j!} \prod_{i=0}^{j-1} \binom{n-2i}{2}$$

and

$$d_m = \frac{1}{\lfloor \frac{m}{2} \rfloor!} \prod_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-2i}{2}$$

where $m = n$ if n is even, and $m = n - 1$ if n is odd.

We point out that the union of the Boolean posets could share more elements than e and w_0^W . For instance, consider $SP(B_3)$ below.

Tab. 1: Finite coxeter groups W , $\text{rank}(SP(W))$, and the number of Boolean posets in $SP(W)$

| W | $\text{rank}(SP(W))$ | $\alpha_W = \#$ of Boolean posets in $SP(W)$ |
|-----------|---|---|
| A_{n-1} | $\lfloor \frac{n-1}{2} \rfloor$ | 1 |
| B_n | n | b_n |
| D_n | n if n is even; $n - 1$ if n is odd | d_n |
| $I_2(m)$ | 2 if m is even; 1 if m is odd | $\frac{m}{2}$ if m is even; 1 if m is odd |
| F_4 | 4 | 24 |
| H_3 | 3 | 5 |
| H_4 | 4 | 75 |
| E_6 | 4 | 3 |
| E_7 | 7 | 135 |
| E_8 | 8 | 2025 |

While some elements other than e and $w_0^{B_3}$ are shared by more than one Boolean poset, each maximal chain belongs to a *unique* Boolean poset.

2.2 One rising chain

Since $[u, v]$ is *EL-shellable* (see [BW82] and [Dye93]), then $[u, v]$ has a unique maximal chain that is rising. So it is reasonable to study the structure of $SP(u, v)$ under the assumption that there is a unique rising chain. Even though this seems to be a strong assumption, there are several examples of Bruhat intervals where $SP(u, v)$ has a unique rising chain; for instance, [21435, 53241].

An important tool in our study are the \tilde{R} -polynomials, defined below.

Definition 2.4 (\tilde{R} -polynomials) Let $s \in S$ so that $\ell(vs) < \ell(v)$. Then define $\tilde{R}_{u,v}(\alpha)$ by

$$\tilde{R}_{u,v}(\alpha) = \begin{cases} \tilde{R}_{us,vs}(\alpha) & \text{if } \ell(us) < \ell(u), \\ \tilde{R}_{us,vs}(\alpha) + \alpha \tilde{R}_{u,vs}(\alpha) & \text{if } \ell(us) > \ell(vs). \end{cases}$$

Dyer [Dye01] provided an interpretation of $\tilde{R}_{u,v}(\alpha)$ in terms of the number of rising paths of $B(u, v)$. Namely,

$$\tilde{R}_{u,v}(\alpha) = \sum_{\substack{\Delta \in B(u,v) \\ D(\Delta) = \emptyset}} \alpha^{\ell(\Delta)}.$$

With this interpretation in mind, we have

Proposition 2.5 $\tilde{R}_{u,y}(\alpha)\tilde{R}_{y,v}(\alpha) \leq \tilde{R}_{u,v}(\alpha)$.

We point out, in passing, that the above proposition generalizes Theorem 5.4, Corollary 5.5 and Theorem 5.6 in [Bre97].

Proposition 2.5 yields the following theorem.

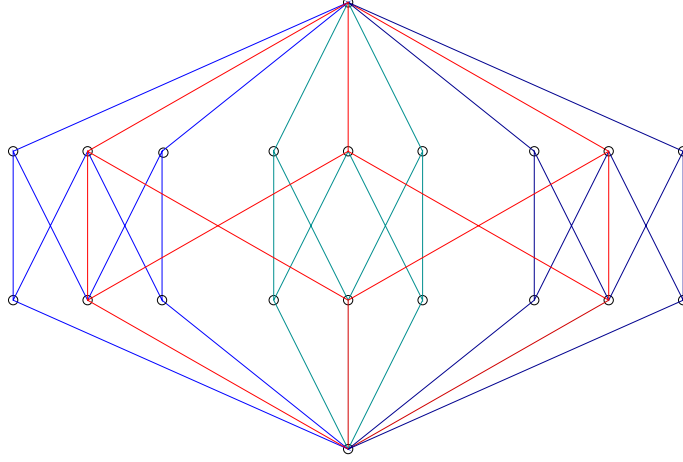


Fig. 2: $SP(B_3)$ has 4 copies of $B(3)$. Notice these copies intersect, but each maximal chain is in a unique Boolean poset.

Theorem 2.6 *If $SP(u, v)$ has a unique rising chain, then*

(a) $SP(u, v)$ is EL-shellable.

(b) $SP(u, v)$ is thin, i.e., every subinterval of length two of $SP(u, v)$ has four elements.

These topological properties will have consequences on the complete \mathbf{cd} -index, and it will be discussed in Section 3.

2.3 FLIP algorithm

Let $k + 1 \stackrel{\text{def}}{=} \text{rank}(SP(u, v))$. An important distinction between $[u, v]$ and $SP(u, v)$ is that $[u, v]$ has a unique maximal, rising chain whereas $SP(u, v)$ could have more than one. So we propose an algorithm that splits the chains of $SP(u, v)$ into $[\mathbf{c}^k]_{u,v}$ posets P_i , $i = 1, \dots, [\mathbf{c}^k]_{u,v}$. The structure of each P_i is easier to understand than $SP(u, v)$. So far we have been shown that the P_i have properties that resemble those of $[u, v]$.

We now follow [BB] to define the *flip* of $\Gamma \in B_2(u, v)$. Let (t_1, t_2) and (r_1, r_2) be in $B_2(u, v)$. We say that $(t_1, t_2) \leq_{lex} (r_1, r_2)$ if $t_1 <_T r_1$ or if $t_1 = r_1$ and $t_2 <_T r_2$, or $t_2 = r_2$. The existence of the complete \mathbf{cd} -index implies that there are as many paths with empty descent set in $B_2(u, v)$ as those with descent set $\{1\}$. Order all the paths in $B_2(u, v)$ lexicographically and let

$$r(\Gamma) = |\{\Delta \in B_2(u, v) : D(\Delta) = D(\Gamma), \Delta \leq_{lex} \Gamma\}|.$$

Definition 2.7 *With everything as above, we define the flip of Γ is the $r(\Gamma)$ -th Bruhat path in $\{\Delta \in B_2(u, v) \mid D(\Delta) \neq D(\Gamma)\}$ ordered by \leq_{lex} . We denote this path by $flip(\Gamma)$.*

Given $\Delta = (t_1, t_2, \dots, t_i, t_{i+1}, \dots, t_k) \in B_k(u, v)$, we denote the path $(t_1, t_2, \dots, t'_i, t'_{i+1}, \dots, t_k)$, where $flip(t_i, t_{i+1}) = (t'_i, t'_{i+1})$, by $FLIP_i(\Delta)$. We are now ready to describe our algorithm.

The pseudocode of FLIP is given in Algorithm 1. In a few words, FLIP returns a (directed) graph G whose vertices are the maximal chains of $SP(u, v)$ and (C, C') is an edge if $FLIP_j(C) = C'$, where

Algorithm 1 FLIP($SP(u, v)$)

$G := (V, E)$, with V is the set of chains of $B(SP(u, v))$ and $E := \emptyset$.
 $T := V$
for C a maximal chain of $SP(u, v)$ **do**
 if $D(C) \neq \emptyset$ **then**
 $i := \min D(C)$
 $C' := \text{FLIP}_i(C)$
 Add edge (C, C') to E .
 end if
end for
return G

$j = \min\{D(C)\}$. Notice that G has $[c^k]_{u,v}$ connected components, say $G_1, G_2, \dots, G_{[c^k]_{u,v}}$. We define P_i to be the poset $SP(u, v)$ with all the chains (represented by vertices) *not* in G_i removed.

Let us illustrate FLIP with the following example. Notice that the chains in $SP(u, v)$ are represented by the labels assigned to the corresponding edges in the $B(u, v)$.

Example 1 Consider the 10 elements of $B_3(1234, 4312)$. Then the output of FLIP is depicted below. In the first column we have the two components of G , and in the right column the posets P_i corresponding to each component.

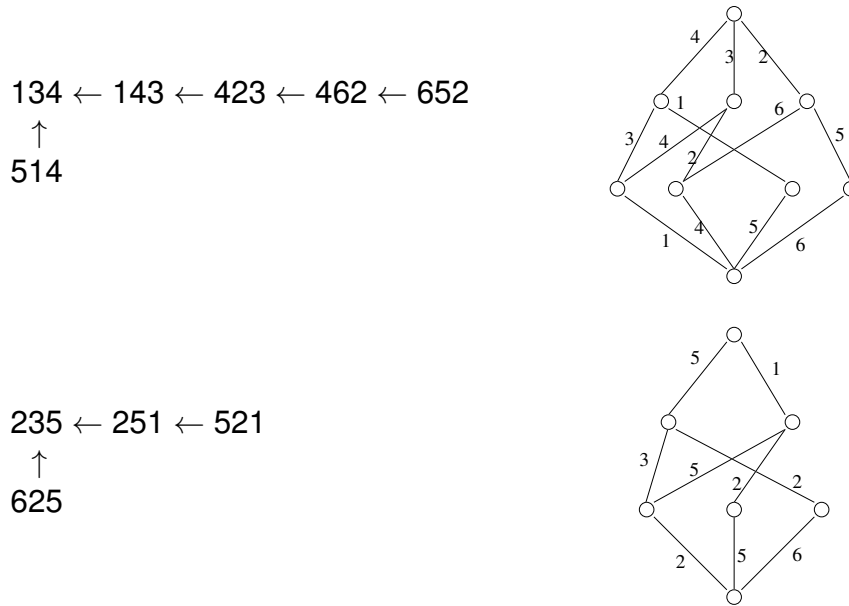


Fig. 3: On the left, we find the output of FLIP: two connected components. On the right the corresponding posets are depicted.

Each P_i satisfies properties resembling those of Bruhat intervals. Concretely, we have

- Proposition 2.8** (a) P_i is graded.
 (b) Every subinterval of P_i has at most one rising chain.
 (c) Every subinterval of length two of P_i has at most two coatoms.

Bruhat intervals satisfy the properties above once we replace “at most” with “exactly”.

2.4 FLIP applied to A_n , B_n and D_n

When applied to A_{n-1} , the output of FLIP is a unique graph G and the corresponding poset P is simply $SP(A_{n-1})$. Furthermore, one can choose a reflection order for the reflections of B_n (see [Bla11]) so that FLIP outputs b_n copies of $B(n)$ (see Table 1). For instance, $\text{FLIP}(SP(B_3))$ separates $SP(B_3)$ into four copies of $B(3)$ (see Figure 2, where the four copies are drawn with different colors). The same holds, mutatis mutandis, for D_n .

So in these cases, FLIP produces the expected results: it divides $SP(W)$ into α_W subposets P_1, \dots, P_{α_W} (where α_W is given in Table 1), and each P_i is a Boolean poset.

3 Connections to the complete cd-index

In [Bla09], it is shown that the lowest-degree terms of $\tilde{\psi}_{e, w_0^W}(\mathbf{c}, \mathbf{d})$ are non-negative. Thus we have the theorem below.

Theorem 3.1 *If W is a finite Coxeter group, then the lowest degree terms of $\tilde{\psi}_{e, w_0^W}(\mathbf{c}, \mathbf{d})$ are nonnegative.*

In fact, these terms can be computed quite easily (see [Bla09] for details).

Now under the assumption of Theorem 2.6, $SP(u, v)$ is EL-shellable and thin. Thus Theorem 3.1.12 in [Wac07] yields the following proposition.

Proposition 3.2 *If $SP(u, v)$ has a unique rising chain, then it is a Gorenstein* poset.*

Now as a consequence of [Kar06, Theorem 4.10], we have the following theorem.

Theorem 3.3 *If $SP(u, v)$ has a unique rising chain, then the lowest degree terms of $\tilde{\psi}_{u, v}(\mathbf{c}, \mathbf{d})$ are non-negative.*

Moreover, in the case $\text{rank}(SP(u, v)) = 2$, the posets P_i described before Example 1 contribute a non-negative quantity to the lowest degree terms of $\tilde{\psi}_{u, v}(\mathbf{c}, \mathbf{d})$. We hope to extend this result to $\text{rank}(SP(u, v)) = 3$.

Acknowledgements

I want to thank Louis Billera for many helpful discussions, as well as kind advice and encouragement throughout the years. I also want to thank Matthew Dyer for his hospitality during my visit to Notre Dame and his willingness to discuss his work.

References

- [BB] Louis J. Billera and Francesco Brenti. Quasisymmetric functions and Kazhdan-Lusztig polynomials. *Israel J. Math.*
- [Bla09] Saúl A. Blanco. Shortest path poset of finite coxeter groups. *DMTCS Proceedings*, 0(01), 2009.
- [Bla11] Saúl A. Blanco. *Shortest path poset poset and the complete cd-index*. PhD thesis, Cornell University, 2011.
- [Bre97] Francesco Brenti. Combinatorial expansions of Kazhdan-Lusztig polynomials. *J. London Math. Soc. (2)*, 55(3):448–472, 1997.
- [BW82] Anders Björner and Michelle Wachs. Bruhat order of Coxeter groups and shellability. *Adv. in Math.*, 43(1):87–100, 1982.
- [Dye91] Matthew J. Dyer. On the “Bruhat graph” of a Coxeter system. *Compositio Math.*, 78(2):185–191, 1991.
- [Dye93] Matthew J. Dyer. Hecke algebras and shellings of Bruhat intervals. *Compositio Math.*, 89(1):91–115, 1993.
- [Dye01] Matthew J. Dyer. On minimal lengths of expressions of Coxeter group elements as products of reflections. *Proc. Amer. Math. Soc.*, 129(9):2591–2595 (electronic), 2001.
- [Kar06] Kalle Karu. The cd -index of fans and posets. *Compos. Math.*, 142(3):701–718, 2006.
- [Wac07] Michelle L. Wachs. Poset topology: tools and applications. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 497–615. Amer. Math. Soc., Providence, RI, 2007.