# Controlling the Katz-Bonacich Centrality in Social Network: Application to gossip in Online Social Networks

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**Abstract**

Recent papers studied the control of spectral centrality measures of a network by manipulating the topology of the network. We extend these works by focusing on a specific spectral centrality measure, the Katz-Bonacich centrality. The optimization of the Katz-Bonacich centrality using a topological control is called the Katz-Bonacich optimization problem. We first prove that this problem is equivalent to a linear optimization problem. Thus, in the context of large graphs, we can use state of the art algorithms. We provide a specific applications of the Katz-Bonacich centrality minimization problem based on the minimization of gossip propagation and make some experiments on real networks.

## I. Introduction

Centrality measures are valuable key measures to understand Social Networks. The most famous ones are the degree centrality, the betweenness centrality, the closeness centrality and finally the eigenvector centrality (see [18]). The popularity of these measures comes from the fact that they are concerned with links between a given node to the overall network (a micro perspective) as opposed to the diameter of a graph, the small world property, etc. (macro perspective). In particular spectral centrality measures of a node depends on the whole topology of the network. This is one of the main differences between spectral centrality measures and the others. Among the huge number of applications of spectral centralities, we will mention two works as example. In [6], the authors characterize the optimal targeted marketing strategies in a Social Network by a spectral centrality measure. The second recent example of application is [2], where authors use a spectral centrality measure to find the delinquent who, once sent in jail, decreases to the maximum the population of delinquency profiles.

The control of the spectral centrality score associated to each node, by modifying the graph topology, is a recent problem. To the best of our knowledge the very first paper on this topic, concerns the maximization of the PageRank, where a webmaster controls multiple pages and hyperlinks between them (see related work of [15]). In the framework of delinquency, the paper [2] proposes to model a network of delinquents with communications between them and design an algorithm to characterize the link to remove that will minimize the overall rate of delinquency. In [25] the authors try to find the minimum set such as the vector of score is fully controlled.

There are several spectral centrality measures: the eigenvector centrality, the alpha centrality, the Katz-Bonacich centrality, the PageRank[1] and the subgraph centrality (see [14]). We propose to restrict ourselves to the Katz-Bonacich centrality. Indeed in this case we are able to derive an efficient algorithm to control this centrality by choosing which nodes to remove. The score given by the Katz-Bonacich centrality to a node is based on a discounted sum of the walks that initially has starting from it. There are applications in Social Network that use it. For instance, in the gossip in social network described, the authors use the Katz-Bonacich centrality to characterize the key node. In other contexts, the Katz-Bonacich centrality has been used in a pricing context [7], to characterize the Nash equilibrium of a Cournot competition over a network [5] and in other economic and social network applications (see [3], [4], [18] and [27]).

The focus in this paper is on deriving an efficient algorithm to optimize the Katz-Bonacich centrality by removing nodes. We prove that it is equivalent to a linear programming formulation and that there is a polynomial solution algorithm. Thus algorithms from [12] can be used in the context of large graphs. Once the linear programming characterization is provided our goal is to propose an application of the control of the Katz-Bonacich Centrality based on the minimization of a gossip process over an online Social Network.

After a short state-of-the-art section, we introduce in section III the Katz-Bonacich centrality optimization problem. We recall the mathematical definition of Katz-Bonacich centrality. We then prove the equivalence between with a linear programming problem. In section IV, we apply the Katz-Bonacich centrality minimization problem to control of a gossip process over a Online Social Network [2]. Finally we compute the solution the Katz-Bonacich centrality minimization problem on real networks in section V.

## II. Related works

The control of the spectral centrality has been studied from different points of view:

**Topology Control.** Lately, the question of how to modify the topology of the network in order to control the centrality score of nodes became a subject of interest.

In [25], an algorithm is proposed to find the minimum controlling centrality subset of nodes in a complex network in the particular case of the eigenvector centrality. The authors propose to only control interactions generated by nodes inside this

[1] see [18] for definitions of the above spectral centrality measures

subset.
In [15] the authors noticed the links between the optimization of the PageRank and the ergodic Markov decision problem. Based on it, they provide an efficient algorithm to optimize the PageRank.

**Key node.** Another question has been also investigated on how does the centrality score of nodes evolve when a node is removed from the network.
In [2], [1], the authors highlight the equivalence between the delinquency effort level and the Katz-Bonacich centrality. Then they propose to find which delinquent has to be in jail in order that the global delinquent profile decreases to the maximum.
In [22], [26], the authors investigate the control of the eigenvalue centrality. Their study is based on the theory of control of linear system and the well-known Kalman's controllability rank condition (see [19]). They propose to find which node can control the whole system based on results from [17].

## III. Control of the Katz-Bonacich centrality

As previously mentioned, the focus of this study is to provide an efficient algorithm to solve the Katz-Bonacich centrality optimization problem. We first recall the definition of the Katz-Bonacich centrality measure. We then, formally, describe the Katz-Bonacich centrality optimization problem.

### *A. The Katz-Bonacich centrality optimization problem*

We begin our analysis by recalling the definition of the Katz-Bonacich centrality. In one of his work [20], made during the 50s, Katz proposed to model the centrality or the prestige of a node in a network in the following manner: the score associated to a node is based on a discounted sum of the walks that initially has started from it. Nowadays, this centrality measure is called Katz-Bonacich centrality because Bonacich proposed in [8] a similar spectral centrality measure. We next provide a formal definition of it.

The social network $\mathscr{G}(\mathscr{I},E)$ is composed of a set $\mathscr{I}:=\{1,\ldots,I\}$ of nodes and the interactions between them is described by an communication matrix $E$. The variable $e_{ij}\in[0,1]$ denotes the $ij^{th}$ entry of $E$ and represents the relative frequency of interaction between node $i\in\mathscr{I}$ and node $j\in\mathscr{I}$. Moreover, for each $i$, we assume that $\sum_j e_{ij}=1$ expect when a node does not communicate with any other nodes and so $\sum_j e_{ij}=0$. The definition of the Katz-Bonacich centrality [20], is given by:

*Definition 1:* Let $\rho\in[0,1]$. The Katz-Bonacich centrality associated to $\mathscr{G}(\mathscr{I},E)$, denoted by $\mathbf{x}^*(E,\rho)\in\mathbb{R}_+^I$, is given by:

$$\mathbf{x}^*(E,\rho):=\sum_{n=0}^{\infty}(\rho E)^n\mathbf{1}_I, \tag{1}$$

where $\mathbf{1}_I$ is the all ones vector of size $I$. Thus the Katz-Bonacich centrality is the limit, as $t$ goes to infinity (whenever it exists), of the following deterministic process, where for each $t\in\mathbb{N}^*$:

$$x_i(t+1)=1+\sum_j\rho e_{ij}x_j(t),\forall i. \tag{2}$$

Moreover if the Perron Frobenius eigenvalue (see [24]), $\lambda_{max}(\rho E)$, associated to the matrix $\rho E$ is smaller than one, then:

$$\mathbf{x}^*(E,\rho)=(Id_I-\rho E)^{-1}\mathbf{1}_I, \tag{3}$$

where $Id_I$ is the identity matrix of size $I\times I$.

The problem we are interested in is the minimization (or the maximization) of the Katz-Bonacich centrality by removing nodes, in other words by controlling the communication matrix $E$. The variable $p_i\in[\underline{p}_i,\overline{p}_i]$ denotes the probability to not remove a node $i$ and $(1-p_i)$ the probability to remove it. Thus the Katz-Bonacich centrality is now in expectation, for each $i$, the limit $\lim_{t\to\infty}\mathbf{E}[x_i(t)]$ (which exists if $\lambda_{max}(\rho E)<1$) of the following dynamical system:

$$\begin{aligned}\mathbf{E}[x_i(t+1)] &= \mathbf{E}[1+\sum_j\rho e_{ij}x_j(t)] & (4)\\ &= p_i(1+\sum_j\rho e_{ij}\mathbf{E}[x_j(t)]). & (5)\end{aligned}$$

Moreover we assume that the Katz-Bonacich cannot be lower that a certain level in some regions of the graph. Let $c\in\{1,\ldots,C\}$ denote a particular region of the graph. Let $N(c)\subset\mathscr{I}$ the subset of nodes that belongs to the region $c$. For each $c$ and $c'$, we assume that $N(c)\cap N(c')=\emptyset$. For each region $c$, the constraint over Katz-Bonacich centrality is:

$$\sum_{j\in N(c)}x_j^*\geq\phi_c, \tag{6}$$

where $\phi_c\geq 0$. These region constraints comes from the fact it is not possible to remove all the nodes of the graph. The Katz-Bonacich centrality minimization problem is therefore defined below:

*Definition 2:* For each $i$, let $p_i \in [\underline{p}_i, \overline{p}_i]$ denotes the probability to remove node $i$ and $\mathbf{p} := (p_1, \ldots, p_I)$ the associated vector. Let $\mathbf{P} := \Pi_i [\underline{p}_i, \overline{p}_i]$ the set of constraints. Let $\rho \in [0,1]$. Let $E$ a sub-stochastic matrix and $\lambda_{max}(\rho E) < 1$. Let $\phi := (\phi_1, \ldots, \phi_C)$ the region constraints. The Katz-Bonacich centrality minimization problem associated to $(E, \rho)$ is defined as:

$$\min_{\mathbf{p}\in\mathbf{P}} \sum_i x_i^*(\mathbf{p}), \tag{7}$$

where for each i, $x_i^*$ is the unique solution of:

$$x_i^*(\mathbf{p}) = p_i(1 + \sum_j \rho e_{ij} x_j^*(\mathbf{p})), \tag{8}$$

such that for each $c$,

$$\sum_{j\in N(c)} x_j^*(\mathbf{p}) \geq \phi_c. \tag{9}$$

### B. Characterization by a linear program

Our main result is to provide an equivalence between the Katz-Bonacich centrality minimization problem and a linear program. Our goal is to first compute, for a given centrality, a closed form of the control that allows us to reach it. Then given a particular matrix $E$ and a scalar $\rho$ we will describe the set of feasible centralities. Finally we will prove the equivalence between the Katz-Bonacich centrality minimization with a linear program. The region constraints will only appear in the formulation of the linear program. We will not use it before.

*Proposition 1:* Let $\mathbf{k} \in \mathbb{R}_+^I$, $E$ and $\rho$. If for each $i$,

$$\underline{p}_i \leq \frac{k_i}{1 + \sum_j \rho e_{ij} k_j} \leq \overline{p}_i, \tag{10}$$

let us define

$$p_i^* = \frac{k_i}{1 + \sum_j \rho e_{ij} k_j}, \forall i \tag{11}$$

then $x_i^* = k_i$ for all $i$, where $x_i^*$ is solution of (8) for each $i$.

*Proof:* If

$$\underline{p}_i \leq \frac{k_i}{1 + \sum_j \rho e_{ij} k_j} \leq \overline{p}_i, \tag{12}$$

then $p_i^* \in [\underline{p}_i, \overline{p}_i]$. This is the reason why $p_i^*$ exists. Moreover for each $i$

$$x_i^* = p_j^*(1 + \sum_j \rho e_{ij} x_j^*) \tag{13}$$

$$x_i^* = \frac{k_i}{1 + \sum_j \rho e_{ij} k_j}(1 + \sum_j \rho e_{ij} x_j^*). \tag{14}$$

This linear system admit a unique solution. And it is easy to check that the unique solution is $x_i^* = k_i$:

$$x_i^* = \frac{k_i}{1 + \sum_j \rho e_{ij} k_j}(1 + \sum_j \rho e_{ij} k_j) = k_i. \tag{15}$$

■

Now we are interested to undersand the set of feasible centralities, in other word to characterize the following set:

$$\mathbf{F} := \left\{ \mathbf{k} \in \mathbf{R}_+^I \mid \exists \mathbf{p} \in \mathbf{P} \text{ such that } \mathbf{k} \text{ sol. of (8)} \right\}. \tag{16}$$

The next proposition characterize this set:

*Proposition 2:* Let $E$ and $\rho$.

$$\mathbf{F} = \left\{ \mathbf{k} \mid \forall i,\, k_i \geq 0,\, \underline{p}_i \leq \frac{k_i}{1 + \sum_j \rho e_{ij} k_j} \leq \overline{p}_i, \right\}. \tag{17}$$

*Proof:* According to the proposition 1, it exists $\mathbf{p}$ such that $x_i^* = k_i$ if for each $i$:

$$\underline{p}_i \leq \frac{k_i}{1 + \sum_j \rho e_{ij} k_j} \leq \overline{p}_i. \tag{18}$$

■

Now according to the two previous propositions we can deduce that the Katz-Bonacich centrality minimization problem is equivalent to:

$$\min_{\mathbf{k}} \sum_i k_i \tag{19}$$

subject to for each $i$ and each $c$:

$$\underline{p}_i \left(1 + \sum_j \rho e_{ij} k_j\right) \quad \leq \quad k_i, \tag{20}$$

$$\overline{p}_i \left(1 + \sum_j \rho e_{ij} k_j\right) \quad \geq \quad k_i, \tag{21}$$

$$\sum_{j \in N(c)} k_j \quad \geq \quad \phi_c, \tag{22}$$

$$0 \quad \leq \quad k_i. \tag{23}$$

Once this linear program is solved the associated control is the one proposed in proposition 1. Because it exists a linear program equivalent to the Katz-Bonacich centrality minimization problem, we can use algorithm proposed in [12] for large graphs.

## IV. Which nodes to remove in a gossip process

The problem we are interested in is the minimization of the propagation of a gossip by removing nodes. We call this problem the Gossip Minimization problem. As proposed in [4] the owner of the Online Social Network (the controller) can block some nodes in the Social Network. For instance, by sending a warning to the friends of a user, the control can disturb communication between users. A more precise exemple, still in the context of Online Social Networks, messages that appear in subscribers' News Feed[2] are control by a content curation algorithm [13], thus the controller can reduce interactions between users. We propose to generalize the model proposed in [4], under realistic considerations.

### A. Interest of Users: A first model

Let $\mathscr{I} := \{1, \ldots, I\}$ the set of users and $i \in \mathscr{I}$ is a user index. A user $i$ of a social network gets news, about the gossip according to a Poisson point process of intensity $\lambda_i \in \mathbb{R}_+$. Thus when an news arrival occurs, concerning the gossip, the probability that it is for user $i$ is:

$$\frac{\lambda_i}{\sum_j \lambda_j}.$$

Let $t_n \in \mathbb{R}_+$ the arrival rate of the $n$-th message. When a user $i$ receives a news, its increases his belief in the gossip. Thus the interest of an average user about the gossip at time $t_n$, is described by a random variable $Y_i(n) \in \mathbb{R}_+$. Let $x_i(n) := \frac{Y_i(n)}{n}$ the time average interest of user $i$ at time $t_n$. A user $i$ updates his interest in the following way: When he receives a news about the gossip at time $t_n$, he increases of one the interest related to it. Thus, for each $i$ the evolution of $Y_i$ is described by:

$$Y_i(n+1) = Y_{i,c}(n) + \zeta_i(n),$$

where the interest update is modeled by:

$$\zeta_i(n) := \begin{cases} 1 & \text{w.p} \quad \dfrac{\lambda_i}{\sum_j \lambda_j}, \\ 0 & \text{w.p} \quad 1 - \dfrac{\lambda_i}{\sum_j \lambda_j}. \end{cases} \tag{24}$$

### B. Imitation between users: a network extension

As an extension of the previous model, we consider that interests imitation can occurs between users. Let $P \in [0,1]^{I \times I}$ the imitation matrix, where the $ij$-entry of $P$, $p_{i,j}$, is the probability that a user $i$ imitates interest of user $j$. For each $i$, assume $\sum_j p_{i,j} = 1$. The time instant when user $i$ decides to imitate one of his neighbors is modeled by a Poisson point process of intensity $\alpha_i \in \mathbb{R}_+$. An event is now the arrival of messages or the activation of a user who wants to imitate someone. When an event occurs, the probability that it is the imitation phase of user $i$ is $\frac{\alpha_i}{\sum_j \lambda_j + \alpha_j}$. At event $n$, when user $i$ imitates user $j$, he will add one to $Y_i(n)$ with probability $x_j(n)$. Thus the new version of $\zeta(n) := [\zeta_1(n), \ldots, \zeta_I(n)]$, associated to the evolution of $Y = [Y_1, \ldots, Y_I]$, is described below:

[2]https://www.facebook.com/help/210346402339221

$$\zeta_i(n) = \begin{cases} 1 & \text{w.p} \quad P_i(\mathbf{x}(n)) := \dfrac{\lambda_i}{\sum_j \lambda_j + \alpha_j} + \alpha_i \dfrac{\sum_j p_{i,j} x_j(n)}{\sum_j \lambda_j + \alpha_j} \\ 0 & \text{w.p} \quad 1 - P_i(\mathbf{x}(n)). \end{cases} \tag{25}$$

### C. Stability Analysis

Following the theory developped in [9], the next theorem provides a sufficient condition about the convergence of the sequence $\{\mathbf{x}^n\}$.

*Theorem 1:* [9] Let $\mathbf{x}^0 \in [0,1]^{C \times I}$ denote the initial conditions. If the matrix $A$ defined as follows,

$$a_{i,j} := \frac{\alpha_i p_{i,j}}{\sum_{i',c''} \lambda_{i',c''} + \alpha_{i'}} - 1_{i=j},$$

is irreducible then the sequence $\{\mathbf{x}(n)\}$ converges almost surely to an unique rest point $\mathbf{x}^* \in [0,1]^{C \times I}$. Moreover

$$x_c^* = B \Lambda \lambda_c \ \forall c, \tag{26}$$

where $\Lambda\lambda_c := [\Lambda\lambda_{1,c}, \ldots, \Lambda\lambda_{I,c}]$, $\Lambda := \frac{1}{\sum_{j,c''} \lambda_{j,c''} + \alpha_j}$ and $B := A^{-1}$.

It is interesting to note that the rest point of the stochastic approximation (25) is the Katz Bonacich centrality of the graph defined by the following matrix $C$, where the $ij$-entry is given by

$$c_{i,j} := \frac{\alpha_i p_{i,j}}{\sum_{i'} \lambda_{i'} + \alpha_{i'}}.$$

**Control description.** In the Gossip minimization problem, the controller can decide which users to block. More precisely, he can reduce the impact of user $i$ over his neighbors using a control $p_i \in [\underline{p}_i, \overline{p}_i]$, such that the interest of each user $i$ is solution of the following linear system:

$$x_i^* = p_i\Big(\frac{\lambda_i}{\sum_j \lambda_j + \alpha_j} + \alpha_i \frac{\sum_j p_{i,j} x_j^*}{\sum_j \lambda_j + \alpha_j}\Big). \tag{27}$$

Moreover because the controller cannot fully control the interest of each user, the sum of the interests of each user cannot be lower that a certain level:

$$\sum_j x_j^* \geq \phi, \tag{28}$$

where $\phi > 0$.

**Utility description.** In the present paper, we assume that the controller wishes to minimize a utility vector depending on the interest of each user. Then the Gossip minimization problem is:

$$\min_{\mathbf{p} \in \mathbf{P}} U(\mathbf{x}^*). \tag{29}$$

where for each i, $x_i^*$ is the unique solution of:

$$x_i^* = p_i\Big(\frac{\lambda_i}{\sum_j \lambda_j + \alpha_j} + \alpha_i \frac{\sum_j p_{i,j} x_j^*}{\sum_j \lambda_j + \alpha_j}\Big), \tag{30}$$

subject to

$$\sum_j x_j^* \geq \phi. \tag{31}$$

We shall be interested in reducing the overall interest, in other words:

$$U(\mathbf{x}) := \sum_i x_i^*. \tag{32}$$

Finally according the previous section III-B, the Gossip minimization problem is equivalent to the following linear programming:

$$\min_{\mathbf{k}} \sum_i k_i \tag{33}$$

| Network | $I$ | $\overline{d}$ | $\sum_i x_i^*$ | $\sum_i x_i^*(\mathbf{p})$ |
|---|---|---|---|---|
| Zachary's karate club [28] | 34 | 9.176471 | 37.77778 | 20 |
| Miserables [21] | 77 | 13.19481 | 85.55556 | 20 |
| Network of American college football games [16] | 115 | 21.32174 | 127.7778 | 20 |
| Dolphin social network [23] | 62 | 10.25806 | 68.88889 | 20 |

TABLE I: Summary table

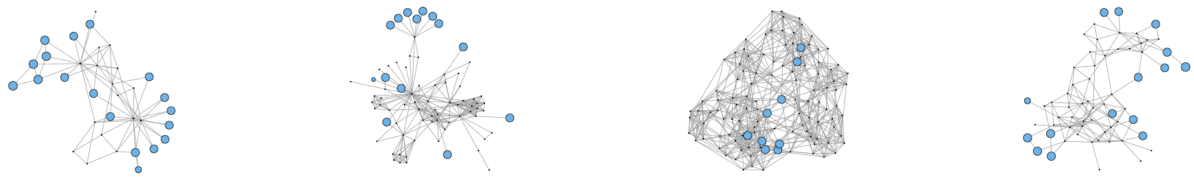

Fig. 1: Visualization of the solution of the Katz-Bonacich centrality minimization problem

subject to for each $i$ and each $c$:

$$\underline{p}_i\left(\frac{\lambda_i}{\sum_j \lambda_j+\alpha_j}+\alpha_i\frac{\sum_j p_{i,j}k_j}{\sum_j \lambda_j+\alpha_j}\right) \leq k_i, \tag{34}$$

$$\overline{p}_i\left(\frac{\lambda_i}{\sum_j \lambda_j+\alpha_j}+\alpha_i\frac{\sum_j p_{i,j}k_j}{\sum_j \lambda_j+\alpha_j}\right) \geq k_i, \tag{35}$$

$$\sum_j k_j \geq \phi, \tag{36}$$

$$0 \leq k_i. \tag{37}$$

It is easy to generalize to a convex utility function. Indeed, as $U(\mathbf{x_i}^*)$ is convex in $\mathbf{x_i}^*$, we can use classical convex optimization algorithm [10] to solve this problem.

## V. Simulations

In this section we propose to compute over different networks the solution of the Katz-Bonacich centrality minimization problem. We study it over 4 different topologies depicted in fig. 1. The summary of experimental results are described in table I. The first column provides the number of nodes, the second column the average degree ($\overline{d}$). We propose to study each network with its normalized adjacency matrix $E$, where the $ij$-entry is given by: $e_{ij} := \frac{a_{ij}}{\sum_j a_{ij}}$, where $A$ is the real adjacency matrix. For each network, the Katz-Bonacich centrality is computed with $\rho = 0.1$. The centrality average value of each network is given in the third column. We applied the Katz-Bonacich centrality minimization problem with, $C = 1$, $\phi_C = 20$ and for each $i$, $\underline{p}_i = 0.1$ and $\overline{p}_i = 1$. The fourth column describes the utility of the Katz-Bonacich centrality minimization problem. The first observation is that for each network, the constraint $\sum_i x_i^* \geq 20$, is saturated. It follows from the fact that, for each network, the aggregated Katz-Bonacich centrality is bigger than 20. We expect that if we increase $\underline{p}_i$ this constraint will not be saturated anymore. When we look at fig. 1, where the size of the node is proportional to the solution of the Katz-Bonacich centrality minimization problem ($x_i^*$ for each $i$), we remark that we do not have a trivial solution like only one node is active or each node has the same centrality.

## VI. Conclusion and Perspectives

The main result of this paper is the equivalence between a linear programming problem and the Katz-Bonacich centrality minimization problem. Once this equivalence is proved, we also describe an application to the control of gossip on Online Social Network.

There is one major follow-up of this work. The extension is to study the Katz-Bonacich centrality minimization problem in an online context, in other words when the structure of the social network changes over the time. The formulation of the problem could be the following: The evolution of the interaction occurs each unit time $t \in \mathbb{N}$. Let $E(t)$ the interaction matrix at time $t$. For each $t$ we assume that $E(t)$ a substochastic matrix. The index $p_i(t) \in [0,1]$ denotes the node that the controller will perturbates at time $t$ between node $i$. The online version of the Katz-Bonacich centrality minimization problem is defined as follow:

**Known parameters:**. Set of nodes $\mathscr{I}$.

For each round $t = 1, 2, \ldots$

1) The controller removes nodes by using a control $\mathbf{p}(t) := (p_1(t), \ldots, p_I(t))$,
2) simultaneously, the adversary select a matrix $E(t)$ of interaction, $\rho(t)$ and $\alpha(t)$,
3) The controller observe $E(t)$, and received

$$\mathbf{1}_I^T (Id_I - \rho(t)E(t))^{-1}\mathbf{1}_I \alpha(t),$$

as an instantaneous vector payoffs.

In order to solve this problem we can use the theory of online convex optimization techniques [11].

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